Eigenstates of a small Josephson junction coupled to a resonant cavity

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We carry out a quantum-mechanical analysis of a small Josephson junction coupled to a single-mode resonant cavity. We find that the eigenstates of the combined junction-cavity system are strongly entangled only when the gate voltage applied at one of the superconducting islands is tuned to certain special values. One such value corresponds to the resonant absorption of a single photon by Cooper pairs in the junction. Another special value corresponds to a *two-photon* absorption process. Near the single-photon resonant absorption, the system is accurately described by a simplified model in which only the lowest two levels of the Josephson junction are retained in the Hamiltonian matrix. We noticed that this approximation does not work very well as the number of photons in the resonator increases. Our system shows also the phenomenon of "collapse and revival" under suitable initial conditions, and our full numerical solution agrees with the two level approximation result.

I. INTRODUCTION

Circuits involving small Josephson junctions have the potential to behave as macroscopic two-level systems which can be externally controlled. Indeed, several recent experiments have demonstrated that such a system can be placed in a coherent superposition of two macroscopic quantum states. The experiments have involved small superconducting loops, and also so-called Cooper pair boxes. In the former case, it was shown experimentally[1,2] that the loop could exist in a coherent superposition of two macroscopic states of different flux through the loop. In the case of the Cooper pair box, the system was placed into a superposition of two states having different numbers of Cooper pairs on one of the superconducting islands.[3–6] Such a coherent superposition had been proposed by several authors.[7,8]

One possible application of such two-state systems is as a quantum bit (qubit) for use in quantum computing.[9] At sufficiently low temperatures, this Josephson qubit will have little dissipation, and hence may be coherent for a reasonable length of time, as is required of a qubit. But in order for it to be potentially useful in computation, it must be possible to prepare entangled states of two such qubits—that is, states of the two qubits which cannot be expressed simply as product states of the two individual qubits.

Such entangled states have now been created and manipulated in systems involving atoms in a high-Q cavity.[10] Only recently has there been any theoretical study of entangled states involving Josephson junction circuits. Shnirman *et al.*[11] have considered low-capacitance Josephson junctions coupled to an inductor-capacitor (LC) circuit in the two level approximation. Buisson and Hekking[12] have studied the states of a single Cooper pair box coupled to the photon mode in an electromagnetic resonant cavity. Both Refs. 11 and 12 considered junctions such that the charging energy is much larger than the Josephson energy. Everitt *et al.*[13] have carried out a similar investigation of the states of a small superconducting quantum interference device (a SQUID—i.e., a small superconducting loop) coupled to a resonant cavity. In all of these studies, entangled states of the photon and the Josephson junction emerged naturally from the quantum-mechanical analysis.

In the present paper, we carry out a full quantum-mechanical investigation of a model for a Josephson junction coupled to a single-mode resonant photonic cavity. Our model Hamiltonian is, as we demonstrate here, equivalent to that studied in Ref. 12, via a canonical transformation. However, we consider a different parameter region where the charging and Josephson energy of the junction are comparable in magnitude. Also we go beyond the analysis of that paper by including a large number of eigenstates of both the Josephson and the photon Hamiltonian, in calculating the energy levels of the coupled system. When the system is at resonance, we find that the entangled states are indeed, as expected, well described by a basis of two states of the combined system. We also find a variety of entangled states, depending on the value of the control parameter (an offset voltage), including, at certain values, an eigenstate which is a superposition of two states in which the number of photons differs by 2. We have also verified that our system shows the phenomenon of "collapse and revival" similar to that seen in quantum optics.[14] Our approach is similar to that applied by Everitt *et al.*[13] to a SQUID ring in a resonant cavity, though our Hamiltonian is not equivalent.

The remainder of this paper is organized as follows. In Sec. II, we give the model Hamiltonian, present several of its properties, and describe our method of solving it. The following section gives numerical results obtained from the model. The last section contains a concluding discussion, an estimate for some of the parameters used in our model, and a description of possible problems for future work.

II. MODEL

A. Model Hamiltonian

We consider an underdamped Josephson junction in a large-Q electromagnetic cavity which can support a single photon mode. The system is assumed to be described by the following model Hamiltonian:

$$\mathcal{H} = \mathcal{H}_{\text{photon}} + \mathcal{H}_J + \mathcal{H}_C. \tag{1}$$

Here $\mathcal{H}_{\text{photon}}$ is the Hamiltonian of the cavity mode, which we express in the form $\mathcal{H}_{\text{photon}} = \hbar \omega(a^\dagger a + 1/2)$, where $a^\dagger$ and a are the usual photon creation and annihilation operators. $\mathcal{H}_J$ is the Josephson coupling energy, which we write in the form $\mathcal{H}_J = -J \cos \gamma$, where J is the Josephson energy of the junction and γ is the gauge-invariant phase across the junction (defined more precisely below). J is related to I_c, the critical current of the junction, by $J = \hbar I_c /(2|e|)$. Finally, $\mathcal{H}_C$ is the capacitive energy of the junction, which we write as $\mathcal{H}_C = \frac{1}{2} U(n - \bar{n})^2$, where $U = 4e^2/C$ is the charging energy of the junction with capacitance C, n is an operator which represents the difference between the number of Cooper pairs on the two superconducting islands which make up the junction, and $\bar{n}$ is a tunable experimental parameter related to the gate voltage which is applied at one of the superconducting islands.

The gauge-invariant phase difference γ is the term which couples the Josephson junction to the cavity. It may be written as $\gamma = \phi - (2\pi/\Phi_0)\int \mathbf{A} \cdot dl \equiv \phi - A$, where $\Phi_0 = hc/(2e)$ is the flux quantum, ϕ is the phase difference across the junction in a particular gauge, $\mathbf{A}$ is the vector potential, and the line integral is taken across the junction. We assume that $\mathbf{A}$ arises from the electromagnetic field of the cavity normal mode. In Gaussian units, and assuming the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$), this vector potential takes the form $\mathbf{A} = \sqrt{hc^2/\omega V}(a + a^\dagger)\hat{\epsilon}$, where $\hat{\epsilon}$ is the unit polarization vector of the cavity mode and V is the volume of the cavity. Here we assumed that the junction dimensions are much smaller than the wavelength of the resonant cavity mode, so that the cavity electric field is approximately uniform within the junction. Given this representation for $\mathbf{A}$, the phase factor A can be written as $A = (g/\sqrt{2})(a + a^\dagger)$, where

$$g = \frac{4\sqrt{\pi}|e|}{\sqrt{\hbar \omega V}} l_\parallel \tag{2}$$

and $l_\parallel = \hat{\epsilon} \cdot \vec{l}$ is the effective length across the junction parallel to $\hat{\epsilon}$. The definition of the Hamiltonian is completed by writing down the commutation relations for the various operators. The only nonzero commutators are $[a, a^\dagger] = 1$, and $[n, \phi] = -i$.

It is useful to make a change of variables $p = i\sqrt{\hbar\omega/2}(a^\dagger - a); q = \sqrt{\hbar/(2\omega)}(a^\dagger + a)$. Then p and q satisfy the canonical commutation relation $[p, q] = -i\hbar$, and $\mathcal{H}_{\text{photon}}$ can be expressed in a form which makes clear that it represents the Hamiltonian of a harmonic oscillator

$$\mathcal{H}_{\text{photon}} = \frac{1}{2}(p^2 + \omega^2 q^2). \tag{3}$$

In terms of these new variables, the gauge-invariant phase difference is given by $\gamma = \phi - g\sqrt{\omega/\hbar}q$.

The Hamiltonian $\mathcal{H}$ can now be expressed conveniently as the sum of three parts:

$$\mathcal{H} = \mathcal{H}_{\text{JJ}} + \mathcal{H}_{\text{photon}} + \mathcal{H}_{\text{int}}, \tag{4}$$

where

$$\mathcal{H}_{\text{JJ}} = \frac{1}{2}U(n - \bar{n})^2 - J \cos \phi \tag{5}$$

and

$$\mathcal{H}_{\text{int}} = -J(\cos \gamma - \cos \phi). \tag{6}$$

In the absence of $\mathcal{H}_{\text{int}}$ the Hamiltonian is the sum of two independent parts, corresponding to the Josephson junction and the resonant photon mode. $\mathcal{H}_{\text{int}}$ couples these two together.

B. Method of solution

It is convenient to diagonalize $\mathcal{H}$ in a complete basis consisting of the direct product of the eigenfunctions of $\mathcal{H}_{\text{JJ}}$ and $\mathcal{H}_{\text{photon}}$. The eigenfunctions of $\mathcal{H}_{\text{photon}}$ are, of course, harmonic oscillator eigenstates; the nth eigenstate has wave function

$$h_n(q) = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}} \exp(-y^2/2) H_n(y), \tag{7}$$

where $H_n(y)$ is a Hermite polynomial of order n, and $y = (\omega/\hbar)^{1/2}q$. An eigenfunction $\psi(\phi)$ of $\mathcal{H}_{\text{JJ}}$ with eigenvalue $E_J(\bar{n})$ can be written as $\psi(\phi) = \exp(i\bar{n}\phi)\eta(\phi)$, and satisfies the Schrödinger equation $\mathcal{H}_{\text{JJ}}\psi(\phi) = E_J\psi(\phi)$. This equation can be written out explicitly using the representation $n = -i(d/d\phi)$, which follows from the commutation relation between n and ϕ, as

$$\frac{d^2 Y(x)}{dx^2} + \left(\frac{8E_J}{U} + 2Q \cos 2x \right) Y(x) = 0, \tag{8}$$

where $x = \phi/2$, $Y(x) = \eta(\phi/2)$, and $Q = 4J/U$. This is a Mathieu equation with characteristic value $a = 8E_J/U$, and potential of strength Q. The eigenvalues E_J are determined by the requirement that the wave function, $\psi(\phi)$, should be single valued, i.e., that $\psi(\phi) = \psi(\phi + 2\pi)$, or equivalently $Y(x + \pi) = \exp(-2i\bar{n}\pi)Y(x)$. We therefore define $Y_k(x)$ as a Floquet solution of the Mathieu equation (8), with Floquet exponent $\nu = 2k - 2\bar{n}$, where $k = 0, \pm 1, \pm 2, \ldots$. We denote the corresponding eigenvalue of $\mathcal{H}_{\text{JJ}}$ by $E_{J,k}(\bar{n})$. For $0 < \bar{n} < 0.5$, the lowest eigenvalue corresponds to $k = 0$, followed in order by $k = 1, -1, 2, -2, \ldots$. The eigenvalues of $\mathcal{H}_{\text{JJ}}$ are periodic in $\bar{n}$ with period unity, and are also symmetric about $\bar{n} = 1/2$.

Finally, any eigenstate $\Psi(\phi, q)$, of the Hamiltonian $\mathcal{H}$ can be expressed as a linear combination of product wave functions consisting of eigenfunctions of $\mathcal{H}_{\text{JJ}}$ and $\mathcal{H}_{\text{photon}}$:

$$\Psi(\phi, q) = \sum_{k,n} A_{kn} \psi_k(\phi) h_n(q), \tag{9}$$

where $\psi_k(\phi) = \exp(i\bar{n}\phi)Y_k(\phi/2)$ and A_{kn} are the expansion coefficients. The only term in $\mathcal{H}$ which is not diagonal in this product basis is the interaction term $\mathcal{H}_{\text{int}}$. The eigenfunc-

tions and eigenvalues of the Hamiltonian of Eq. (4) follow from the Schrödinger equation $\mathcal{H}\Psi(\phi,q)=E\Psi(\phi,q)$.

C. Canonical transformation

Before proceeding to the solutions of this Schrödinger equation, we show that $\mathcal{H}$ is equivalent, via a canonical transformation, to another Hamiltonian which has sometimes been discussed in the literature.[11,12] A similar transformation has been demonstrated in Ref. 11 for the case of one and two junctions. The required transformation is as follows:

$$\begin{cases} \phi'=\phi-g\sqrt{\omega/\hbar}q, \\ n'=n, \\ p'=p+g\sqrt{\hbar\omega}n, \\ q'=q. \end{cases} \quad (10)$$

It is readily found that the new operators satisfy the commutation relations $[\phi',n']=i$, $[q',p']=i\hbar$, with all other commutators vanishing. The Hamiltonian $\mathcal{H}$ can readily be expressed in terms of the new variables. The result is

$$\mathcal{H}'=\frac{1}{2}U'(n'-\bar{n}')^2-J\cos\phi'+\frac{1}{2}(p'^2+\omega^2q'^2)$$

$$-g\sqrt{\hbar\omega}p'n'+\frac{U\hbar\omega\bar{n}^2g^2}{2(U+\hbar\omega g^2)}, \quad (11)$$

where $\bar{n}'=U\bar{n}/(U+\hbar\omega g^2)$ and $U'=U+\hbar\omega g^2$.

If we interpret the photonic mode as an LC resonator, then this Hamiltonian describes a junction which is *capacitively* coupled to that mode. Obviously, this transformed Hamiltonian will have the same spectrum of eigenvalues as the original one. Although it is not obvious from the form of Eq. (11), we have numerically confirmed that the spectrum is symmetric with respect to $\bar{n}=1/2$, as is implied by the original Hamiltonian.

III. NUMERICAL RESULTS

We have diagonalized $\mathcal{H}$ [Eq. (4)] numerically, using the basis discussed in Sec. II B. We used a truncated basis which is composed of the direct product of the lowest five energy levels of the junction ($k=0,\pm1,\pm2$), and the lowest ten of the electromagnetic field ($n=0,1,\ldots,9$). We have also confirmed numerically that increasing the number of states in this basis has little effect on the eigenvalues at least up to $E=6U$.

Figure 1 shows the calculated eigenvalues of $\mathcal{H}$, plotted as a function of $\bar{n}$ up to an energy of U, for coupling constants $g=0$, $g=0.15$ and $g=1.5$. In all three cases, we have arbitrarily chosen $Q=0.7$ and $\hbar\omega/U=0.3$, corresponding to a case where the charging energy, Josephson energy, and quantum $\hbar\omega$ of radiation energy are all comparable in magnitude. The degeneracy of some of the energy levels in Fig. 1(a) is broken in Fig. 1(b) (the numerical value of the second and third energy levels splitting at $\bar{n}=0.258$ is given in the caption); the degeneracy is more noticeably broken in Fig. 1(c),
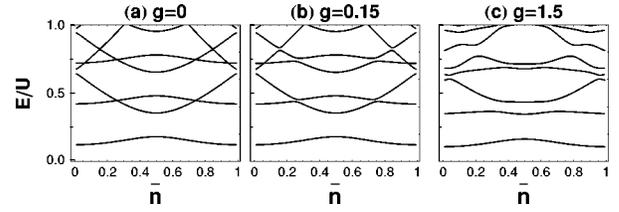

FIG. 1. Lowest eigenvalues E/U of the system described by the Hamiltonian (6), calculated as a function of $\bar{n}$ for the parameters $\hbar\omega/U=0.3$ and $Q=0.7$, and (a) $g=0$, (b) $g=0.15$, and (c) $g=1.5$. All eigenvalues are shown up to $E/U=1$. Note in particular the breaking of the degeneracy of the second and third energy levels around $\bar{n}=0.26$, seen clearly in (c). The separation ΔE between these levels at $\bar{n}=0.258$ is $\Delta E/U=0$, 0.01, and 0.06 in (a), (b), and (c), respectively.

where the coupling constant g is much larger. The degeneracy breaking is, of course, caused by the perturbation $\mathcal{H}_{\text{int}}$. Moreover, the magnitude of the energy splitting also tends to increase with increasing g. This behavior is a characteristic feature of "level repulsion" predicted by standard degenerate perturbation theory in quantum mechanics.

Next, we discuss the time evolution of the system described by Hamiltonian (4), given its state at time $t=0$. The expectation value of any operator $\mathcal{O}$ at time t is

$$\langle\mathcal{O}(t)\rangle=\text{Tr}[\rho(t)\mathcal{O}], \quad (12)$$

where $\rho(t)$ is the density matrix of the system at time t. $\rho(t)$ is given by $\rho(t)=U(t)\rho(0)U^\dagger(t)$, where $U(t)=\exp(-i\mathcal{H}t/\hbar)$ is the evolution operator and $\rho(0)$ is determined by the initial state of the system. The time average of the operator $\mathcal{O}$ is then given by

$$\langle\langle\mathcal{O}\rangle\rangle=\lim_{T\to\infty}\frac{1}{T}\int_0^T\text{Tr}[\rho(t)\mathcal{O}]dt, \quad (13)$$

where the inner and outer set of triangular brackets denote, respectively, a quantum mechanical and a time average.

As an illustration, we have calculated $\langle\langle\mathcal{H}_{\text{JJ}}\rangle\rangle$ and $\langle\langle\mathcal{H}_{\text{photon}}\rangle\rangle$. For each operator, we carried out the calculation making the arbitrary assumption that the state of the system at time $t=0$ is $|\alpha=i\sqrt{2};k\rangle$. Here $|\alpha=i\sqrt{2}\rangle$ is a coherent state of the electromagnetic field, i.e., an eigenstate of the annihilation operator, a such that $a|\alpha\rangle=\alpha|\alpha\rangle$. (We have confirmed numerically that the time averages are unchanged if the initial state is an eigenstate of the number operator $a^\dagger a$.) $|k\rangle$ is an eigenstate of $\mathcal{H}_{\text{JJ}}$ with quantum number k, and we have considered two different initial states, corresponding to $k=0$ and $k=1$. The density matrix $\rho(0)$ at time $t=0$ is easily calculated once the initial state is specified; in obvious notation it is $\rho(0)=|\alpha=i\sqrt{2};k\rangle\langle\alpha=i\sqrt{2};k|$. We have calculated the time evolution using the parameters $Q=0.7$, $\hbar\omega/U=0.3$, and $g=0.15$.

The time-averaged energy $\langle\langle\mathcal{H}_{\text{JJ}}\rangle\rangle$ for these parameters is shown in Fig. 2(a) for $k=0$ and $k=1$. Each curve shows a strong structure near $\bar{n}=0.26$ (and $\bar{n}=0.74$). At these values
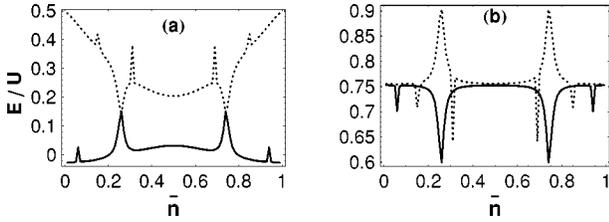
FIG. 2. Time and quantum-mechanical average of the operator $\mathcal{H}_{JJ}$ (a), and of $\mathcal{H}_{photon}$ (b), plotted as a function of $\bar{n}$ for two different assumptions about the initial state of the system: $|\alpha = i\sqrt{2}; k=0\rangle$ (solid curves) and $|\alpha = i\sqrt{2}; k=1\rangle$ (dashed curves). The states are defined explicitly in the text. In both cases, we used the parameter values $\hbar\omega/U=0.3$, $Q=0.7$, and $g=0.15$.

FIG. 3. (a) Time-dependent value of $\langle\mathcal{H}_{JJ}\rangle$ (full curve) and of $\langle\mathcal{H}_{photon}\rangle$ (dashed curve) for $\bar{n}=0.258$. The system is prepared at time $t=0$ in state $|n=1; k=0\rangle$. The other parameters are $Q=0.7$, $\hbar\omega/U=0.3$, and $g=0.15$. (b) Same as (a) except that we show the time-dependent probability for finding the Josephson junction (full curve) and the photonic resonator (dashed curve) in their first excited states.

of $\bar{n}$, according to Fig. 1, the difference in energy between the ground state energy, and the first excited energy is $\sim 0.3U=\hbar\omega$. The structure thus corresponds, we believe, to a *resonant absorption* of a photon by a Cooper pair in the junction—that is, at this value of $\bar{n}$, Cooper pairs of electrons move from the ground state $|k=0\rangle$ of the junction to the first excited state $|k=1\rangle$. The corresponding structure for $k=1$ (dotted line) arises from stimulated emission, in which a Cooper pair falls from the excited state to the ground state of the junction with the emission of a photon. The exchange of energy between the junction and the resonator can also be seen in Fig. 2(b), where we show the time-averaged energy contained in the photon mode $\langle\langle\mathcal{H}_{photon}\rangle\rangle$ for the same two initial states. $\langle\langle\mathcal{H}_{photon}\rangle\rangle$ for both states is close to the unperturbed value $(\langle n\rangle + 1/2)\hbar\omega = 5\hbar\omega/2$, which is the energy contained in the photon field for the coherent state $|\alpha = i\sqrt{2}\rangle$, but one increases sharply, while the other decreases, near the resonance at $\bar{n}=0.26$ (and $\bar{n}=0.74$).

The curves in Fig. 2 also show weaker structure near $\bar{n} \approx 0.06$ (and near 0.94). This structure corresponds to a process in which *two photons* are absorbed by the junction Cooper pairs. Specifically, the junction is excited from its ground state to its second excited state at this value of $\bar{n}$, with a consequent loss of two photons from the electromagnetic field. This is discussed further below.

We have also studied the time evolution of this system under conditions such that $\bar{n}$ is fixed at a value of 0.258, near the principal resonance mentioned above. To see the time evolution, we prepared the system at time $t=0$ in the state $|n=1; k=0\rangle$, for which the photon resonator is in state $n=1$ and the junction in state $k=0$. We then allowed the system to evolve in time according to the evolution operator $U(t)=\exp(-i\mathcal{H}t/\hbar)$. In Fig. 3 we show the resulting time-dependent expectation value of $\langle\mathcal{H}_{JJ}\rangle$, and of $\langle\mathcal{H}_{photon}\rangle$. Also shown in the same figure are the time-dependent probabilities for finding the junction in state $k=1$ at time t and the resonator in state $n=1$ at time t, given this initial state. Clearly, the system is oscillating between the states $|n=1; k=0\rangle$ and $|n=0; k=1\rangle$. This energy exchange is periodic in time: both $\langle\mathcal{H}_{JJ}\rangle$ and $\langle\mathcal{H}_{photon}\rangle$ vary periodically in time, with a period $T_1 \approx 675\hbar/U$ for our particular choice of Q/U and $\hbar\omega/U$. In the first half of this period, Cooper pairs absorb energy from the electromagnetic field and are driven
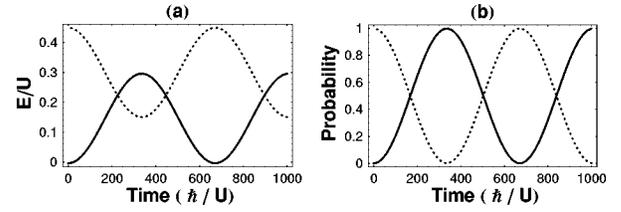
into the first excited state, while in the second half, they lose energy to the electromagnetic field and fall back into the ground state. In the present model, this process continues indefinitely. In a more realistic model which includes dissipation, these oscillations would gradually decay in time.

Note also that the existence of a periodic oscillation between one state and another is not dependent on the choice of the initial state. For example, if the initial state is chosen as $|n=0; k=1\rangle$, the junction would interact with the vacuum fluctuations of the electromagnetic field, and would periodically visit the state $|n=1; k=0\rangle$. We have confirmed this behavior numerically in our model.

In Fig. 4, we show $\langle\mathcal{H}_{JJ}\rangle$ and $\langle\mathcal{H}_{photon}\rangle$ as a function of time for two different initial states $|n=1; k=0\rangle$ (full line) and $|n=1; k=1\rangle$ (dashed line). (The first of these is also shown in Fig. 3.) Evidently both states oscillate periodically in time, but with slightly different periods.

Figure 5 shows the analogous time evolution of a system prepared in a fixed initial state at $\bar{n}=0.061$, where we expect to see a signature of a two-photon absorption. In this case, we prepared the system in the initial state $|n=2; k=0\rangle$. The system oscillates between the $k=0$ (ground state) and $k=-1$ (second excited state) of the Josephson junction, which is separated from the ground state by an energy $2\hbar\omega$ at this value of $\bar{n}$. The number of photons in the system also oscillates periodically in time, between the values $n=2$ and $n=0$, with period $T_2 \approx 7500\hbar/U$. This period is about eleven times longer than the period at $\bar{n}=0.258$ (single-photon ab-
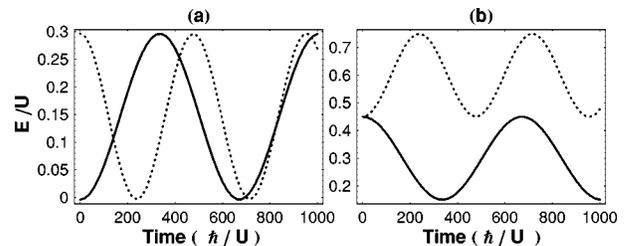

FIG. 4. (a) Time-dependent value of $\langle\mathcal{H}_{JJ}\rangle$ for two different initial states $|n=1; k=0\rangle$ (full curve) and $|n=1; k=1\rangle$ (dashed curve). In both cases, $\bar{n}=0.258$, $Q=0.7$, $\hbar\omega/U=0.3$, and $g=0.15$. (b). Same as (a) except we plot $\langle\mathcal{H}_{photon}\rangle$.
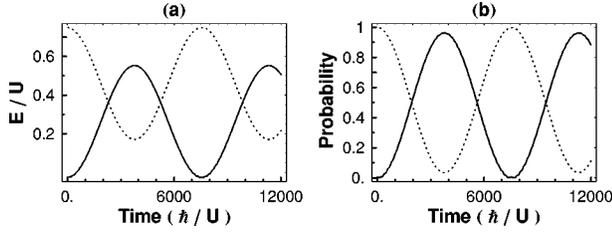
FIG. 5. Same as Fig. 3, except that $\bar{n}$ is fixed at 0.061, and the system is prepared initially in the state $|n=2;k=0\rangle$ at time $t=0$. Other parameters are the same as in Fig. 3.

sorption), because the matrix element connecting the two states at $\bar{n}=0.06$ is much smaller.

IV. DISCUSSION

The problem studied here closely resembles that of a two-level atom interacting quantum-mechanically with a monochromatic electromagnetic field. Specifically, when $\bar{n}$ is such that the field is approximately in resonance with a transition between the lowest and first excited state of the Josephson junction, then it is a good approximation to consider only four states in the basis. We now show that this approximation gives results in good agreement with those obtained using the full direct product basis.

We consider a basis constructed out of the two lowest energy states of the junction, i.e., for $k=0$, 1 and of the electromagnetic field, for $n=0$, 1. Using a four-dimensional basis formed from the direct product of these states $|n=0,1;k=0,1\rangle$, we can express the total Hamiltonian of Eq. (4) as a four times four-dimensional matrix. If we retain in the basis only the two states $|0;1\rangle$ and $|1;0\rangle$ which are approximately degenerate, the Hamiltonian can be further reduced to the 2×2 matrix

$$\mathcal{H}=\epsilon_{0,1}|0;1\rangle\langle0;1|+\epsilon_{1,0}|1;0\rangle\langle1;0|+\xi_1|0;1\rangle$$
$$\times\langle1;0|+\xi_1^*|1;0\rangle\langle0;1|. \tag{14}$$

The diagonal matrix elements take the values $\epsilon_{0,1}=E_{J,1}(0.258)+\hbar\omega/2$; $\epsilon_{1,0}=E_{J,0}(0.258)+3\hbar\omega/2$, where $E_{J,k}(\bar{n})$ is the kth eigenenergy of $\mathcal{H}_{\mathrm{JJ}}$ at $\bar{n}$. ξ_1 is given by $\xi_1=\langle0,1|\frac{1}{2}\mathcal{H}_{\mathrm{int}}|1,0\rangle$, where $\mathcal{H}_{\mathrm{int}}$ is given by Eq. (6). To first order in the coupling strength g, $\mathcal{H}_{\mathrm{int}}\approx-Jg\sqrt{(\omega/\hbar)}q\sin\phi$, and hence, to the same order in g, $\xi_1\approx-(Jg/\sqrt{8})\langle k=0|\sin\phi|k=1\rangle$. A similar two-level approximation can be carried out for coupling between the states $|n;1\rangle$ and $|n-1;0\rangle$; in this case, the coupling energy is found to be $\xi_1\sqrt{n}$. However, as expected, we have found that the two-level approximation becomes progressively more inaccurate as n is increased.

To illustrate this approximation, we consider the case where the energy levels $\epsilon_{0,1}$ and $\epsilon_{1,0}$ are degenerate. Diagonalizing the reduced Hamiltonian matrix, Eq. (14), gives us the two energy levels $E_{\pm}=\epsilon_0\pm|\xi_1|$. These levels correspond to two entangled photon-Josephson states. The Rabi period of oscillation between them is $T_1=\pi\hbar/|\xi_1|$. Using the pa-
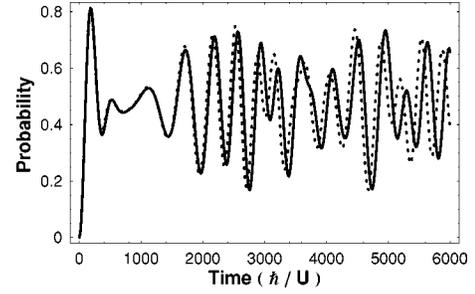

FIG. 6. Time dependent probability that the junction is in the first excited state given that initially the resonator is in the coherent state $|\alpha=i\sqrt{3}\rangle$ and the junction in the ground state. Also shown (dotted line) is the probability $\mathcal{P}(t)$ [Eq. (16)] calculated using the two level approximation. Here we used $\bar{n}=0.258$, $Q=0.7$, and $\hbar\omega/U=0.3$.

rameters and wave functions corresponding to our calculation of Fig. 3, we obtain $T_1\approx667\hbar/U$, which is in excellent agreement with the value found from the full numerical solution ($675\hbar/U$), based on the complete basis and shown in Fig. 3. We conclude that indeed the reduced Hamiltonian, defined in Eq. (14), is an excellent description of the system, at least when the two lowest levels are close to resonance.

The two-photon process discussed in Fig. 6 can also be understood using the reduced Hamiltonian:

$$\mathcal{H}=\epsilon_{2,0}|2;0\rangle\langle2;0|+\epsilon_{0,-1}|0;-1\rangle\langle0;-1|+\xi_2|2;0\rangle$$
$$\times\langle0;-1|+\xi_2^*|0;-1\rangle\langle2;0|. \tag{15}$$

Here we consider only the two degenerate levels with energy, $\epsilon_{2,0}=E_{J,0}(0.061)+5\hbar\omega/2$; $\epsilon_{0,-1}=E_{J,-1}(0.061)+\hbar\omega/2$. The coupling strength ξ_2 is given by $\xi_2=\langle2,0|\frac{1}{2}\mathcal{H}_{\mathrm{int}}|0,-1\rangle$, where $\mathcal{H}_{\mathrm{int}}$ is given by Eq. (6). To second order in the coupling strength g, $\mathcal{H}_{\mathrm{int}}\approx-(1/2)Jg^2(\omega/\hbar)q^2\cos\phi$, and hence, to the same order in g, $\xi_2\approx-(Jg^2\sqrt{2}/8)\langle k=-1|\cos\phi|k=0\rangle$. Using the same parameters as before, we find that the Rabi oscillation period is $T_2=\pi\hbar/|\xi_2|\approx5800\hbar/U$, compared to the value $7500\hbar/U$ obtained from the exact numerical solution. Thus the two-level approximation is not quite as good as in the one-photon case, at least for these parameters.

As mentioned in the Introduction, our Hamiltonian model shows the phenomenon of "collapse and revival," provided that the resonator is prepared initially in a coherent state.[14] This behavior arises from the interference between the different Rabi oscillations which exist when the initial state is a superposition of different photon number states. In Fig. 6. we show the time dependent probability that the junction is in the excited state $k=1$ when the initial state of the system is $|\alpha=i\sqrt{3};k=0\rangle$. Also on the same figure we show (dotted line) the same probability calculated using the reduced Hamiltonian of Eq. (14):[14]

$$\mathcal{P}(t)=\frac{1}{2}-\frac{1}{2}e^{-|\alpha|^2}\sum_{s=0}^{\infty}\frac{|\alpha|^{2s}}{s!}\cos(2\xi_1\sqrt{s}t). \tag{16}$$

The agreement between the two results is very good, especially for short times. For longer times or for larger α, the agreement between the two level approximation and the full numerical solution is less good, primarily because the number of photons in the cavity becomes large.

Before concluding, we make some order-of-magnitude estimates for some of the parameters entering our model, and compare them with those in plausible junctions and cavities. First, the parameter g may be estimated using Eq. (2). If we arbitrarily assume a cylindrical cavity of radius r and length d, the frequency of the lowest mode is approximately $2c/r$, and may be either a TE or TM mode, depending on the ratio d/r. Substituting into Eq. (2), we obtain $g \sim l_\parallel \sqrt{8/(137rd)}$ $\sim 0.24 l_\parallel / \sqrt{rd}$, where $e^2/(\hbar c) \approx 1/137$ is the fine structure constant. To estimate $\hbar \omega/U$, we use the same expression for $\omega \approx 2c/r$ and $U = 4e^2/C$, where C is the junction capacitance. Taking $C = x$, where x is a characteristic length, we obtain $\hbar \omega/U \sim 137 x/r$. Thus, a value of 0.3 for this parameter implies $x/r \sim 0.002$ or less. For a capacitance of 10^{-16} F, readily achievable within current technology,[11] we have $x \sim 10^{-3}$ cm, or $U \sim 1 \times 10^{-15}$ erg ~ 8 K, and hence $r \sim 0.5$ cm. The associated frequency would be $\omega \sim 100$ GHz, which is in the range used in some recent experiments.[15] Of course the quantum of radiation is limited by the superconducting gap Δ of the superconductor used. We also note that our choice $Q = 0.7$ corresponds to $J \sim 3$ K for the above parameters, or equivalently, a critical current I_c of about 0.1 μA. The value of J should be of the same order as a quantum of radiation $\hbar \omega$.

Using the above estimates, and assuming a realistic junction with an insulating layer of thickness $l \sim 100$ nm, the coupling strength g will be $g \sim 10^{-5}$ if $r \sim d \sim 0.5$ cm. If we make $d \ll r$, as in a disklike microcavity, we can presumably increase g substantially, but it will still remain much below the value of 0.15 used in our calculations. Such a value of $g = 0.15$, or even larger, is preferable because this would make the Rabi period of the same order or less than the coherence time in Josephson junctions. In a high-Q cavity, the dissipation-induced decoherence time may be as large as 2 ns,[5] much smaller than would be produced by $g \sim 10^{-5}$. A larger g would decrease the Rabi time, and technological improvements will no doubt increase the junction coherence time, making these times more comparable. Another way to increase the effective coupling, and hence decrease the Rabi period, would be to use many identical junctions located in a region small compared to a wavelength. The effective coupling is then found to increase as the square root of the number of junctions in the cavity.[16] On the basis of all these estimates, we conclude that the two-level approximation tested in this paper should be excellent for the range of parameters likely to be achievable experimentally. Of course, the precise relevant parameters for a given experiment depend on the details of the geometry, which have only been very crudely estimated in the above discussion.

It is of interest to compare the present quantum problem to another system which bears some similarities, namely an *array* of Josephson junctions placed in a resonant cavity.[15] This system involves large numbers of junctions, each coupled to a single-mode cavity such as is considered here. While the appropriate Hamiltonian for treating such an array resembles that of the present problem, there are important differences. First, the observed array behavior involves large numbers of photons in the microcavity, whereas the present model is suitable to the case of one or a few photons. Secondly, the *IV* characteristics of the array involve junctions which are biased onto either the resistive or the superconducting branch, whereas the present model considers only the latter branch. Finally, on the basis of the above estimates, the junctions to which the present calculation applies are probably substantially smaller, with smaller critical currents, than those studied in Ref. 15. Thus, a rather different approach, such as those described in Refs. 17–21 is called for.

To summarize, we have considered the interaction between a small Josephson junction and the photon field in a resonant cavity, in the limit when dissipation can be neglected. We find that there is a strong interaction between the junction and the resonant photon mode at special values of the gate voltage, as defined by the variable $\bar{n}$. At such values of $\bar{n}$, the lowest two eigenstates of the system correspond to *entangled states* involving states of both the photon field and the Josephson junction. The Rabi period of oscillation between these states is inversely proportional to a certain coupling strength g. We also found evidence for a two-photon absorption process at another resonant value of the variable $\bar{n}$. Finally, we found that the lowest entangled eigenstates are accurately described by a simple two-level model which is a truncation of the full model Hamiltonian, Eq. (4).

We conclude with a brief discussion of possible improvements in the present calculations. First, of course, the effects of dissipation need to be included. In principle such dissipation can be included by coupling the Hamiltonian degrees of freedom to a bath of harmonic oscillators. If the density of states of this bath has the right frequency dependence, this coupling leads to Ohmic damping.[22–24] Secondly, it would be of great interest to extend this work to more than one junction coupled to the same resonant cavity.[16] If the two-level approximation continues to be accurate in this case, the study of such a group of Josephson junctions would be greatly simplified.

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