Glassy phase in an array of Josephson junctions

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We study the critical behavior of an irrationally frustrated XY model on a square lattice as a model for the Josephson-junction array in an incommensurate magnetic field. Approximate arguments are presented suggesting that the system will exhibit successive orderings at larger and larger length scales with fewer and fewer defects as the temperature is lowered. This low-temperature phase appears to display features very similar to those in the replica-symmetry-breaking formulation of spin glasses, which strongly suggest a glassy phase without any disorder. We propose a Parisi-like order parameter for this state.

Two-dimensional (2D) periodic arrays of coupled Josephson junctions show periodic variation of many properties with external magnetic field.[1] In the high-capacitance limit, these are reasonably well described by uniformly frustrated 2D XY models,[2] in which the commensurate-incommensurate effects due to external magnetic fields are reflected in the gauge-invariant quantity "frustration." A number of studies[2-7] on these models reveal a wide variety of critical behavior as a function of the frustration f $(0 \leqq f < 1)$.

The present paper is concerned with the incommensurate system, which is of particular interest since the rational numbers form only a set of measure zero.[8] Recent work has suggested the interesting possibility of a metastable glassy phase (without any disorder).[6] Numerical simulations indeed seem to favor the existence of such a glassy state.[7] In this Brief Report, we present an approximate analysis of the proposed glassy phase, based on a Landau-Ginzburg-Wilson (LGW) formalism, and we also define an order parameter appropriate to this phase. We suggest that the ordering in the glassy phase may involve a series of quasitransitions involving ordering at larger and larger length scales at successively lower temperatures. We also briefly consider how the ordering might be influenced by the finite sizes inherent in real experimental samples.

We consider a class of uniformly frustrated XY models described by the Hamiltonian (Boltzmann constant $=1$)

$$-H/T=\sum_{\mathbf{r},\mathbf{r}'} K_{\mathbf{rr}'}\cos(\phi_{\mathbf{r}}-\phi_{\mathbf{r}'}-A_{\mathbf{rr}'}) \ ,$$

$$K_{\mathbf{rr}'}=\begin{cases} K \text{ for nearest neighbors} \\ 0 \text{ otherwise} \ , \end{cases} \tag{1a}$$

where $\mathbf{r}$ is the position vector of the site at (x,y) on an $L\times L$ square lattice, T is the temperature, and $A_{\mathbf{rr}'}$ is a bond angle describing uniform frustration. Thus the plaquette sum is constant over the whole lattice, $\sum A_{\mathbf{rr}'}=2\pi f$. In the case of Josephson-junction arrays $A_{\mathbf{rr}'}$ is given by

$$A_{\mathbf{rr}'}=\frac{2\pi}{\Phi_0}\int_{\mathbf{r}}^{\mathbf{r}'}\mathbf{A}\cdot d\mathbf{l} \ , \tag{1b}$$

where $\Phi_0\equiv hc/2e$ is the flux quantum, and the vector potential $\mathbf{A}$ may be taken as that of a uniform transverse magnetic field $\mathbf{B}=B\hat{\mathbf{z}}$ in the limit of large penetration depth. The uniform frustration f is simply the flux per plaquette in units of the flux quantum, i.e.,

$$f=\frac{Ba^2}{\Phi_0} \ . \tag{1c}$$

The lattice constant a will be henceforth set equal to unity. With the Landau gauge $\mathbf{A}=Bx\hat{\mathbf{y}}$,

$$A_{\mathbf{rr}'}=\begin{cases} \pm 2\pi f x \text{ for } \mathbf{r}'=\mathbf{r}\pm\hat{\mathbf{y}} \\ 0 \text{ for } \mathbf{r}'=\mathbf{r}\pm\hat{\mathbf{x}} \ . \end{cases}$$

In the commensurate system, where $f=m/n$ with m and n relatively prime, a unit cell of the Hamiltonian consists of n basic cells (plaquettes).[5] In the incommensurate system, however, the unit cell is infinite in extent along the x direction,[6] and correspondingly the usual periodic boundary conditions cannot be used on a finite $L\times L$ lattice. Instead it is plausible to use a free-end boundary condition along the x direction, while along the y direction either a free-end boundary condition or a periodic boundary condition may be used without affecting the essential physics. The free-end boundary condition imposed along the x direction on the system of finite size L may be regarded as the limiting case of a periodic boundary condition imposed on the system of size $L_0\to\infty$. Thus instead of the original $L\times L$ lattice we consider an $L\times L_0$ square lattice with periodic boundary conditions along both directions. This corresponds to the requirement that q_1 (the x component of the wave vector) as well as the frustration f scale as L_0^{-1} while q_2 (the y component) scales as L^{-1}. The Hamiltonian (1a)–(1c) originally defined on the $L\times L$ lattice with mixed boundary conditions is then regarded as being defined on the $L\times L_0$ lattice with periodic boundary conditions. It is important, however, to note that $K_{\mathbf{rr}'}$ still vanishes unless both the sites $\mathbf{r}$ and $\mathbf{r}'$ are members of the original $L\times L$ lattice.

To obtain the critical modes we consider the Fourier transform of the interaction matrix $P_{\mathbf{rr}'}\equiv K_{\mathbf{rr}'}e^{-iA_{\mathbf{rr}'}}$. This quantity is well defined on an $L\times L_0$ lattice and, to the first order in $\Delta_k\equiv L\delta_k\equiv L(f-f_k)$, takes the form

$$P_{\mathbf{qq}'}\equiv(LL_0)^{-1}\sum_{\mathbf{r},\mathbf{r}'}\exp[-i(\mathbf{q}\cdot\mathbf{r}-\mathbf{q}'\cdot\mathbf{r}')]P_{\mathbf{rr}'}=\sum_k P_{\mathbf{qq}'}^{(k)} \ , \tag{2}$$

where the summation is to be performed over all rational f_k's close enough to f, i.e., $|\Delta_k| < 1$.[9] Except for the shift $q_2 \to q_2 - \pi\Delta_k$, $P^{(k)}_{\mathbf{qq}'}$ is exactly the interaction matrix of the commensurate system with frustration f_k.[5]

Transforming Eq. (2) back to the real space, we obtain the corresponding interaction matrix on the $L \times L_0$ lattice also in the form of a summation. Note that to the leading order this interaction matrix defined on the $L \times L_0$ lattice is equivalent to an effective form on the $L \times L$ lattice[10]

$$P_{\mathbf{rr}'} = \sum_k P^{(k)}_{\mathbf{rr}'} , \tag{3}$$

where $P^{(k)}_{\mathbf{rr}'}$ is given by

$$P^{(k)}_{\mathbf{rr}'} = K_{\mathbf{rr}'} \exp[-i\bar{A}^{(k)}_{\mathbf{rr}'}] ,$$

$$\bar{A}^{(k)}_{\mathbf{rr}'} = \begin{cases} \pm(2\pi f_k x + \pi\Delta_k) \text{ for } \mathbf{r}' = \mathbf{r} \pm \hat{\mathbf{y}} \\ 0 \text{ for } \mathbf{r}' = \mathbf{r} \pm \hat{\mathbf{x}} . \end{cases}$$

In Eq. (3) and hereafter the summation over k implies that over all rational approximants $f_k = m_k/n_k$ such that

$$n_k \ll L \ll |\delta_k|^{-1} . \tag{4}$$

Note that the first inequality in Eq. (4) simply implies that the system size should be sufficiently larger than the size of a unit cell, while the second one is the criterion that the defects $\delta_k = f - f_k$ are negligible since the phase change across the system due to those defects is given by $2\pi L \delta_k$. This observation allows us to come back to the original $L \times L$ lattice. Henceforth we will consider the $L \times L$ lattice on which the interaction matrix in Eq. (3) is defined.

It is tedious but straightforward to derive an effective LGW Hamiltonian from the interaction matrix given by Eq. (3).[5] In the incommensurate case there is no coupling between fluctuations of the critical modes associated with different k's. This results from the fact that the critical modes due to $p^{(k)}$—the critical modes of the kth "level"—are given by

$$\mathbf{Q}_{\alpha_k} = (0, 2\pi f_k \alpha_k + \pi\Delta_k), \ (\alpha_k = 1, 2, \ldots, n_k) . \tag{5}$$

The fact that $\mathbf{Q}_{\alpha_k} \neq \mathbf{Q}_{\alpha_{k'}}$ unless $k = k'$ and that $\mathbf{Q}_{\alpha_k} + \mathbf{Q}_{\alpha_{k'}} \neq \mathbf{Q}_{\alpha_{k''}} + \mathbf{Q}_{\alpha_{k'''}}$ unless $k = k' = k'' = k'''$ leads to the absence of coupling between different levels in the LGW Hamiltonian. A lengthy calculation finally gives the LGW Hamiltonian in the desired form

$$F(\psi) = \sum_k F^{(k)}(\psi^{(k)}) , \tag{6}$$

where $F^{(k)}$ is the LGW Hamiltonian for the commensurate system with frustration f_k. That commensurate system in general possesses n_k degenerate critical modes given by Eq. (7), and we need n_k (complex) order parameters $\psi_{\alpha_k} (\alpha_k = 1, 2, \ldots, n_k)$ to describe fluctuations from those n modes. Then $F^{(k)}$ takes the form in the real space

$$F^{(k)} = \int d^2 r f^{(k)} ,$$
$$f^{(k)} \sim \sum_\alpha (|\psi_\alpha|^2 + |\nabla\psi_\alpha|^2) + \sum_{\alpha,\beta,\gamma,\delta} \psi^*_\alpha \psi^*_\beta \psi_\gamma \psi_\delta \delta_{\mathbf{Q}_\alpha + \mathbf{Q}_\beta, \mathbf{Q}_\gamma + \mathbf{Q}_\delta} , \tag{7}$$

where it is understood that all $\alpha, \beta, \ldots$ have the same subscript k.

Since there is no interlevel coupling in Eq. (6), each piece of Hamiltonian $F^{(k)}$ behaves independently of the others, and gives rise to the phase transition which would be present in the commensurate system with frustration f_k. That transition temperature depends on n_k, and is expected to decrease with it.[4,6] This implies that the whole Hamiltonian would display a succession of transitions each of which corresponds to that in the commensurate system with the frustration given by a rational approximant as long as the condition in Eq. (4) is satisfied. In such a succession of phase transitions, the one which corresponds to the frustration given by the rational approximant $f_k = m_k/n_k$ can be interpreted as that into an ordering at the scale n_k in the presence of defects δ_k. At a lower temperature the transition corresponding to the frustration given by the approximant

$$f_{k+1} = m_{k+1}/n_{k+1} (n_k < n_{k+1})$$

would then drive the system into an ordered state at the larger scale n_{k+1} with, presumably, fewer defects δ_{k+1}, and so on. Thus the system is expected to exhibit successive orderings at larger and larger scales with fewer and fewer defects as the temperature gets lower. At sufficiently low temperatures, therefore, the system would be in a mixed state where orderings at various length scales coexist.

Heretofore our consideration has been confined to the first order in Δ_k. To this order, Eq. (2) shows that each $P^{(k)}$ is identical to the interaction matrix of the commensurate system with frustration f_k. Thus $P^{(k)}$ yields the n_k degenerate critical modes given by Eq. (5). Higher-order terms break the degeneracy between the n_k critical modes, thus possibly generating metastable states. If Δ_k is small, the splitting is also small and the critical modes are still almost degenerate. In this case the overall critical behavior is expected to be the same.[10]

Another point which should be noted is the finiteness of the system. Each successive transition in the finite system is, in fact, a quasitransition since it would be rounded off due to the finite-size effects. These effects are, however, rather well understood[11] and will not be discussed in this paper. In the thermodynamic limit it is obvious from Eq. (4) that δ_k must equal zero, which implies the absence of the phase transition corresponding to any rational approximants except the irrational f itself.

The remaining task is how, for a given irrational f, to find the sequence of the rational approximants $\{f_k = m_k/n_k\}$ which are better than others, i.e., for every rational $m/n (\neq m_k/n_k)$ with $1 \leqq n \leqq n_k$, we have $|\delta_k| \equiv |f - f_k| < |f - m/n|$.

An obvious choice is the continued-fraction expansion,[12] which is uniquely determined for a given real number f between 0 and 1:

$$f = [p_1, p_2, p_3, \ldots] \equiv 1/[p_1 + 1/p_2 + 1/(p_3 + \cdots)] , \tag{8}$$

with each p_i a positive integer. If f is a rational (an irrational), this expansion is finite (infinite). The kth convergent f_k to an irrational f is defined to be

$$f_k=[p_1,p_2,\ldots,p_k] \ . \tag{9}$$

It is well known that these convergents are rational approximants better than others. In general, however, the convergents form merely a subsequence of the desired sequence, and there can exist better approximants which are not convergents. All of these approximants as well as all the convergents can be accommodated if we use the concept of the Farey sequence.[12]

We now consider the condition that there exist many successive transitions. Equation (4) gives the condition that the transition corresponding to the approximant $f_k=m_k/n_k$ exists. For the existence of many successive transitions, Eq. (4) should be satisfied over many successive k's for a given L. To check this we consider a simple quadratic irrational

$$f=(\sqrt{p^2+4}-p)/2=[\bar{p}] \ , \tag{10}$$

whose convergents $\{f_k=m_k/n_k\}$ are determined by

$$\begin{aligned} m_k&=n_{k-1} \ , \\ n_k&=pn_{k-1}+n_{k-2} \ \ (n_0\equiv 1, n_{-1}\equiv 0) \ . \end{aligned} \tag{11}$$

It is trivial to solve Eq. (11) and obtain

$$n_k=\frac{x^{k+1}-y^{k+1}}{x-y} \ , \tag{12}$$

where $x,y=[p\pm(p^2+4)^{1/2}]/2$. Through the use of the known property

$$|\delta_k|<(n_k n_{k+1})^{-1} \ , \tag{13}$$

it is then straightforward to show that the condition for the existence of at least l (at most pl) successive transitions

$$n_{k+1}\ll L\ll|\delta_k|^{-1} \tag{14}$$

is indeed satisfied if l is sufficiently smaller than k. Therefore as k gets larger, the number of successive transitions also increases. In particular, the critical points form a dense set in the limit $k\rightarrow\infty$. In this case, however, n_k also becomes arbitrarily large implying that the (highest) transition temperature would also become arbitrarily low. This is just the thermodynamic limit ($L\rightarrow\infty$) in which no finite-temperature transition has been already concluded for the incommensurate system. Although we have considered only the simple case of quadratic irrationals, we believe the above features to be valid for any irrationals.

Finally, to characterize the low-temperature "glassy" phase,[13] we introduce a set of order parameters

$$q^{\alpha\beta}=\sum_{\gamma,\delta}{}' P(\gamma)P(\delta)q^{\alpha\beta;\gamma\delta}\delta_{\mathbf{Q}_\alpha+\mathbf{Q}_\beta,\mathbf{Q}_\gamma+\mathbf{Q}_\delta} \ , \tag{15}$$

where $P(\gamma)$, the weight of the mode ("valley") γ, is related to its free energy $F(\gamma)$ via[14]

$$P(\gamma)=Z^{-1}e^{-F(\gamma)/T}, \ \ Z\equiv\sum_\alpha e^{-F(\alpha)/T} \ , \tag{16}$$

and $q^{\alpha\beta;\gamma\delta}$ is related to the coupling term in the LGW Hamiltonian given by Eq. (7),

$$q^{\alpha\beta;\gamma\delta}=N^{-1}\sum_{\mathbf{r}}\langle\psi_\alpha^*(\mathbf{r})\psi_\beta^*(\mathbf{r})\psi_\gamma(\mathbf{r})\psi_\delta(\mathbf{r})\rangle \ . \tag{17}$$

In Eq. (15) the prime in the summation implies that neither γ nor δ equal either α or β, and $\mathbf{Q}_\alpha$ is a critical mode of any level given by Eq. (5). We also suggest the order parameter measuring the spin-glass order

$$q=\sum_{\alpha,\beta}P(\alpha)P(\beta)q^{\alpha\beta} \ , \tag{18}$$

which increases from zero at the critical temperature (the highest-transition temperature in the succession) and reaches its maximum value of one at the zero temperature.

With the assignment

$$\frac{dz}{d\tilde{q}}=\sum_{\alpha,\beta}P(\alpha)P(\beta)\delta(\tilde{q}-q^{\alpha\beta}) \ , \tag{19}$$

which is the probability for correctly weighted modes to have overlap $\tilde{q}$, it is straightforward to obtain the relation between q and $\tilde{q}(z)$:

$$q=\int_0^1\tilde{q}(z)dz \ . \tag{20}$$

Thus similarity to the replica-symmetry-breaking formalism of spin glasses[15] is now obvious. It is also easy to show that for any three modes α,β,γ,

$$q^{\alpha\beta}=q^{\beta\gamma}\leqq q^{\gamma\alpha} \ , \tag{21}$$

since $q^{\alpha\beta}>0$ if and only if α,β belong to the same level; $q^{\alpha\beta}=0$ otherwise. Equation (21) implies that this low-temperature phase possesses a peculiar ultrametric topology, which is believed to be a characteristic of spin glasses.[16] This property is, in fact, a direct consequence of the absence of the interlevel coupling.

All of these properties, which are surprisingly similar to those of spin glasses, strongly suggest that this low-temperature phase is a kind of glassy phase. This is a very interesting possibility since there is no quenched disorder in this system. However, it appears that the glass transition temperature becomes arbitrarily low as the thermodynamic limit is approached.

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