Several small Josephson junctions in a resonant cavity: Deviation from the Dicke model

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We have studied quantum mechanically a system of several small identical Josephson junctions in a lossless single-mode cavity for different initial states, under conditions such that the system is at resonance. This system is analogous to a collection of identical atoms in a cavity, which is described under appropriate conditions by the Dicke model. We find that our system can be well approximated by a reduced Hamiltonian consisting of two levels per junction. The reduced Hamiltonian is similar to the Dicke Hamiltonian, but contains an additional term resembling a dipole-dipole interaction between the junctions. This extra term can be understood as a natural consequence of degenerate second-order (Löwdin) perturbation theory. For typical, physically reasonable values of the junction-cavity coupling, we find that this perturbation treatment is an adequate way to include the junction energy levels beyond the lowest two. As in the Dicke model, we find that, when N junctions are present in the cavity, the junction-cavity interaction is enhanced by $\sqrt{N}$, with a corresponding decrease in the Rabi oscillation period. We find that this enhancement survives even if the junctions differ slightly from one another, as expected in a realistic system. Since coherence effects thus reduce the Rabi period, it may become smaller than the decoherence time due to dissipation, making these oscillations observable.

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I. INTRODUCTION

In a previous paper,[1] we numerically studied a small Josephson junction capacitively coupled to the single mode of a resonant cavity. We found that the quantum states of the combined system are strongly entangled only at certain special values of the gate voltage across the junction, corresponding to a one- or two-photon absorption or emission process by the junction Cooper pairs. We also showed that at a gate voltage corresponding to the one-photon process, our results, obtained from exact diagonalization, agree well with those obtained from a reduced Jaynes-Cummings Hamiltonian[2] which includes only two-junction states and two-photon states.

In the present paper, we extend this work to several similar junctions placed in a resonant cavity within a radiation wavelength. Each junction is assumed to be voltage biased by an independent voltage source, but all bias voltages are assumed to be equal. As in the single-junction case, we solve for the eigenstates of this coupled system by direct diagonalization and compare the results to those obtained from a reduced Hamiltonian which includes only two energy levels per junction.

A major goal of our work is to compare the junction-cavity interaction to that in a system first discussed by Dicke,[3] consisting of an assembly of identical two-level atoms in a single-mode cavity. We find that, for typical values of our system parameters, the effects of the energy levels beyond the lowest two per junction are included adequately by a straightforward second-order perturbation theory. These higher-level effects manifest themselves as an additional effective interaction between the junctions (beyond the Dicke Hamiltonian) which has the form of a dipole-dipole coupling.

Before proceeding further, we note some distinctions between the system considered here and another system which has recently been studied experimentally.[4] These experiments involve a two-dimensional array of junctions, such that each junction is coupled to a single-mode resonant cavity. For suitable junction and cavity parameters, the cavity interaction causes the array to phase lock and to radiate coherently into the cavity. The models used to describe such arrays[5–8] typically involve a semiclassical approximation, appropriate for large numbers of photons in the cavity. By contrast, our system involves only a few photons and needs to be studied quantum mechanically. Another distinction is that, in Ref. 4, the junctions are biased onto either the resistive or the superconducting branch of the I-V characteristic, whereas in the present model, only the latter branch is considered.

The present work may also be relevant to quantum computation.[9] Josephson devices have been proposed as possible quantum bits (qubits) in computing systems, in which quantum logic operations could be performed by controlling gate voltages or magnetic fields.[9–14] Thus far, two alternative realizations of qubits have been proposed for Josephson junctions: a charge qubit and a flux qubit. In the first case, the charging energy of the junction is much larger than the Josephson energy, so that the charge on one of the superconducting islands is nearly a good quantum number. Qubits of this type have been studied experimentally by Nakamura *et al.*[10,11] In the second case, studied experimentally by Mooij *et al.*[13] and Feigel'man *et al.*,[14] the Josephson energy predominates, and the phase difference across the junction is well defined. In the present paper, the system is constructed from junctions in which the charging and Josephson energies are comparable in magnitude; in other respects, it somewhat resembles that studied in Ref. 12.

The findings of this paper may also be useful for studying "quantum leakage"[15] in Josephson-junction-based multiqubits. Quantum leakage occurs when the computational space of the physical system is a subspace of a larger Hilbert space.

Such leaking is an intrinsic source of decoherence, even in the absence of dissipation, and imposes a time limit beyond which the system can no longer perform quantum computations. Thus, comparing the results of numerically diagonalizing the full Hamiltonian and the reduced Hamiltonian provides a means of studying this decoherence. Such a comparison was done in Ref. 15 and is also done here for a different Hamiltonian and a different parameter region.

Finally, because of the analogy between our system and a group of two-level atoms in a single-mode cavity, the present work may be relevant to the field of cavity quantum electrodynamics.[16] Specifically, using the system considered here, it may be possible to ''engineer'' a so-called Einstein-Podolsky-Rosen or Greenberger-Horne-Zeilinger state comprised of Josephson junctions. It may also be feasible to perform a ''quantum nondemolition'' measurement on a one-photon resonator much as is done using a two-level atom in a cavity.[17] We hope to develop these ideas in a future publication.

The remainder of this paper is organized as follows. In Sec. II, we introduce our model Hamiltonian and its physical realization; we also outline the method used to solve for its eigenstates. Section III contains our principal results. We describe the reduced Hamiltonian, and we compare the time evolution of the system as predicted by the full Hamiltonian and by the two-level approximation. In the final section, the main points of the paper are summarized, and some of the physical parameters entering the calculations are estimated.

II. MODEL HAMILTONIAN AND ITS DIAGONALIZATION

A. Hamiltonian

We consider a group of underdamped Josephson junctions in a lossless electromagnetic cavity which can support a single-photon mode of angular frequency ω. The system is assumed to be described by the following model Hamiltonian:

$$\mathcal{H}=\mathcal{H}_{\rm photon}+\sum_j \mathcal{H}_J(j). \tag{1}$$

Here $\mathcal{H}_{\rm photon}$ is the Hamiltonian of the cavity mode, which we express in the form $\mathcal{H}_{\rm photon}=\hbar\omega(a^\dagger a+1/2)$, where $a^\dagger$ and a are the photon creation and annihilation operators with commutation relation $[a,a^\dagger]=1$. $\mathcal{H}_J(j)$ is the Hamiltonian of the jth junction, which implicitly contains the coupling to the cavity; it can be written as

$$\mathcal{H}_J(j)=\frac{1}{2}U(n_j-\overline{n_j})^2-J\cos\gamma_j, \tag{2}$$

where the first and second terms represent the charging and Josephson energies of a junction. $U=4e^2/C$ is the junction charging energy, $E_J=\hbar I_c/(2e)$ is the Josephson coupling energy, C is the effective junction capacitance, and I_c the junction critical current. n_j is an operator which represents the difference between the number of Cooper pairs on the two grains comprising the junction, and $\overline{n_j}$ is related to the gate voltage applied across the junction. Although $\overline{n_j}$ could, in principle, be controlled separately for each junction, we assume that all the $\overline{n_j}$'s are equal. The last parameter γ_j is the gauge-invariant phase of the junction, as defined below. We assume that all the junctions have the same J and U, and also that J and U are comparable in magnitude. Furthermore, we neglect the current through the shunt resistance. Finally, we consider only junctions such that both J and U are small compared to the superconducting gap Δ, which is thus the largest energy scale in the system.

The interaction between the junctions and the electromagnetic field is contained in the gauge-invariant phase difference

$$\gamma_j=\phi_j-(2\pi/\Phi_0)\int_j \mathbf{A}(\mathbf{x},t)\cdot dl. \tag{3}$$

Here ϕ_j is the phase difference across the junction in a particular gauge; it satisfies the commutation relation $[n_k,\phi_j]=-i\delta_{jk}$. $\Phi_0=hc/(2e)$ is the flux quantum, $\mathbf{A}(\mathbf{x},t)$ is the vector potential, and the line integral is taken across the jth junction. We consider only a vector potential arising from the electromagnetic field of the cavity normal mode. We also assume that the junctions are all located within a distance small compared to the radiation wavelength, so that $\mathbf{A}(\mathbf{x})$ is approximately uniform throughout the region of the junctions. If the junctions all have the same orientation, this condition implies that each junction has the same coupling to the cavity electromagnetic field. The generalization to nonuniform fields is straightforward, however.

In Gaussian units and in the Coulomb gauge ($\nabla\cdot\mathbf{A}=0$), the vector potential takes the form $\mathbf{A}=\sqrt{hc^2/\omega V}(a+a^\dagger)\hat{\boldsymbol{\epsilon}}$, where $\hat{\boldsymbol{\epsilon}}$ is the unit polarization vector of the cavity mode and V is the volume of the cavity. Thus we can define the coupling g such that $(2\pi/\Phi_0)\int\mathbf{A}\cdot dl=(g/\sqrt{2})(a+a^\dagger)$ or

$$g=4e\sqrt{\frac{\pi}{\hbar\omega V}}\hat{\boldsymbol{\epsilon}}\cdot\vec{l}, \tag{4}$$

where l is the thickness of the insulating layer in a junction. With the change of variables $p=i\sqrt{\hbar\omega/2}(a^\dagger-a)$, $q=\sqrt{\hbar/(2\omega)}(a^\dagger+a)$, γ_j can be rewritten

$$\gamma_j=\phi_j-g\sqrt{\omega/\hbar}q, \tag{5}$$

where p and q satisfy the canonical commutation relation $[p,q]=-i\hbar$.

Finally, we decompose the Hamiltonian (1) as follows:

$$\mathcal{H}=\mathcal{H}_{\rm photon}+\sum_j \mathcal{H}_J^0(j)+\mathcal{H}_{\rm int}, \tag{6}$$

where

$$\mathcal{H}_J^0(j)=\frac{1}{2}U(n_j-\overline{n_j})^2-J\cos\phi_j \tag{7}$$

and

$$\mathcal{H}_{\rm int}=-J\sum_j(\cos\gamma_j-\cos\phi_j). \tag{8}$$

In this form, the photon-junction interaction is entirely contained in the last term.

B. Method of solution

As in Ref. 1, we diagonalize $\mathcal{H}$ in a complete basis formed from the direct product of the eigenfunctions of $\mathcal{H}_{\rm photon}$ and $\mathcal{H}_{\rm J}^0(j)$. The eigenfunctions of $\mathcal{H}_{\rm photon}$ are, of course, harmonic oscillator eigenstates; the normalized nth eigenstate has wave function

$$h_n(q)=\frac{1}{\sqrt{\sqrt{\pi}2^n n!}}\exp(-y^2/2)H_n(y),$$

where $H_n(y)$ is a Hermite polynomial of order n and $y=(\omega/\hbar)^{1/2}q$. An eigenfunction $\psi(\phi_j)$ of $\mathcal{H}_{\rm J}^0(j)$ with eigenvalue $E^{(j)}(\bar{n}_j)$ can be written as $\psi(\phi_j)=\exp(i\bar{n}_j\phi_j)\eta(\phi_j)$ and satisfies the Schrödinger equation $\mathcal{H}_{\rm J}^0(j)\psi(\phi_j)=E^{(j)}\psi(\phi_j)$. Using the representation $n_j=-i\partial/\partial\phi_j$, which follows from the commutation relation $[n_j,\phi_j]=-i$, we write the Schrödinger equation as

$$\frac{d^2Y(x)}{dx^2}+\left(\frac{8E^{(j)}}{U}+2Q\cos 2x\right)Y(x)=0, \tag{9}$$

where $x=\phi_j/2$, $Y(x)=\eta(\phi_j/2)$ and $Q=4J/U$. This is a Mathieu equation with characteristic value $a^{(j)}=8E^{(j)}/U$ and potential of strength Q. The eigenvalues $E^{(j)}$ are determined by the requirement that $\psi(\phi_j)=\psi(\phi_j+2\pi)$ or, equivalently, $Y(x+\pi)=\exp(-2i\bar{n}_j\pi)Y(x)$. Thus we are interested only in the Floquet solutions $Y_{k_j}(x)$ of the Mathieu equation (9), with Floquet exponent $\nu_j=2k_j-2\bar{n}_j$, where $k_j=0,\pm1,\pm2,\ldots$. We denote the corresponding eigenvalue of $\mathcal{H}_{\rm J}^0(j)$ by $E_{k_j}^{(j)}(\bar{n}_j)$. It is sufficient to consider only $0\leqslant\bar{n}_j\leqslant0.5$, because the eigenvalues of $\mathcal{H}_{\rm J}^0(j)$ are periodic in $\bar{n}_j$ with period unity and are symmetric about $\bar{n}_j=1/2$.

Now, any eigenstate $\Psi(\phi_1,\ldots,\phi_N,q)$ of the Hamiltonian $\mathcal{H}$ can be expressed as a linear combination of product wave functions consisting of eigenfunctions of $\mathcal{H}_{\rm photon}$ and the $\mathcal{H}_{\rm J}^0(j)$'s, i.e.,

$$\Psi(\phi_1,\ldots,\phi_N,q)=\sum_{k_1\ldots k_N,n}A_{k_1\cdots k_N,n}h_n(q)\prod_{j=1}^{N}\psi_{k_j}(\phi_j), \tag{10}$$

where $\psi_{k_j}(\phi_j)=\exp(i\bar{n}_j\phi_j)Y_{k_j}(\phi_j/2)$ and N is the number of junctions in the resonator. The only term in $\mathcal{H}$ which is not diagonal in this product basis is the interaction term $\mathcal{H}_{\rm int}$. We can obtain the product coefficients and eigenvalues by directly diagonalizing the Hamiltonian of Eq. (1) in this product basis, i.e., by solving the Schrödinger equation $\mathcal{H}\Psi(\phi,q)=E\Psi(\phi,q)$, where we use the shorthand notation $\phi\equiv(\phi_1,\ldots,\phi_N)$. Once the eigenvalues E and expansion coefficients $A_{k_1\cdots k_N,n}$ are known, we can study the time evolution of the system given its state at time $t=0$ and hence obtain the time-dependent expectation value of any operator $\mathcal{O}$ at time t using

$$\langle\mathcal{O}(t)\rangle=\mathrm{Tr}[\rho(t)\mathcal{O}]. \tag{11}$$

Here $\rho(t)$ is the density matrix of the system at time t, defined by $\rho(t)=U(t)\rho(0)U^\dagger(t)$, where $U(t)=\exp(-i\mathcal{H}t/\hbar)$ is the evolution operator and $\rho(0)$ is determined by the initial state of the system.

We have carried out extensive calculations for the Hamiltonian (1), using the product basis described above. In our calculations we have arbitrarily used $Q=0.7$ and $\hbar\omega/U=0.3$, representing a case where the charging energy, Josephson coupling, and photon energy are all comparable. The attainability of these parameters in realistic systems is discussed further below. Most of our calculations have also been carried out near the value $\bar{n}=0.258$, corresponding to the case where the energy difference between the two lowest junction levels equals a single quantum of photon energy. In our single-junction calculations,[1] this choice leads to resonant absorption of a single photon and to maximum entanglement between the junction and the photon states. For larger number of junctions ($N\geqslant4$) we found that we have to tune $\bar{n}$ slightly away from the resonance condition of the single junction to get the system on resonance again.

In all of our calculations, we use a truncated basis of the eigenstates of $\mathcal{H}_{\rm photon}$ and of $\mathcal{H}_{\rm J}^0(j)$. For $N=1,2,3$ junctions, we include the first nine photon eigenstates and the first five levels of each junction, i.e., a total of 9×5^N states. For $N=4$, we use fewer eigenstates: the lowest eight photon states and the first four junction levels. In all cases, we confirmed the numerical stability of our results by increasing the dimensionality of the basis and finding that there is almost no change in our results. As another check, we have calculated $\mathrm{Tr}(\rho)$, which should be time independent and equal to unity for our normalization. We find that this trace does remain equal to unity to high accuracy for all times considered, thus providing evidence that we have correctly diagonalized the Hamiltonian.

III. RESULTS AND DISCUSSION

A. Reduced Hamiltonian

The main goal of this paper is to study how well the results of this exact diagonalization procedure can be matched by a reduced Hamiltonian which includes only the lowest two energy levels from each junction. For the one-junction case, the two-level approximation was already tested in Ref. 1 at $\bar{n}=0.258$. At this value of $\bar{n}$, the splitting between the two lowest levels equals the single-photon energy $\hbar\omega$. The reduced Hamiltonian then takes the Jaynes-Cummings form[2]

$$\mathcal{H}_{JC}=\hbar\omega(a^\dagger a+S_z)+\xi a^\dagger S_-+\xi^* a S_+. \tag{12}$$

Here the junction is represented as a quantum spin 1/2, with $[S_+,S_-]=2S_z$, and the junction energy zero is shifted so that the lowest two levels are $\pm\hbar\omega/2$. The matrix element coupling the junction and the field is given by $\xi=\langle f|\frac{1}{2}\mathcal{H}_{\rm int}|i\rangle$, where $\mathcal{H}_{\rm int}$ is the one-junction version of expression (8); the ket $|i\rangle\equiv|n=1;k=0\rangle$ is a wave function containing one photon and the junction in the ground state

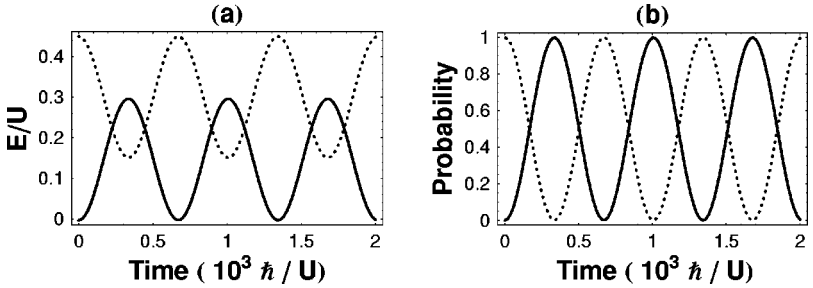

FIG. 1. (a) Time dependence of $\langle \mathcal{H}_J^0 \rangle$ (solid curve) and of $\langle \mathcal{H}_{\rm photon} \rangle$ (dashed curve), plotted for one junction with $\bar{n}=0.258$ (on-resonance condition). The system is prepared at time $t=0$ in the state $|n=1;k=0\rangle$. The other parameters are $Q=0.7$, $\omega/U=0.3$, and $g=0.15$. (b) Same as (a) except that we show the time-dependent probability for finding the Josephson junction (solid curve) and the photonic resonator (dashed curve) in their first excited states.

$(k=0)$, while $|f\rangle \equiv |n=0;k=1\rangle$ is a state with zero photons and the junction in its first excited state $(k=1)$. To first order in g, $\mathcal{H}_{\rm int} \approx -Jg\sqrt{\omega/\hbar}q \sin\phi$ and, hence, to the same order in g,

$$\xi \approx -(Jg/\sqrt{8})\langle k=0|\sin\phi|k=1\rangle. \tag{13}$$

This completes the definition of the reduced Hamiltonian for one junction and establishes its relation to the full Hamiltonian of Eq. (1).

A naive extension of the reduced Hamiltonian (12) to N junctions would be[17]

$$\mathcal{H}_D = \mathcal{H}_D^0 + \mathcal{H}_D^I, \tag{14}$$

where

$$\mathcal{H}_D^0 = \hbar\omega a^\dagger a + \hbar\omega \sum_{j=1}^{N} S_z^{(j)},$$

$$\mathcal{H}_D^I = \sum_{j=1}^{N} (\xi a^\dagger S_-^{(j)} + \xi^* a S_+^{(j)}).$$

Here $S_-^{(j)}$, $S_+^{(j)}$, and $S_z^{(j)}$ are the lowering, raising, and inversion operators of the jth junction, which satisfy the commutation relations

$$[S_z^{(j)}, S_\pm^{(k)}] = \pm S_\pm^{(j)} \delta_{jk},$$

$$[S_+^{(j)}, S_-^{(k)}] = 2S_z^{(j)} \delta_{jk}.$$

The Hamiltonian (14) resembles the one-mode Dicke model,[3,18] which describes N identical two-level atoms within a radiation wavelength of each other, all interacting with a single-mode resonator. The initial-state problem considered later in this section has been solved exactly for this Hamiltonian in a cavity containing one or more accessible electromagnetic modes.[19,20]

For the present system, when the resonator contains more than one photon, the Hamiltonian (14) does not reproduce the full time-dependent solution to the Schrödinger equation for $N \geqslant 2$, at least for our numerical parameters. Nevertheless, as shown below, the full numerical solution is still well approximated by another reduced Hamiltonian in which each junction is represented by a basis of two levels only, namely,

$$\mathcal{H}_{D'} = \mathcal{H}_{D'}^0 + \mathcal{H}_{D'}^I, \tag{15}$$

where the zeroth-order Hamiltonian $\mathcal{H}_{D'}^0$ is the same as the Dicke $\mathcal{H}_D^0$ appearing in Eq. (14), but the interaction part of the Hamiltonian $\mathcal{H}_{D'}^I$ is now

$$\mathcal{H}_{D'}^I = \sum_{j=1}^{N} (\xi a^\dagger S_-^{(j)} + \xi^* a S_+^{(j)}) + \Omega \sum_{\substack{j,k=1 \\ (j\neq k)}}^{N} (S_+^{(j)} S_-^{(k)} + S_-^{(j)} S_+^{(k)}). \tag{16}$$

The physics behind this reduced Hamiltonian[21] is discussed further below, in Sec. III D 2.

B. Review of the single-junction solution

We first review the time-dependent solution to the Schrödinger equation for a single junction in a resonator.[1] If the initial junction-cavity state is $|n=1;k=0\rangle$, then the system evolves in time according to the evolution operator $U(t) = \exp(-i\mathcal{H}t/\hbar)$. Figure 1 shows the resulting time-dependent expectation values $\langle \mathcal{H}_J^0 \rangle$ and $\langle \mathcal{H}_{\rm photon} \rangle$, as calculated using Eq. (11) and the full Hamiltonian $\mathcal{H}$. Figure 1 also shows the time-dependent probabilities for finding the junction and the resonator in their first excited states, given this initial state. The system wave function is evidently an entangled state oscillating between $|n=1;k=0\rangle$ and $|n=0;k=1\rangle$, with a period $T \approx 675\hbar/U$ for our particular choice of g, Q/U, and $\hbar\omega/U$. When the Rabi period for oscillations between the Josephson-photon entangled states is calculated using the two-level approximation, it is found to be $\pi\hbar/|\xi| = 667\hbar/U$, in almost perfect agreement with the result obtained from the full Hamiltonian.

C. Several junctions, asymmetric initial state

Next, we consider N junctions such that initially (at time $t=0$) one junction is in the first excited state, while the remaining $(N-1)$ junctions are in the ground state, and the cavity initially contains zero photons. Thus all the junctions interact with the vacuum fluctuations of the cavity mode via the Hamiltonian $\mathcal{H}_{\rm int}$.

Our primary interest is to compare the time evolution calculated from the exact Hamiltonian with that obtained from

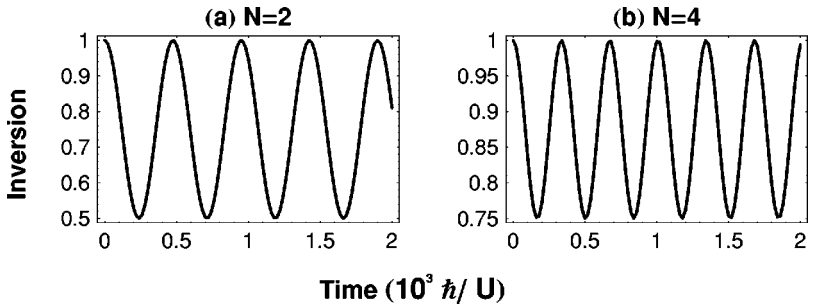

FIG. 2. Time-dependent inversion $\mathcal{S}(t)$ [Eq. (17)] for (a) $N=2$ and (b) $N=4$ junctions in a resonator, calculated using the full Hamiltonian. Initially, one junction is in its first excited state, while all the other junctions and the resonator are in their ground states.

the two reduced Hamiltonians discussed above. We will focus our attention on the ''inversion'' $\mathcal{S}(t)$ of the system, defined as

$$\mathcal{S}(t)=\left\langle S_z(t)+\frac{1}{2}\right\rangle, \tag{17}$$

where the average is taken with respect to the quantum state of the system. $\mathcal{S}(t)$ is the sum of the probabilities that the junctions are in their first excited states, and satisfies $0 \leqslant \mathcal{S}(t) \leqslant N$. This quantity is of interest because $\hbar\omega\, d\mathcal{S}/dt = I$ is the rate at which energy is transferred from the junctions to the cavity or vice versa.

For the initial state considered in this section, it has been proved[19] for the Dicke model [Eq. (14)] that

$$\mathcal{S}(t)=1-\frac{1}{N}\sin^2(|\xi|\sqrt{N}t/\hbar). \tag{18}$$

Thus, the frequency of energy transfer between the photon field and the junctions is proportional to $\sqrt{N}$. The group of N junctions behaves somewhat like a single junction with a coupling to the cavity mode of $\sqrt{N}\xi$ instead of ξ. It also follows from Eq. (18) that the photon field never gains more than $1/N$ of the energy initially stored in the excited junction. Thus, as N increases, it becomes progressively less likely that a photon will be emitted into the cavity, the energy remaining largely trapped within the junctions (actually, within the initially excited junction). The existence of $N-1$ unexcited junctions (or ''atoms'') has the effect of preventing the emission of a photon from the initially excited junction,[19] a phenomenon known as ''radiation suppression.''[20]

When we calculate $\mathcal{S}(t)$ using our direct diagonalization scheme, we find that the numerical results are in excellent agreement with the predictions of the two-level approximation (18) for $N=1$, 2, 3, and 4 junctions in the resonator, at least for several Rabi periods of oscillation. In Fig. 2, we show the inversion as calculated using the full basis for $N=2$ and $N=4$.

D. Dicke-symmetric states

Next, we consider an initial state in which all the junctions are in the ground state and there are n photons in the system ($n=1$, 2, 3). This is a simple case in which the initial state of the system is symmetric under the interchange of any two junctions. We have solved for the time-dependent state of the system using the full Hamiltonian and compared the results to the approximate Hamiltonians (14) and (15). Results of our calculations are shown in Figs. 3–5.

As can be seen from these figures, the one-mode Dicke model defined in Eq. (14) does not reproduce the full numerical solution for $n \geqslant 2$. But the numerical solution is well approximated by the reduced model Hamiltonian (15), in which each junction is still represented by only two levels. This Hamiltonian differs from the Dicke model only in the last term, which is proportional to Ω and which represents an effective *direct* interaction between the junctions. This term is needed in order to obtain a good fit between the predictions of the reduced model and the exact Hamiltonian, even though the junctions in the original model Hamiltonian [Eq. (1)] have only an indirect coupling with each other through the electromagnetic field.

A similar model Hamiltonian to Eq. (15) has been studied in Ref. 22 for two atoms in a cavity. This model differs from

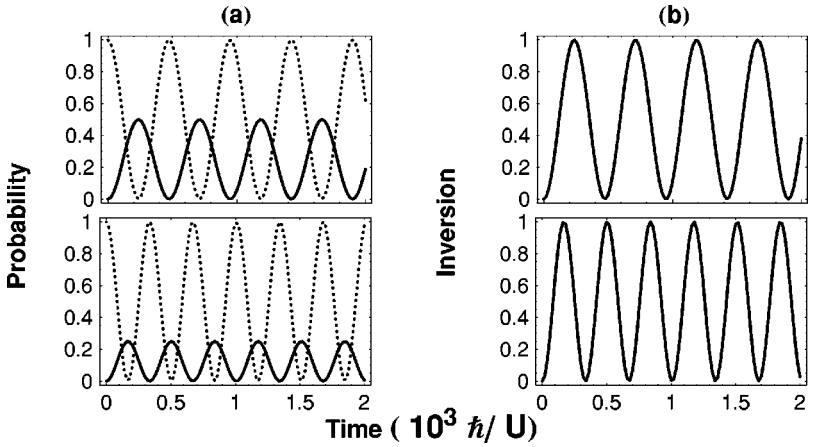

FIG. 3. (a) Time-dependent probability for finding any one of the junctions (solid curve) or the photonic resonator (dashed curve) in its first excited state for $N=2$ (top) and $N=4$ (bottom). The system is prepared at time $t=0$ such that there is one photon in the system and all the junctions are in their ground states. (b) Same as (a) except we show the inversion $\mathcal{S}(t)$ of the system as defined in Eq. (17).

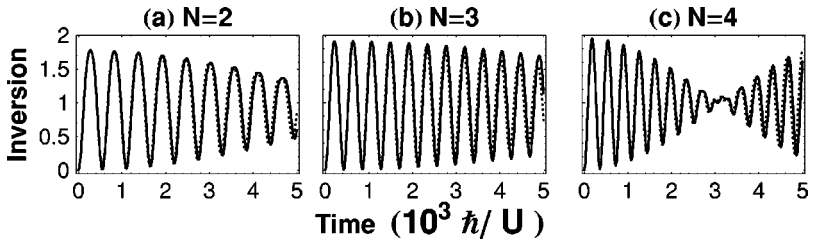

FIG. 4. Time-dependent inversion $\mathcal{S}(t)$ for $m=2$ excitations and $N=2$, 3 and 4 junctions in the resonator [(a), (b), and (c), respectively]. The initial state has two photons in the resonator, and all the junctions in their ground states. The solid line is the full numerical solution, and the dotted line follows from Eqs. (28). Note in particular the decrease in the Rabi period of oscillation as N increases. Other parameters are the same as in Fig. 1 except that $\overline{n_j}=0.2598$ for $N=4$.

the present one, however, in that the direct coupling in Ref. 22 represents a specific term in the Hamiltonian, namely, an effective dipole-dipole interaction between the atoms. In the calculations to be presented below, the coupling strength Ω in the Hamiltonian of Eq. (15) is treated simply as a parameter determined by best fitting to the numerical data. However, it can also be derived explicitly from the full Hamiltonian, as we show below.

There is one further point to be made about these numerical results. Namely, as long as $n \leqslant N$, we find that the interaction term ξ which best fits the computed time evolution is numerically equal to that which enters the single-junction reduced Hamiltonian [Eq. (13)] when there is no more than one photon present. This result is intuitively reasonable, because in this symmetric case all the junctions in the resonator share equally in the photons, and thus, for $n \leqslant N$, none will absorb more than one photon.

Now we will discuss the basis used to study the Hamiltonian (15). The Hilbert space of N two-level junctions is 2^N dimensional. Thus the system is analytically intractable even for a small value of N unless there is some symmetry of the Hamiltonian which confines the evolution of the system to some lower invariant subspace. The Hamiltonian (15) has two useful symmetries of this kind. First, a symmetric initial state is an eigenstate of the permutation operator $\mathcal{P}_{jk}$, which exchanges the jth and kth junctions. Since $\mathcal{P}_{jk}$ commutes with the Hamiltonian of Eq. (15), a symmetric initial state evolving under $\mathcal{H}_{D'}$ will remain symmetric at later times. The second symmetry involves the excitation number operator

$$\mathcal{M}=a^{\dagger}a+\sum_{j=1}^{N}\left(S_z^{(j)}+\frac{1}{2}\right), \tag{19}$$

which also commutes with $\mathcal{H}_{D'}$ and thus is also a constant of the motion. In this case it is convenient to study the Hamiltonian $\mathcal{H}_{D'}$ using the basis vectors formed from the direct product of the eigenstates of $\mathcal{M}$ and $a^{\dagger}a$. We denote these states as $|m;n\rangle$, where

$$\mathcal{M}|m;n\rangle=m|m;n\rangle,$$

$$a^{\dagger}a|m;n\rangle=n|m;n\rangle, \tag{20}$$

and $0 \leqslant n \leqslant m$. Note that the time evolution of the system is restricted to a $(m+1)$-dimensional subspace spanned by the basis vectors $|m;0\rangle, \ldots, |m;m\rangle$.[23]

It is convenient to write out the state $|m;n\rangle$ explicitly. There are n photon excitations in the cavity and $m-n$ excitations distributed symmetrically among the N junctions. Thus we may write $|m;n\rangle=|n\rangle\otimes|m-n\rangle_J$, where $|n\rangle$ is a Fock state and $|m-n\rangle_J$ is a normalized symmetric Dicke state of the junctions, described by

$$|k\rangle_J=\left[\frac{N!}{k!(N-k)!}\right]^{-1/2}\sum_{\mathcal{P}}\prod_{l=1}^{k}|+\rangle_{j_l}\prod_{m\neq 1\cdots k}|-\rangle_{j_m}, \tag{21}$$

where $|+\rangle_{j_l}\equiv|k_l=1\rangle$ $(|-\rangle_{j_l}\equiv|k_l=0\rangle)$ is the spin-up (-down) state of the lth junction and the summation is over permutations $\mathcal{P}$ among the N junctions.

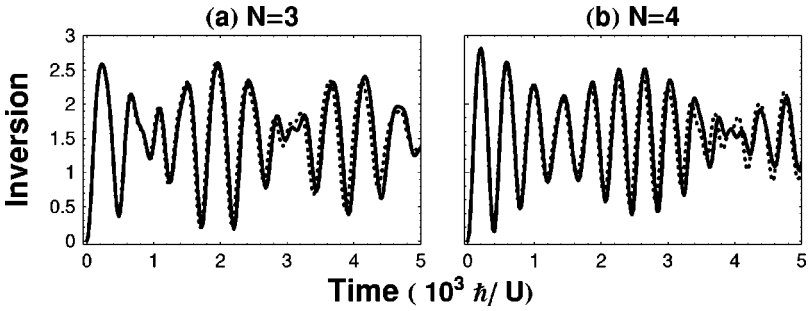

FIG. 5. Same as Fig. 3, except that the initial state has three photons in the resonator and calculations are carried out for $N=3$ and $N=4$ junctions, as indicated in the figure.

We now discuss the cases of $m=1,2,3$ excitations and compare the time-dependent states obtained from the full numerical solution to the predictions of the two level approximation, using the basis $|m;n\rangle$.

1. One excitation $(m=1)$

In this case, our numerical results are well described by the Dicke Hamiltonian [Eq. (14)], at least for $N\leqslant 4$. This behavior is expected, since the junctions cannot interact directly with each other in the presence of only one photon. For one excitation in the system, an exact solution for a two-level system can be found even in a damped cavity.[24] In the absence of damping, the wave function of the system in the interaction picture can be written in the two-level approximation as

$$|\Psi^{\mathrm{I}}(t)\rangle = C_1(t)|1;1\rangle + C_0(t)|1;0\rangle. \tag{22}$$

From the time-dependent Schrödinger equation (taking $\hbar = 1$ henceforth)

$$\mathcal{H}_{\mathrm{D}}^{\mathrm{I}}|\Psi^{\mathrm{I}}(t)\rangle = i\frac{d}{dt}|\Psi^{I}(t)\rangle, \tag{23}$$

we obtain

$$i\frac{dC_1(t)}{dt} = \xi\sqrt{N}C_0(t),$$

$$i\frac{dC_0(t)}{dt} = \xi^*\sqrt{N}C_1(t).$$

Taking the junctions initially to be in their ground states and assuming $n=1$ photons in the system [i.e., $C_1(t)|_{t=0}=1$], we find that

$$C_1(t) = \cos(|\xi|\sqrt{N}t),$$

$$C_0(t) = \sin(|\xi|\sqrt{N}t).$$

Thus the system oscillates between the states $|m=1;n=1\rangle$ and $|m=1;n=0\rangle$—that is, the junctions alternately absorb the photon of the field, then reemit it, in a reversible evolution, as expected in a lossless cavity. The oscillation period $T\propto N^{-1/2}$. From this solution, the inversion of the system [Eq. (17)] is

$$\mathcal{S}(t) = \sin^2(|\xi|\sqrt{N}t), \tag{24}$$

and hence the rate of radiation into the cavity is

$$I(t) = -\omega\frac{d\mathcal{S}(t)}{dt} = -\omega|\xi|\sqrt{N}\sin(2|\xi|\sqrt{N}t). \tag{25}$$

Both the oscillation frequency and the amplitude of the rate of radiation are proportional to $\sqrt{N}$, a signature of a cooperative phenomenon. For very short times, the rate of radiation into the cavity is

$$I(t) \approx -2\omega|\xi|^2 Nt, \tag{26}$$

that is, N times the rate of a single junction. Thus the presence of N junctions in the cavity gives rise to a collective behavior.

Figure 3(a) shows the time-dependent probability that the Josephson junctions (solid curve) and the photonic resonator (dashed curve) are in their first excited states for $N=2$ and $N=4$. The system is prepared at time $t=0$ with all the junctions in the ground state and the resonator in the number state $|n=1\rangle$. The system develops into an entangled state such that the photon is shared equally among all the junctions. This behavior can be seen both in the two-level approximation and in the results of an exact calculation. In Fig. 3(b) we show the inversion $\mathcal{S}(t)$ calculated using the full Hamiltonian for $N=2$ and $N=4$ junctions and one excitation $(m=1)$. The numerical results are clearly in excellent agreement with the predictions of the two-level approximation [Eq. (24)].

2. Two excitations $(m=2)$

For $m=2$ we need consider only three eigenstates of the Hamiltonian. In the interaction picture the system wave function is

$$|\Psi^{\mathrm{I}}(t)\rangle = C_2(t)|2;2\rangle + C_1(t)|2;1\rangle + C_0(t)|2;0\rangle, \tag{27}$$

where the coefficients satisfy

$$i\frac{dC_2(t)}{dt} = \xi\sqrt{2N}C_1(t),$$

$$i\frac{dC_1(t)}{dt} = \xi^*\sqrt{2N}C_2(t) + \Omega C_1(t) + \xi\sqrt{2(N-1)}C_0(t),$$

$$i\frac{dC_0(t)}{dt} = \xi^*\sqrt{2(N-1)}C_1(t).$$

Assuming an initial state $|2;2\rangle$, i.e., a state containing two photons, with all the junctions in their ground states, we find that the solution of these coupled equations is

$$C_2(t) = \frac{1}{\zeta(2N-1)}\{\zeta(N-1) + Ne^{-i\Omega t/2}[\zeta\cos(\zeta t) + i\Omega\sin(\zeta t)]\},$$

$$C_1(t) = -i\sqrt{2N}(\xi/\zeta)e^{-i\Omega t/2}\sin(\zeta t),$$

$$C_0(t) = \frac{\sqrt{N(N-1)}}{2(2N-1)\zeta} \times\{e^{-i\Omega t/2}[2\zeta\cos(\zeta t) + i\Omega\sin(\zeta t)] - 2\zeta\}, \tag{28}$$

where $\zeta = \frac{1}{2}\sqrt{\Omega^2 + 8|\xi|^2(2N-1)}$. The probability of a two-junction excitation of the system is $P_2(t) = |C_0(t)|^2$, which can be reduced to

$$P_2(t)=\frac{N(N-1)}{(2N-1)^2\zeta^2}[2\zeta^2-2(2N-1)|\xi|^2\sin^2(\zeta t) -2\zeta^2\cos(\zeta t)\cos(\Omega t/2)-\Omega\zeta\sin(\zeta t)\sin(\Omega t/2)]. \tag{29}$$

For $\Omega=0$, this becomes

$$P_2(t)=\frac{N(N-1)}{(2N-1)^2}[\cos(\omega_2 t)-1]^2, \tag{30}$$

so that the Rabi period of oscillation is now $\omega_2=[2(2N-1)]^{1/2}|\xi|$. Since $|P_2(t)|\leqslant N(N-1)/(2N-1)^2\leqslant 1/4$, the system is unlikely to contain two excitations simultaneously. Even if $\Omega\neq 0$, the same statement is still valid, but the time evolution is no longer perfectly sinusoidal, but instead is characterized by beats with a period inversely proportional to Ω.

The inversion of the system is given by

$$\mathcal{S}(t)=|C_1(t)|^2+2|C_0(t)|^2. \tag{31}$$

A full expression for the inversion readily calculated from Eqs. (28) is quite complicated and is not given here. Figure 4 shows the results obtained numerically for $N=2$, 3, and 4, using the full basis (solid line), as well as the inversion calculated from Eq. (31) using the modified Dicke model with $\Omega\neq 0$ (dashed line).

We now discuss the origin of the extra direct interaction in the effective Hamiltonian (15), in light of these numerical results. Equation (27) expresses the wave function for two excitations as a linear combination of three states, which we will denote compactly as $|a\rangle$, $|b\rangle$, and $|c\rangle$ respectively. In the absence of $\mathcal{H}_{\rm int}$, these states are degenerate. According to the Dicke model, $\mathcal{H}_{\rm int}$ breaks this degeneracy because some of the matrix elements of $\mathcal{H}_{\rm int}$ between these states are nonzero. This is the *first-order* effect of $\mathcal{H}_{\rm int}$ (or, equivalently, in g).

However, there is also a *second-order* effect. Namely, as is well known from degenerate second-order Löwdin perturbation theory,[25] the *effective* Hamiltonian matrix elements between, for example, states $\langle a|$ and $|b\rangle$ should be written

$$\langle a|\mathcal{H}_{\rm eff}|b\rangle=\langle a|\mathcal{H}_{\rm int}|b\rangle+\sum_r{}' \frac{\langle a|\mathcal{H}_{\rm int}|r\rangle\langle r|\mathcal{H}_{\rm int}|b\rangle}{E_a-E_r}. \tag{32}$$

Here the sum runs over all states $|r\rangle$ *other than* $|a\rangle$, $|b\rangle$, and $|c\rangle$. These are the terms which give rise to the extra interaction proportional to Ω in Eq. (15).

To confirm this hypothesis, we have calculated these matrix elements, including the terms second order in $\mathcal{H}_{\rm int}$, through order g^2. We do find that these terms produce a direct interaction between the junctions of the form (15), as found numerically. Specifically, the matrix element $\langle b|\mathcal{H}_{\rm eff}|b\rangle$ becomes nonzero when these second-order terms are included. Furthermore, we find that the value of Ω found from this direct calculation is comparable to that found numerically. Specifically, for the the cases mentioned above, we find $\Omega/(|\xi|^2/\omega)=4.89$, 9.65, and 14.42 for $N=2$, 3, and 4. For comparison, the best-fit values of Ω to the numerical results of Fig. 4 are $\Omega/(|\xi|^2/\omega)\approx 5.74$, 4.46, and 13.40 for $N=2$,3, and 4, respectively. We do not ascribe great significance to the discrepancy between the analytically obtained and best-fit numerical values for N = 3, since the latter is inferred from the shape of the envelope of the oscillating inversion curve of Fig. 3(b).

3. Three excitations ($m=3$)

In this case the wave function at any time t in the interaction picture is

$$|\Psi^{\rm I}(t)\rangle=\sum_{i=0}^{3} C_i(t)|3;i\rangle, \tag{33}$$

where the expansion coefficients satisfy

$$i\frac{dC_3(t)}{dt}=\xi\sqrt{3N}C_2(t),$$

$$i\frac{dC_2(t)}{dt}=\xi^*\sqrt{3N}C_3(t)+\Omega C_2(t)+2\xi\sqrt{N-1}C_1(t),$$

$$i\frac{dC_1(t)}{dt}=2\xi^*\sqrt{N-1}C_2(t)+\Omega C_1(t)+\xi\sqrt{3(N-2)}C_0(t),$$

$$i\frac{dC_0(t)}{dt}=\xi^*\sqrt{3(N-2)}C_1(t).$$

The inversion of the system in this case is

$$\mathcal{S}(t)=|C_2(t)|^2+2|C_1(t)|^2+3|C_0(t)|^2. \tag{34}$$

In Fig. 5 we show the results obtained from the full numerical solution (solid line) and from Eq. (34) (dashed line) for $N=3$ and $N=4$ junctions. Here the best-fit values of Ω are given such that $\Omega/(|\xi|^2/\omega)\approx 7$ and 13.4, respectively. For comparison, the second-order degenerate perturbation calculation mentioned above gives $\Omega/(|\xi|^2/\omega)=10$ and 15.5. Once again, the agreement between the analytical and numerical best-fit values is quite good, especially considering that the numerical values are extrapolated from the envelope of a curve which is itself obtained numerically and, hence, are subject to some uncertainty.

E. Coherent state

As a last nontrivial case we consider the system such the resonator is initially in a coherent state. This case is of special interest because the number eigenstates of the electromagnetic field are generally difficult to generate. Nevertheless, as we will see below, our main conclusion—namely, the enhancement of the junction-resonator interaction with increasing N—still holds in this case.

A coherent state $|\beta\rangle$ with an average number of photons $\langle n\rangle=|\beta|^2$ can be written as a linear superposition of number states $|s\rangle$ with a Poisson distribution:

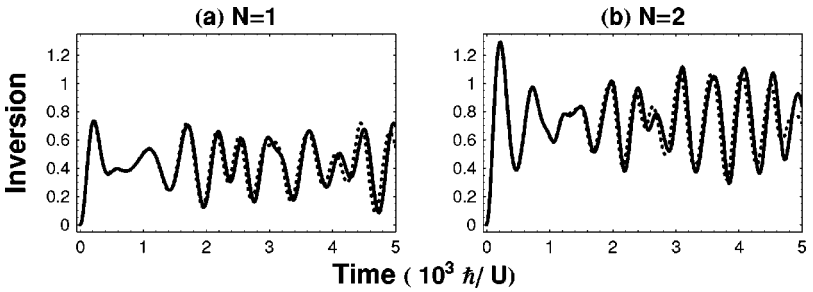

FIG. 6. Time-dependent inversion $\mathcal{S}(t)$ for (a) $N=1$ and (b) $N=2$ junctions given that the initial state of the resonator is a coherent state with $\langle n\rangle=2$ and the junctions are initially in their ground states. The solid line denotes the full numerical solution, and the dotted line follows from the two-level approximation.

$$|\beta\rangle=\sum_s e^{-|\beta|^2/2}\frac{\beta^s}{\sqrt{s!}}|s\rangle. \tag{35}$$

For $N=1$, it is easily proved that the inversion of the system is given by

$$\mathcal{S}(t)=\frac{1}{2}-\frac{1}{2}e^{-|\beta|^2}\sum_{s=0}^{\infty}\frac{|\beta|^{2s}}{s!}\cos(2\xi\sqrt{s}t), \tag{36}$$

which shows the phenomenon of ''collapse and revival''; i.e., the envelope of the Rabi oscillations periodically ''collapses'' to a half with a subsequent ''revival'' at later times.[1,26] For $N>1$, the analysis of the previous subsection can be readily generalized to treat a coherent initial state.

As an example, we show in Fig. 6 the inversion $\mathcal{S}(t)$ for $N=1$ and $N=2$ assuming $\beta=\sqrt{2}$. The solid lines follow from the full Hamiltonian while the dashed lines are from the reduced Hamiltonians of Eqs. (12) and (15). The best-fit value of Ω in this case is $\Omega\approx 5.74|\xi|^2/\omega$. This value of Ω is the same as for the number state considered before with $m=2$ excitations. The enhancement of the junction-cavity interaction can be seen from the decrease of the period of oscillation in Fig. 6(b) compared to Fig. 6(a).

IV. SUMMARY AND CONCLUSIONS

The present paper extends our previous publication[1] to the case of $N>1$ identical junctions interacting with a single lossless cavity mode. Although oversimplified, this model has the advantage that we can obtain detailed, numerically exact dynamical solutions with a range of initial conditions. Our treatment of the Hamiltonian is entirely quantum mechanical.

We have diagonalized the Hamiltonian exactly, using a basis large enough to ensure accuracy, and compared our results to those of a reduced Hamiltonian [Eq. (14) or (15)], in which each junction is represented by two levels only. The reduced Hamiltonian resembles the one-mode Dicke Hamiltonian but contains a direct interaction term between the junctions. For short times (of the order of several Rabi periods) the agreement between results obtained from the two-level Hamiltonian of Eq. (14) or (15) and those found from the full Hamiltonian [Eq. (1)] is excellent, but decreases somewhat for longer times. Also, as expected for a Dicke-like model, the system of several identical junctions behaves just like a single junction for short times, but with an enhanced junction-cavity coupling strength $\xi\sqrt{N}$ and a correspondingly decreased Rabi period. This reduced Rabi period might be smaller than the coherence time, making these oscillations experimentally observable.

A striking feature of our numerical results is the presence of an effective *direct* interaction between the junctions, resulting from the cavity. This interaction has the form of a dipole-dipole interaction between the ''spins'' representing the two lowest levels of each junction. This extra interaction must be included in the effective two-level Hamiltonian in order for it to reproduce the numerical results obtained from the full Hamiltonian at long times. While the analogous term in atomic physics is a true dipole-dipole interaction, in our model it arises through the effects of the higher-energy levels on the two lowest states, as calculated through Löwdin second-order perturbation theory.

Our numerically calculated strengths of this effective interaction agree reasonably well with those predicted by second-order perturbation theory. The slight discrepancies between the two probably arise from uncertainties in the numerically calculated strengths, which are read off a solution to coupled time-dependent differential equations. We conclude that second-order perturbation theory is adequate to treat departures from the Dicke model for the junction-cavity interaction strengths we have considered. In reality, we have probably considered a worst-case scenario for the Dicke model—i. e., we have considered an interaction strength which is larger than is likely to be encountered in a real system (see below)—the perturbation treatment is likely to be even more accurate than what we have found.

Before concluding, we briefly discuss the case of nonidentical junctions. Obviously, this is an important consideration for Josephson junctions since, in contrast to atoms, it is impossible to fabricate perfectly identical Josephson junctions. To test whether the coherence survives when the junctions are nonidentical, we have repeated the two-junction calculations shown in Figs. 2–4 using junctions which differ slightly from one another. For convenience, we considered two junctions with slightly different values of J, but the same values of U. With these parameters, the resonant values of $\overline{n}$ are slightly different for the two junctions in the same cavity. We find that, if the two J's differ only slightly (by only 1%–3% for our choice of J/U), the coherent behavior, as signaled by the $\sqrt{2}$ enhancement of the Rabi oscillation frequency, survives. For greater differences between the J's, the coherence disappears and one finds instead two distinct resonances, each with an unenhanced Rabi oscillation frequency.

We have also carried out similar calculations in other parameter regimes—specifically, for Q varying between 0.01 and about 3 (with g remaining unchanged). The qualitative

results—that is, the presence of Rabi oscillations and the characteristic frequency changes associated with coherence—are unchanged.

Finally, we briefly discuss whether the present system could be constructed in the laboratory. The values of the parameters entering the model have been estimated previously.[1] For a cylindrical cavity of length d and radius r, a typical g [Eq. (4)] was found to be $g \sim 3l\sqrt{\alpha/(rd)}$, where $\alpha = e^2/(\hbar c) \sim 1/137$ is the fine structure constant and $l_\parallel$ is the junction thickness. Achievable values of the other parameters were estimated as $U \sim 10^{-15}$ erg ~ 8 K, and $\omega \sim 2c/r$. In order to attain $Q \sim 0.7$, as used in our calculations, we need $J/k_B \sim 3$ K or a critical current $I_c \sim 0.1 \mu$A. To obtain $\hbar\omega/U \sim 0.3$, these estimates give $x \sim 10^{-3}$ cm and $r \sim 0.1-1$ cm, where x is a typical junction dimension (e.g., island size, junction radius). These values, which would give $\omega \sim 100$ GHz, are achievable with present technology. The corresponding maximum power radiated into the cavity [Eq. (25)] will be of the order of $10^{-11}\sqrt{N}$ W.

However, the value $g = 0.15$, used in our calculations, is much larger than the value $g \sim 10^{-5}$ which was previously estimated[1] to result from the parameters just described. Such a small g would certainly cause the two-level approximation to be even better than found in our calculations here. But it would have the disadvantage of producing longer Rabi oscillation time and hence of making coherence over a Rabi period more difficult to achieve; it would also lead to a much lower power radiated. Thus, the largest possible g is clearly desirable in order to detect this effect. The Rabi period could also be decreased by junctions with larger critical currents than those considered here. However, the critical current is limited by the requirement $\hbar\omega \ll \Delta$, where Δ is the superconducting gap. Thus, the best hope for observing these oscillations may be to reduce the decoherence effects.

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