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Role of long-range Coulomb interactions in granular superconductors

R. S. Fishman* and D. Stroud
Physics Department, The Ohio State University, Columbus, Ohio 43210
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This paper uses the self-consistent phase-phonon approximation to study the effect of long-range Coulomb interactions in a superconducting array. We find that crucial features of the results of mean-field theory are confirmed: long-range interactions enhance the superconducting state and lower the critical value of the grain diameter, below which superconductivity is impossible. However, the reentrant phase transition found in the mean-field solution is absent from the self-consistent results. This may be because the self-consistent approximation is invalid when the superconducting state is suppressed and phase fluctuations are large, such as in the reentrant regime of the mean-field theory.

I. INTRODUCTION

Over the past ten years, many experiments[1-8] on granular superconductors have observed nonmonotonic behavior of the resistivity, characterized by a dip in the resistance below the bulk superconducting transition temperature T_c^0 followed by a rise at still lower temperatures. Early theoretical work[9] on a mean-field (MF) Hamiltonian for superconducting arrays predicted the existence of a reentrant phase transition from the superconducting to the normal state with decreasing temperature. Presumably, the disorder present in a granular superconductor would change this true reentrant transition into the quasireentrant transition that is observed. However, further work[10,11] revealed that when the correct boundary conditions on the phase variables of the superconducting grains were imposed, the predicted reentrance did not appear.

Assuming the temperature is low enough that fluctuations of the magnitude of the order parameters can be neglected, the Hamiltonian for a superconducting array can be written $H=H_0+H_1$, where

$$H_0=2\sum_{i,j} U_{ij}\hat{n}_i\hat{n}_j \ , \tag{1}$$

$$H_1=J\sum_{\langle i,j\rangle}[1-\cos(\phi_i-\phi_j)] \ . \tag{2}$$

Here $\hat{n}_i=-id/d\phi_i$ is the excess Cooper-pair number operator conjugate to the phase ϕ_i of the ith grain. The positive matrix elements U_{ij} are the Coulomb interactions between the ith and jth grains, and the positive quantity J is the coupling strength for Cooper pair tunneling between nearest-neighbor grains. For simplicity, we do not include a magnetic field and the effects of dissipation. We also assume that the superconducting array is translationally invariant so that $U_{ij}=U(\mathbf{x}_i-\mathbf{x}_j)$, where $\mathbf{x}_i$ is the position of the ith grain. In the simplest model for the Coulomb interactions, the diagonal component U_{11} is inversely proportional to the diameter of the grains.[12] The charging energy H_0 tends to inhibit Cooper pair tunneling between grains while the Josephson energy H_1 favors the phase-coherent state. When the elements U_{ij} become too large compared to J, the phase-coherent state becomes energetically prohibited and the array remains normal down to zero temperature. Most theoretical work on the Hamiltonian of Eqs. (1) and (2) has assumed either that $\underline{U}$ is diagonal[11,13,14] ($U_{ij}=U\delta_{ij}$) or that $\underline{U}$ contains only diagonal and nearest-neighbor matrix elements.[9,10,15]

In a previous paper[16] (hereafter FS), working in the MF approximation, we found that for sufficiently long-ranged U_{ij}, a superconducting array can reenter the normal state with decreasing temperature. The long-range interactions enhance the phase coherence by reducing the cost in charging energy for a Cooper pair to tunnel between the ith and jth grains from $4U_{11}$ to $4U_{11}-4U_{ij}$. We speculated that the observed behavior of the resistivity in granular superconductors was the consequence of long-range Coulomb interactions acting in a disordered system.

The present paper reexamines the possibility of reentrance and the role of long-range Coulomb interactions by using the self-consistent phase-phonon[17] (SCPP) approach. Several crucial features of the MF results are confirmed. Again, Coulomb interactions act to increase the transition temperature T_c of the array and to decrease the critical value of the grain diameter below which $T_c=0$. Unfortunately, perhaps due to the limitations of the SCPP theory near criticality, the reentrant features of the MF phase diagram are not duplicated. Within the region of validity of the SCPP approach, however, there is qualitative agreement with the MF results. Although we have not verified the reentrance found in the MF solution, neither can we reject the possibility that reentrance follows from an exact solution of the Hamiltonian of Eqs. (1) and (2).

This paper is divided into four parts. First, we review the MF solution of Eqs. (1) and (2), which yields reentrance for sufficiently long-ranged interactions. Then we discuss the SCPP formulation, generalizing the previous work of Wood and Stroud[13] for diagonal interactions. Using these two approximations to study a model for the interaction matrix $\underline{U}$ in three dimensions, we find that long-range Coulomb interactions enhance the phase-coherence of a superconducting array. Finally, we discuss the significance of the absence of reentrance within the SCPP approximation, concluding that the SCPP theory is invalid in the region where the MF theory finds reentrance.

II. MEAN-FIELD THEORY

The MF theory approximates H_1 by neglecting the coupling of phase fluctuations on different grains. Aside from constant terms, which do not change the results, the MF Hamiltonian is $H^{\mathrm{MF}}=H_0+H_1^{\mathrm{MF}}$, where

$$H_1^{\mathrm{MF}}=-zJ\langle\cos\phi\rangle\sum_i\cos\phi_i\ . \tag{3}$$

In zero field, $\langle e^{i\phi_i}\rangle=\langle e^{i\phi_j}\rangle\equiv\langle e^{i\phi}\rangle$ for all i,j and $\langle\sin\phi\rangle$ can be set to zero. The number of nearest neighbors is given by z. One of the central results of FS is the scaling expression

$$\frac{T_c(\{U_{ij}\})}{T_c(0)}=G(\{U_{ij}\})\ , \tag{4}$$

where[18]

$$G=\frac{T_c}{Z_0U_{11}}\sum_{\{n_i\}}\left\{e^{-E_0(\{n_i\})/T_c}\left[1-4\left[\sum_k U_{1k}n_k/U_{11}\right]^2\right]^{-1}\right\}\ , \tag{5}$$

$$Z_0=\sum_{\{n_i\}}e^{-E_0(\{n_i\})/T_c}\ . \tag{6}$$

In Eqs. (5) and (6), $E_0(\{n_i\})=2\sum_{i,j}U_{ij}n_in_j$ is the eigenvalue of H_0 for a given charge configuration $\{n_i\}$. The coupling strength J enters this result only through $T_c(0)=zJ/2$, which is the transition temperature evaluated for all $U_{ij}=0$.

Equations (4)–(6) can be solved perturbatively for T_c in the regime $T_c/U_{11}\ll 1$. As shown in FS, the normal state will reenter below the superconducting transition if $S>0$, where

$$S=\sum_j\frac{U_{1j}^2}{U_{11}^2-4U_{1j}^2}\ . \tag{7}$$

For a diagonal interaction matrix, $S=-\frac{1}{3}$ and no reentrance occurs. If only the diagonal and nearest-neighbor matrix elements of $\underline{U}$ are nonzero, reentrance cannot occur in greater than one dimension[10] without first destabilizing the neutral ground state $\{n_i\}=0$. On the other hand, for sufficiently long-ranged U_{ij}, S can be positive and the system can be reentrant, while the charging energy E_0 remains positive definite.

In FS we also derived an expression for G that is useful for perturbative expansions in the parameter U_{11}/T_c. By interpolating between the results at small and large T_c/U_{11}, we can estimate $T_c^*=T_c(\{U_{ij}\})/zJ$ for any $\alpha=zJ/U_{11}$ and interaction matrix $\underline{U}$. An interesting result of this analysis is that in the limit $T_c^*\to 0$, when the superconducting transition is suppressed, $\alpha\to 2$ regardless of the values of the off-diagonal elements of $\underline{U}$. This happens because nontrivial charge configurations are energetically prohibited near $T_c^*=0$, so that the off-diagonal interactions have no effect in this limit. Reentrance occurs if Eqs. (4)–(6) have a nontrivial solution for T_c^* when $\alpha<2$.

III. SELF-CONSISTENT PHASE-PHONON THEORY

The SCPP theory was first applied to superconducting arrays by Simanek[19] and later by Wood and Stroud,[13] all of whom studied diagonal interactions. In this approach, the exact free energy is replaced by the trial free energy

$$F_t=F_h+\sum_{\langle i,j\rangle}\{J[1-\langle\cos(\phi_i-\phi_j)\rangle]-\tfrac{1}{2}K\langle(\phi_i-\phi_j)^2\rangle\}\ , \tag{8}$$

where F_h is constructed from the harmonic Hamiltonian $H^h=H_0+H_1^h$ with

$$H_1^h=\tfrac{1}{2}K\sum_{\langle i,j\rangle}(\phi_i-\phi_j)^2\ . \tag{9}$$

All expectation values are evaluated in an ensemble defined by H^h. The variational parameter K is determined by minimizing F_t, with the result

$$\frac{K}{J}=\exp[-\tfrac{1}{2}\langle(\phi_i-\phi_j)^2\rangle]\ , \tag{10}$$

where i and j are nearest neighbors. The appeal of the SCPP approach is that the rotation symmetry of the exact Hamiltonian is preserved. The small amplitude phase oscillations of the harmonic and exact Hamiltonians have the same dispersion relations, with the variational parameter K replacing the coupling strength J. The MF theory, on the other hand, replaces the phase phonons of the exact Hamiltonian with a set of Einstein oscillators.

As phase fluctuations increase K/J decreases, and the exact Hamiltonian becomes increasingly anharmonic. Qualitatively, the SCPP approximation should break down when $\langle(\phi_i-\phi_j)^2\rangle\approx(\pi/2)^2$ or $K/J\approx 1/e$. An artifact of the SCPP approximation, when applied to any system,[17] is that the phase transition becomes first order, characterized in this case by a jump in K/J and $\langle\cos\phi\rangle$ at T_c^*.

Generalizing the method of Wood and Stroud to treat long-range interactions, we can solve for K/J and $\langle\cos\phi\rangle$ in terms of the Fourier-transformed variables

$$U_{\mathbf{q}}=\sum_j U_{1j}e^{-i\mathbf{q}\cdot\mathbf{x}_j}\ , \tag{11a}$$

$$\hat{n}_{\mathbf{q}}=\frac{1}{\sqrt{N}}\sum_j \hat{n}_j e^{-i\mathbf{q}\cdot\mathbf{x}_j} , \tag{11b}$$

$$\phi_{\mathbf{q}}=\frac{1}{\sqrt{N}}\sum_j \phi_j e^{-i\mathbf{q}\cdot\mathbf{x}_j} , \tag{11c}$$

where N is the number of grains and $\mathbf{x}_1$ is set equal to 0 in Eq. (11a). Transformed to these variables, the harmonic Hamiltonian becomes

$$H^h=2\sum_{\mathbf{q}} U_{\mathbf{q}}\hat{n}_{\mathbf{q}}\hat{n}_{-\mathbf{q}}+\tfrac{1}{2}\sum_{\mathbf{q}} v_{\mathbf{q}}^2\phi_{\mathbf{q}}\phi_{-\mathbf{q}} , \tag{12}$$

where

$$v_{\mathbf{q}}^2=\frac{4K}{N}\sum_{\langle i,j\rangle}\sin^2[\tfrac{1}{2}\mathbf{q}\cdot(\mathbf{x}_i-\mathbf{x}_j)] , \tag{13}$$

and the sums over momentum are restricted to the first Brillouin zone. Defining the canonical variables $P_{\mathbf{q}}=2\sqrt{U_{\mathbf{q}}}\hat{n}_{\mathbf{q}}$ and $Q_{\mathbf{q}}=\phi_{\mathbf{q}}/2\sqrt{U_{\mathbf{q}}}$, so that $[P_{\mathbf{q}},Q_{\mathbf{q}'}]=-i\delta_{\mathbf{q},-\mathbf{q}'}$, we can bring Eq. (12) into the familiar harmonic form

$$H^h=\tfrac{1}{2}\sum_{\mathbf{q}}(P_{\mathbf{q}}P_{-\mathbf{q}}+\omega_{\mathbf{q}}^2Q_{\mathbf{q}}Q_{-\mathbf{q}}) , \tag{14}$$

where $\omega_{\mathbf{q}}=2v_{\mathbf{q}}\sqrt{U_{\mathbf{q}}}$ are the frequencies of the phase phonons of the harmonic Hamiltonian. Applying the equipartition theorem[20] to Eq. (14), we readily find that

$$\langle(\phi_i-\phi_j)^2\rangle=\frac{8}{N}\sum_{\mathbf{q}}\frac{U_{\mathbf{q}}}{\omega_{\mathbf{q}}}\sin^2[\tfrac{1}{2}\mathbf{q}\cdot(\mathbf{x}_i-\mathbf{x}_j)]\coth\frac{\beta\omega_{\mathbf{q}}}{2} \tag{15}$$

and

$$\langle\cos\phi\rangle=\exp\left[-\frac{1}{N}\sum_{\mathbf{q}}\frac{U_{\mathbf{q}}}{\omega_{\mathbf{q}}}\coth\frac{\beta\omega_{\mathbf{q}}}{2}\right] , \tag{16}$$

where $\beta=1/T$. Given the interaction matrix $\underline{U}$, Eqs. (10) and (15) can be solved for K/J, which is then used in Eq. (16) to obtain $\langle\cos\phi\rangle$. From the form of these equations it can be shown that the system is nonreentrant for any set of interactions. Also, as $T_c^*\to 0$, the limiting value of α depends on the off-diagonal elements of $\underline{U}$. These results will become clearer in the next section, when we examine a specific model for the interaction matrix.

IV. RESULTS

The model we now study assumes that the capacitance matrix $\underline{C}=\underline{U}^{-1}$ contains only diagonal and nearest-neighbor components. The charging Hamiltonian for this model is

$$H_0=\tfrac{1}{2}\sum_{i,j}C_{ij}\hat{V}_i\hat{V}_j$$
$$=\frac{C_{11}}{2z(1+\lambda)}\sum_{\langle i,j\rangle}(\hat{V}_i-\hat{V}_j)^2+\frac{C_{11}\lambda}{2(1+\lambda)}\sum_i\hat{V}_i^2 , \tag{17}$$

where $\hat{V}_i=-2\sum_j U_{ij}\hat{n}_j$ is the operator for the voltage induced on the ith grain by the charge configuration of Cooper pairs (assuming the electron charge e is set equal to 1). Notice that when $\lambda\to\infty$, the capacitance and interaction matrices become diagonal with $U_{11}=1/C_{11}$. As shown in FS, Eq. (17) yields a screened logarithmic interaction matrix for a two-dimensional square array and a screened $1/|\mathbf{x}_i-\mathbf{x}_j|$ interaction matrix for a three-dimensional simple cubic array. The dimensionless parameter λ corresponds to an inverse screening length which cuts off these interactions. In the remainder of this paper, we restrict ourselves to the physically interesting case of three dimensions. We then find that $U_{11}=I(\lambda)/C_{11}$, where $I(\lambda)$ is a monotonic function with the limiting values $I(0)=1.516$ and $I(\infty)=1$. The integral form of $I(\lambda)$ is given in FS.

This model for the charging Hamiltonian is useful for two reasons. First, it allows us to study the effect of long-range interactions by adjusting only the two theoretical parameters λ and $\alpha=6J/U_{11}$. For an array with intergrain distance s and grain diameter d, the self-interaction takes the functional form[12] $U_{11}=2e^2F(s/d)/d$, where $F(x)$ is a dimensionless, monotonically increasing function of x. Given the function $F(x)$, we can separately examine the effects of changing the range of the Coulomb interactions and of changing the parameters of the array. Second, this model allows us to study the effects of possible screening mechanisms in the array. At finite temperatures, Efetov[18] found that normal electron tunneling between grains provides one such screening mechanism, which increases the value of λ.

Using the MF theory to study this model, we found previously[16] that the array becomes reentrant when $\lambda<\lambda^*=0.18$. If s/d is a constant, then the array remains superconducting down to zero temperature only for $d\geq Fe^2/6J$. If the screening parameter is less than λ^*, arrays with grain diameters just below $Fe^2/6J$ will reenter the normal state.

The perturbative expansions discussed in FS can be used to plot the phase diagram of the array in $\alpha-T^*$ space for a given λ. In Fig. 1 we plot T_c^* versus α for $\lambda=0.04<\lambda^*$ and for $\lambda=\infty$. The reentrance for $\lambda=0.04$, seen by the existence of a double-valued solution for T_c^* in the narrow region $1.99\leq\alpha<2$, is possible only for a very limited range of grain diameters. However, as $\lambda\to 0$, $S\to\infty$ and the reentrance should become much more pronounced. Unfortunately, the perturbative expansion for $T_c/U_{11}\ll 1$, used to obtain part of Fig. 1, becomes invalid as $\lambda\to 0$, so that we are restricted to larger values of λ with less pronounced reentrant bulges.

In applying the SCPP method to diagonal interaction matrices, it is customary[17] to adopt the Debye approximation for the phase-phonon frequencies $\omega_{\mathbf{q}}$ and to replace the cubic Brillouin zone by a sphere. To generalize this approach to the case of long-range interactions, we expand both $U_{\mathbf{q}}$ and $\omega_{\mathbf{q}}$ lowest order in $\mathbf{q}$, with the results

$$U_{\mathbf{q}}=\frac{6}{C_{11}}\frac{1+\lambda}{6\lambda+q^2a^2} , \tag{18}$$

$$\omega_{\mathbf{q}}^2=\frac{24K}{C_{11}}\frac{(1+\lambda)q^2a^2}{6\lambda+q^2a^2} , \tag{19}$$

where $a=s+d$ is the nearest-neighbor distance. Notice that as λ decreases from ∞ to 0, the phase-phonon spectrum changes drastically. At $\lambda=\infty$, $\omega_q^2=4Kq^2a^2/C_{11}$ takes the usual Debye form. But at $\lambda=0$, $\omega_q^2=24K/C_{11}$ is finite and independent of **q**. Thus the Goldstone modes of an array with diagonal interactions are replaced by the dispersionless plasma wave of an array with unscreened Coulomb interactions.[21]

With a spherical Brillouin zone, the governing equations of the array can be written as

$$\frac{K}{J}=\exp\left[-\frac{3}{\pi^2}(1+\lambda)^{1/2}\left[\frac{J}{K\alpha I}\right]^{1/2}\int_0^{q_Da}dx\left\{\frac{x}{(6\lambda+x^2)^{1/2}}\left[1-\frac{\sin x}{x}\right]\coth\left[\frac{x}{T^*}\left[\frac{1+\lambda}{6\lambda+x^2}\right]^{1/2}\left[\frac{K}{J\alpha I}\right]^{1/2}\right]\right\}\right], \tag{20}$$

and

$$\langle\cos\phi\rangle=\exp\left[-\frac{3}{2\pi^2}(1+\lambda)^{1/2}\left[\frac{J}{K\alpha I}\right]^{1/2}\int_0^{q_Da}dx\left\{\frac{x}{(6\lambda+x^2)^{1/2}}\coth\left[\frac{x}{T^*}\left[\frac{1+\lambda}{6\lambda+x^2}\right]^{1/2}\left[\frac{K}{J\alpha I}\right]^{1/2}\right]\right\}\right], \tag{21}$$

where $q_D=(6\pi^2)^{1/3}/a$ is the Debye wave vector. Equation (20) must be implicitly solved for K/J as a function of α, λ, and T^*.

In the $\alpha\to\infty$ limit, Eqs. (20) and (21) become independent of λ, as expected when the charging energy vanishes. In that limit Wood and Stroud obtained $T_c^*=0.57$, somewhat increased from the MF value of 0.50. As shown in Fig. 2 for three values of λ, T_c^* is a monotonically increasing, single-valued function of α, so that the array never reenters the normal state. If α^* is defined as the value of α in the limit $T_c^*\to 0$, then for the case of diagonal interactions studied by Simanek,[19] $\alpha^*=0.98$, decreased from the MF value of 2. Thus, in arrays with diagonal interactions, the SCPP theory suggests that superconductivity can persist for smaller grains than indicated by MF theory. Because of numerical errors, neither Simanek[19] nor Wood and Stroud[13] noticed this effect.

When $\lambda=0$, the Coulomb interactions are unscreened and α^* drops even further to 0.51. For general λ, α^* can be calculated by taking the $T^*=0$ limit of Eq. (20), with the result

$$\alpha^*=\frac{9e^2(1+\lambda)}{4\pi^4 I}\left[\int_0^{q_Da}dx\frac{1}{(6\lambda+x^2)^{1/2}}(x-\sin x)\right]^2, \tag{22}$$

which is plotted in Fig. 3. The strong dependence of α^* on λ within the SCPP approximation contrasts within the MF result that $\alpha^*=2$ for all λ.

Finally, in Fig. 4, the order parameter $\langle\cos\phi\rangle$ is plotted versus T^* for several values of α and for three values of λ. In all cases the phase transition is first order; this feature is an artifact[17] of the SCPP approximation. Notice that as $\alpha\to\alpha^*$, $\langle\cos\phi\rangle$ approaches a nonzero value, but that $T_c^*\to 0$ continuously.

Of course, the quantitative results discussed above are approximate, even within the SCPP theory, because only the lowest order **q** dependence of U_q and ω_q has been

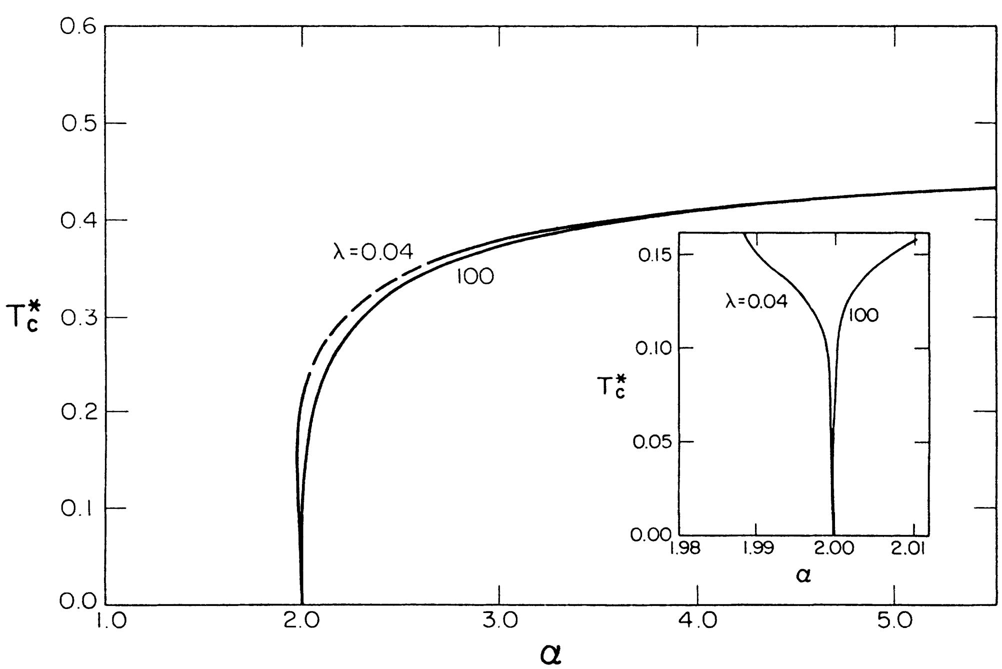

FIG. 1. T_c^* vs α for $\lambda=100$ and 0.04 using MF theory. The dashed line is an interpolation between the results for $T_c/U_{11}>1$ and $T_c/U_{11}\ll 1$. The inset figure contains the perturbative, MF results for $T_c/U_{11}\ll 1$. Reentrance occurs when $\lambda<0.18$.

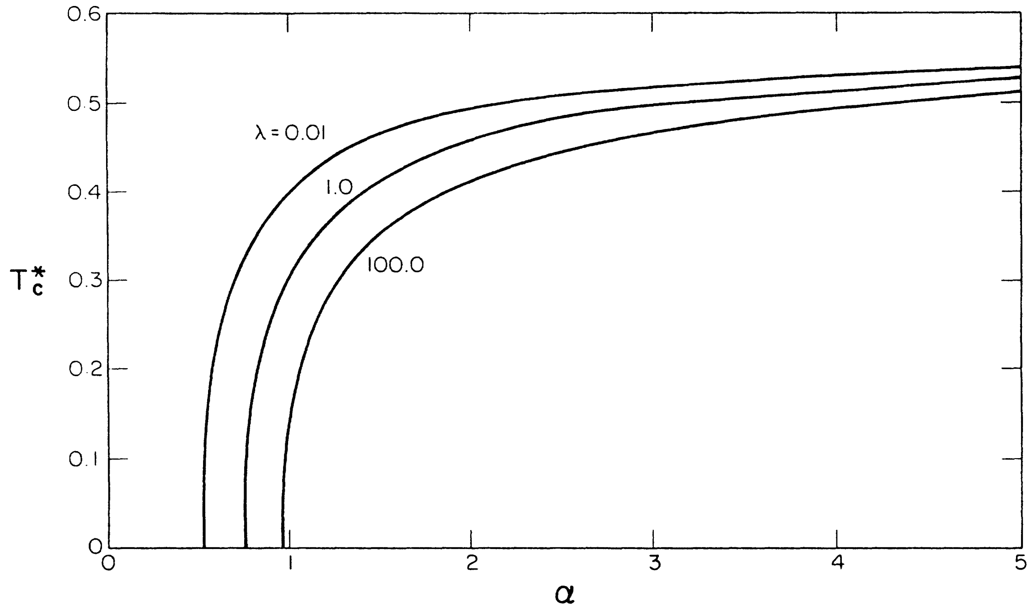

FIG. 2. T_c^* vs α for $\lambda = 100$, 1, and 0.01 using the SCPP theory. No reentrance occurs.

considered and because a spherical Brillouin zone has been used. If the cubic Brillouin zone is retained, we find that the results change only slightly. The $\alpha \to \infty$ limit of T_c^* decreases from 0.57 to 0.56, the $\lambda \to \infty$ limit of α^* increases from 0.98 to 1.11, and the $\lambda \to 0$ limit of α^* increases from 0.51 to 0.54. If in addition the full momentum dependence of $U_{\mathbf{q}}$ and $\omega_{\mathbf{q}}$ is included, the changes are much more dramatic. The $\alpha \to \infty$ limit of T_c^* decreases to $1/e = 0.37$, the $\lambda \to \infty$ limit of α^* increases to 1.75, and the $\lambda \to 0$ limit of α^* increases to 1.22. The values of α^* obtained from the full solution are still lower than the MF result $\alpha^* = 2$. More importantly, the qualitative features of the "Debye" results, including the absence of reentrance, are preserved in the full solution. Note that there is really no justification for truncating the Fourier transforms $U_{\mathbf{q}}$ and $\omega_{\mathbf{q}}$. The SCPP approximation

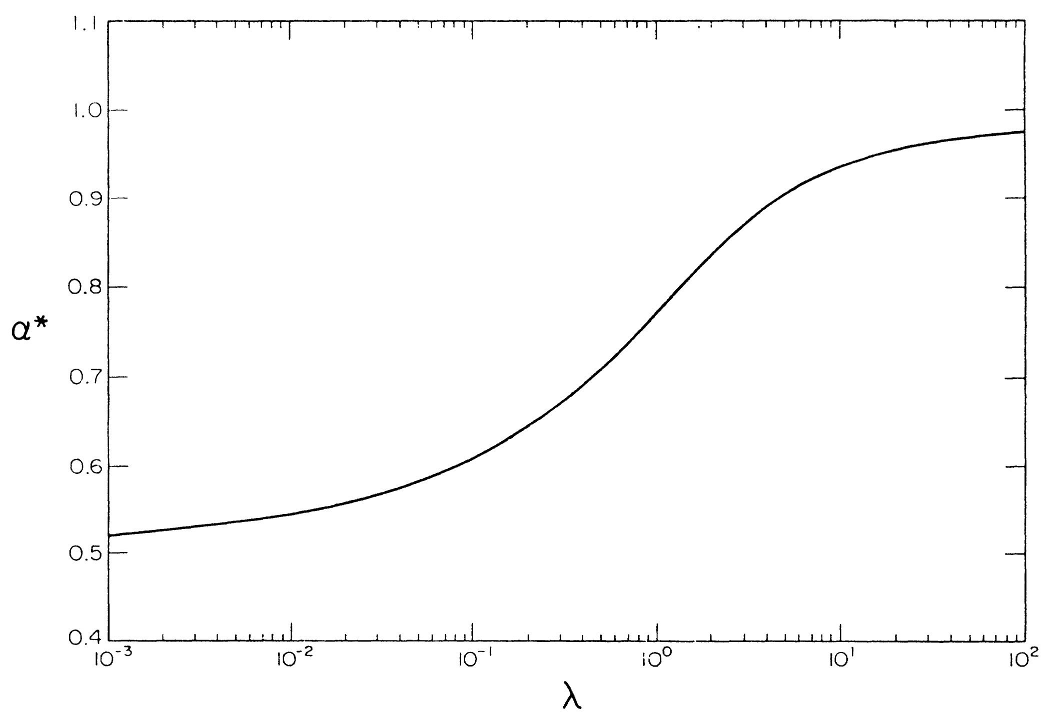

FIG. 3. α^* vs λ in the SCPP theory. We defined α^* as the value of α in the limit $T_c^* \to 0$.

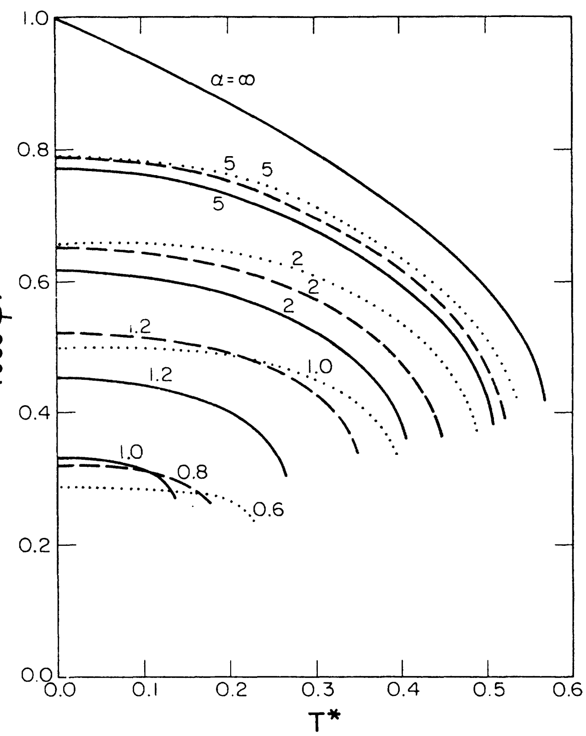

FIG. 4. $\langle\cos\phi\rangle$ vs T^* for several values of α and for $\lambda=100$ (solid), 1 (dashed), and 0.01 (dotted) within the SCPP theory.

requires only that the amplitudes of the phase phonons are small so that $\langle(\phi_i-\phi_j)^2\rangle \ll 1$, not that their wave-vectors also are small. The only reason to prefer the "Debye" approximation is calculational convenience.

V. DISCUSSION AND CONCLUSION

The most important result of this analysis is that in both the MF and SCPP theories, long-range interactions enhance superconductivity in the $\alpha-T^*$ phase diagram. Restricting our attention for the moment to values of α greater than 2, we see that T_c^* always decreases with increasing λ. A comparison of Figs. 1 and 2 reveals that T_c^* is more sensitive to the effect of long-range interactions within the SCPP theory for $\alpha\geq 2$.

If $\alpha<2$, of course, the MF theory yields a reentrant transition for $\lambda<\lambda^*$, while the SCPP theory yields no reentrance but does permit superconductivity in the regime $2>\alpha>1.75$ for all λ and in the regime $1.75>\alpha>1.22$ for $\lambda=0$. Both approximations agree, however, that the upper-critical value of T_c^* is a decreasing function of λ, so that the superconducting transition temperature is enhanced by long-range interactions for all values of α.

Both approximations also agree that the lowest value of α in the superconducting state, which we denote as α_c, is an increasing function of λ. In the absence of reentrance, $\alpha_c=\alpha^*$ since superconductivity is impossible for $\alpha<\alpha^*$. In the presence of reentrance, however, $\alpha_c<\alpha^*$ since a double-valued solution for T_c^* exists in some regime $\alpha_c<\alpha<\alpha^*$. For example, when the MF theory is used for $\lambda=0.04$, as in Fig. 1, a double-valued reentrant solution exists in the regime $1.99<\alpha<2$ so that $\alpha_c=1.99$. In the SCPP theory, on the other hand, $\alpha_c=\alpha^*$ for all values of λ. Notice that if s/d is constant, then $\alpha_c=3d_cJF/e^2$ is proportional to the critical grain diameter d_c. Arrays of grains with diameters less than d_c cannot become superconducting. The results of FS provide no lower bound to d_c because the sum S diverges and the perturbative expansions break down as $\lambda\rightarrow 0$. But the SCPP theory does place a lower bound on the grain diameter in a superconducting array: $d_c>0.41e^2/JF$. Both approximations agree that d_c is an increasing function of λ. Thus, long-ranged Coulomb interactions act to decrease the critical grain diameter, below which superconductivity is not allowed.

The absence of reentrance in the SCPP theory may have a very natural explanation. As indicated earlier, the SCPP approximation becomes worse as K/J decreases. For values of K/J less than $1/e$, $\langle(\phi_i-\phi_j)^2\rangle\geq 2$ and the exact Hamiltonian is very anharmonic. In this regime, the trial free energy of Eq. (8) provides a very poor lower bound to the exact free energy. Notice, however, that for $\alpha=\alpha^*, K/J=1/e^2$ and $\langle(\phi_i-\phi_j)^2\rangle=4$. For $\alpha\rightarrow\infty$, on the other hand, K/J varies from 1 at $T^*=0$ to $1/e$ at $T^*=T_c^*$. Therefore, the SCPP theory should not be trusted in the critical region near T_c^*, where phase fluctuations become important. Qualitative results of the theory for T_c^* as a function of α become increasingly suspect as α approaches α^*. In the region of small T_c^*, near α^*, the SCPP theory is no long valid, and the absence of reentrance should not be taken too seriously. Indeed, in the extremely anharmonic region near α^*, it seems likely that the MF theory is more accurate.

As mentioned earlier, experiments[1–8] do not observe a true reentrant phase transition in either two or three dimensions. The observed quasireentrant transitions are characterized by a dip in the normal state resistance below T_c^0, followed by a rise at lower temperatures. If this transition is identified with the reentrant transition of MF theory, then the results of FS can be used to estimate the value of α which separates the quasireentrant and superconducting regimes. As discussed in FS, superconductivity actually persists for values of α that are two orders of magnitude smaller than this estimate. One possible explanation[22] for this discrepancy is that dissipative effects allow superconductivity to exist for all $\alpha>0$ if the junction resistance R is less than a critical value R_c. For $R>R_c$, the observed quasireentrance may be a consequence of long-range Coulomb interactions acting in a dissipative, disordered environment.

Though study of the SCPP theory has been inadequate to verify the existence of reentrance, several other crucial feature of the MF theory have been confirmed. We have verified that long-range interactions enhance superconductivity in the $\alpha-T^*$ phase diagram. For a constant s/d, long-range interactions act to decrease the critical value of grain diameter, below which superconductivity is impossible. Within the range of validity of the SCPP theory, the qualitative features of the MF results have been confirmed.

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*Present address. Solid State Division, Oak Ridge National Laboratory, P. O. Box 2008, Oak Ridge, TN 37831.

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