

Model for a Josephson junction array coupled to a resonant cavity

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We describe a simple Hamiltonian for an underdamped Josephson array coupled to a single photon mode in a resonant cavity. Using a Hartree-like mean-field theory, we show that, for any given strength of coupling between the photon field and the Josephson junctions, there is a transition from incoherence to coherence as a function of N , the number of Josephson junctions in the array. Above that value of N , the energy in the photon field is proportional to N^2 , suggestive of coherent emission. These features remain even when the junction parameters have some random variation from junction to junction, as expected in a real array. Both of these features agree with recent experiments by Barbara and co-workers.

I. INTRODUCTION

Researchers have long sought to cause Josephson junction arrays to radiate coherently.^{1,2} To achieve this goal, a standard approach is to inject a dc current into an overdamped array. If this current is sufficiently large, it generates an ac voltage V_{ac} across the junctions, of frequency $\omega_J = 2eV_{dc}/\hbar$, where V_{dc} is the time-averaged voltage across the junction.³ Each junction then radiates (typically at microwave frequencies). If the junctions are coherently phase locked, the radiated power $P \propto N^2$, where N is the number of phase-locked junctions. This N^2 proportionality is a hallmark of phase coherence. But many difficulties inhibit phase coherence in practice. For example, the junctions always have a disorder-induced spread in critical currents, which produces a distribution of Josephson frequencies and makes phase locking difficult.⁴⁻⁷ Furthermore, in small-capacitance (and underdamped) Josephson junctions, quantum phase fluctuations inhibit phase locking.⁸⁻¹³ Thus, until recently, the most efficient coherent emission was found in two-dimensional arrays of *overdamped* Josephson junctions, where quantum fluctuations are minimal.

Recently, Barbara and co-workers have reported a remarkable degree of coherent emission in arrays of *underdamped* junctions.^{14,15} Their arrays were placed in a microwave cavity, so as to couple each junction to a resonant mode of the cavity. If the mode has a suitable frequency and is coupled strongly enough to the junction, it can be excited by a Josephson current through the junction. The power in this mode then feeds back into the other junctions, causing the array to phase lock and inducing a total power $P \propto N^2$. For a given coupling, Barbara and co-workers found that there is a threshold number of junctions, N_c , below which no emission was observed. The coupling, and hence N_c , could be varied by moving the array relative to the cavity walls.

Barbara and co-workers interpreted their results by analogy with the Jaynes-Cummings model^{16,17} of two-level atoms interacting with a radiation field in a single-mode resonant cavity. In this case, each Josephson junction acts as a two-level atom; the coupling between the “atoms” is provided by the induced radiation field. A dynamical calculation based on a model similar to that of Jaynes and Cummings

has been carried out by Bonifacio and collaborators^{18,19} for Josephson junction arrays in a cavity. Their model does produce spontaneous emission into the cavity above a threshold junction number, provided that the Heisenberg equations are treated in a certain semiclassical limit appropriate to large numbers of photons in the cavity.

In this paper, we present a simple model for the onset of phase locking and coherent emission by an underdamped Josephson junction array in a resonant cavity. We also calculate the threshold for the onset of phase coherence, using a form of mean-field theory. Our model derives from more conventional models of Josephson junction arrays, but treats the interaction with the radiation field quantum mechanically. Within the mean-field theory, we find that for *any* strength of that coupling, there exists a threshold number of junctions, N_c , in a linear array above which the array is coherent. Above that threshold, the energy in the photon field is *quadratic* in the number of junctions, as found experimentally.¹⁴ The model is easily generalized to two-dimensional arrays. Furthermore, as we show, the threshold condition and N^2 dependence of the energy in the radiation field are both preserved even in the presence of the disorder which will be present in any realistic array. Finally, the coupling constant between junctions and radiation field can, in principle, be calculated explicitly, given the geometry of the array and the resonant cavity.

The remainder of this paper is organized as follows. In Sec. II, we describe our model and approximations. Our numerical and analytical results are presented in Sec. III. Section IV presents a brief discussion and suggestions for future work.

II. MODEL

A. Hamiltonian

We consider a Josephson junction array containing N junctions arranged in series, placed in a resonant cavity, arranged in a geometry shown schematically in Fig. 1. It is assumed that there is a total time-averaged voltage Φ across the chain of junctions; this boundary condition is discussed further below. The Hamiltonian for this array is taken as the sum of four parts:

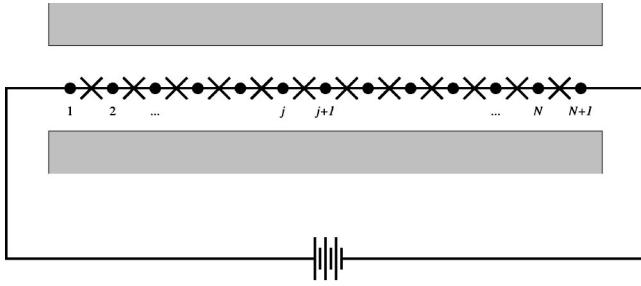


FIG. 1. Schematic of the geometry used in our calculations, consisting of an underdamped array of Josephson junctions coupled to a resonant cavity and subjected to an applied voltage Φ . The array consists of $N+1$ grains, represented by the dots, coupled together by Josephson junctions, represented by the crosses. Nearest-neighbor grains j and $j+1$ are connected by a Josephson junction, and the cavity is assumed to support a single resonant photonic mode. For the specific calculations carried out in this paper, we assume a one-dimensional array, as shown, and a specific form for the capacitive energy, as discussed in the text.

$$H = H_J + H_C + H_{\text{phot}} + H_{\text{int}}. \quad (1)$$

Here $H_J = -\sum_{j=1}^N E_{Jj} \cos \phi_j$ is the Josephson coupling energy, where ϕ_j is the gauge-invariant phase difference across the j th junction, $E_{Jj} = \hbar I_{cj}/q$, the critical current of the j th Josephson junction is I_{cj} , and $q = 2|e|$ is the magnitude of a Cooper pair charge. H_C is the capacitive energy of the array, which we assume can be written in the form $H_C = \sum_{j=1}^N E_{Cj} n_j^2$, where $E_{Cj} = q^2/(2C_j)$, the capacitance of the j th junction is C_j , and n_j is the difference in the number of Cooper pairs on the two grains connected by the j th junction. The field energy may be written as $H_{\text{phot}} = \hbar \Omega (a^\dagger a + 1/2)$, where Ω is the frequency of the cavity resonant mode (assumed to be the only mode supported by the cavity), and a^\dagger and a are the usual photon creation and annihilation operators, satisfying the commutation relations $[a, a^\dagger] = 1; [a, a] = [a^\dagger, a^\dagger] = 0$. We assume that the number operator n_j and phase ϕ_k have commutation relations $[n_j, \exp(\pm i \phi_k)] = \pm \exp(\pm i \phi_k) \delta_{jk}$, which implies that n_j can be represented as $-i \partial / (\partial \phi_j)$.

The crucial term in the Hamiltonian for phase locking is the interaction term H_{int} . We write this in the form $H_{\text{int}} = (1/c) \int \mathbf{J} \cdot \mathbf{A} d^3x$, where \mathbf{J} is the Josephson current density, \mathbf{A} is the vector potential corresponding to the electric field of the cavity mode, c is the speed of light, and the integral is carried out over the cavity volume. Since \mathbf{J} is comprised of the Josephson currents $I_{cj} \sin \phi_j$ passing through the junctions, we may write this last term as $H_{\text{int}} = \sum_{j=1}^N E_{Jj} A_j \sin \phi_j$:

$$A_j = \frac{q}{\hbar c} \int_j^{j+1} \mathbf{A} \cdot d\mathbf{s}, \quad (2)$$

where the integral is across the j th junction, i.e., between the j th and $(j+1)$ th superconducting grain (see Fig. 1).²⁰ The phase factor A_j may be expressed in terms of the creation and annihilation operators for the photon quanta as (in esu) $A_j = i \sqrt{\hbar c^2 / (2\Omega)} (a - a^\dagger) \alpha_j$, where α_j is a suitable coupling constant depending on the polarization and electric field of

the cavity mode.²¹ It is convenient to introduce the notation $\hbar g_j / \sqrt{V} = E_{Jj} \alpha_j \sqrt{\hbar c^2 / (2\Omega)}$, where V is the cavity volume.

Finally, we need to discuss suitable boundary conditions for this linear array. Let Φ_j denote the time-averaged voltage across the j th junction. For our assumed form of the capacitive energy, $\Phi_j = q \langle n_j \rangle / C_j$, where $\langle \dots \rangle$ denotes a quantum-statistical average. We will impose a constant-voltage boundary condition, by requiring that $\Phi = \sum_{j=1}^N \Phi_j$ across the linear array should take on a specified value. Here Φ represents the total, time-averaged voltage across the linear array. It is most convenient to impose the constant-voltage boundary condition by using the method of Lagrange multipliers, adding to the Hamiltonian a term $\mu \sum_{j=1}^N \Phi_j = \mu \sum_{j=1}^N q n_j / C_j$, where the constant μ will be determined later by specifying Φ .

If we combine all these assumptions, we can finally write an explicit expression for H' , the operator whose ground state we seek:

$$\begin{aligned} H' &= H + \mu \sum_{j=1}^N q n_j / C_j \\ &= \hbar \Omega \left(a^\dagger a + \frac{1}{2} \right) + \sum_{j=1}^N \left(-E_{Jj} \cos \phi_j + E_{Cj} n_j^2 \right. \\ &\quad \left. + \mu q n_j / C_j + \frac{\hbar g_j}{\sqrt{V}} i (a - a^\dagger) \sin \phi_j \right). \end{aligned} \quad (3)$$

B. Mean-field approximation

The eigenstates of H' are many-body wave functions, depending on the phase variables ϕ_j and n_j , and the photon coordinates a and a^\dagger . We will estimate the ground-state wave function and energy using a mean-field approximation. To define this approximation, we express H' in the form

$$H' = H_{\text{phase}} + H_{\text{phot}} + H_{\text{int}}, \quad (4)$$

where $H_{\text{phase}} = \sum_{j=1}^N (-E_{Jj} \cos \phi_j + E_{Cj} n_j^2 + \mu q n_j / C_j)$, and $H_{\text{int}} = i (\hbar / \sqrt{V}) (a - a^\dagger) \sum_{j=1}^N g_j \sin \phi_j$. The mean-field approximation consists of writing²²

$$\begin{aligned} H_{\text{int}} &\approx i \frac{\hbar}{\sqrt{V}} \left(\langle a - a^\dagger \rangle \sum_{j=1}^N g_j \sin \phi_j + (a - a^\dagger) \right. \\ &\quad \left. \times \sum_{j=1}^N g_j \langle \sin \phi_j \rangle - \langle a - a^\dagger \rangle \sum_{j=1}^N g_j \langle \sin \phi_j \rangle \right). \end{aligned} \quad (5)$$

With this approximation, H' is decomposed into a sum of one-body terms, each of which depends only on the photon variables or on the phase variables of one junction, plus a constant term. The eigenstates of H' , in this approximation, are of the form $\Psi(a, a^\dagger, \{\phi_j\}) = \psi_{\text{phot}}(a, a^\dagger) \prod_{j=1}^N \psi_j(\phi_j)$, where ψ_{phot} and the ψ_j 's are one-body wave functions.

That part of H' which depends on photon variables may be written $H'_{\text{phot}} = H_{\text{phot}} + i (\hbar / \sqrt{V}) (a - a^\dagger) \sum_{j=1}^N g_j \langle \sin \phi_j \rangle$, where $\langle \sin \phi_j \rangle$ denotes a quantum-mechanical expectation value with respect to $\psi_j(\phi_j)$. With the definition $\lambda_j = \langle \exp(i \phi_j) \rangle$, H'_{phot} takes the form

$$H'_{\text{phot}} = \hbar\Omega \left(a^\dagger a + \frac{1}{2} \right) + i \sum_{j=1}^N \frac{\hbar g_j \tilde{\lambda}_j}{\sqrt{V}} (a - a^\dagger), \quad (6)$$

where $\tilde{\lambda}_j = (\lambda_j - \lambda_j^*) / (2i) = \langle \sin \phi_j \rangle$. This is the Hamiltonian of a *displaced* harmonic oscillator; its ground-state energy eigenvalue $E_{\text{phot};0}$ is readily found by completing the square to obtain

$$H'_{\text{phot}} = \hbar\Omega \left(b^\dagger b + \frac{1}{2} \right) - \frac{\hbar}{\Omega V} \eta^2, \quad (7)$$

where $b^\dagger = a^\dagger + i \sum_{j=1}^N g_j \tilde{\lambda}_j / (\Omega \sqrt{V})$, and we have defined

$$\eta = \sum_{j=1}^N g_j \tilde{\lambda}_j. \quad (8)$$

(Note that b and b^\dagger have the same commutation relations as a and a^\dagger , i.e., $[b, b^\dagger] = 1$.)

The resulting ground-state energy of H'_{phot} is

$$E_{\text{phot};0} = \frac{1}{2} \hbar\Omega - \frac{\hbar}{\Omega V} \eta^2. \quad (9)$$

Similarly in the ground state, since $\langle b^\dagger \rangle = 0$,

$$\langle a^\dagger \rangle = -i \frac{1}{\Omega \sqrt{V}} \eta. \quad (10)$$

Also, the total energy stored in the photon field is

$$E_{\text{phot}} = \hbar\Omega \left\langle a^\dagger a + \frac{1}{2} \right\rangle = \frac{\hbar\Omega}{2} + \frac{1}{\Omega V} \eta^2, \quad (11)$$

where we use Eq. (10) and the fact that $\langle b^\dagger b \rangle = 0$ in the ground state.

The wave function $\psi_j(\phi_j)$ is an eigenstate of the effective single-particle Schrödinger equation $H_j \psi_j = E_j \psi_j$, where

$$H_j = -E_{Jj} \cos \phi_j + \frac{1}{2C_j} (q^2 n_j^2 + 2\mu q n_j) + \frac{\hbar g_j}{\sqrt{V}} i \sin \phi_j (a - a^\dagger), \quad (12)$$

and $\langle a - a^\dagger \rangle = 2i \eta / (\Omega \sqrt{V})$. Using this expression and completing the square, we can write

$$H_j = E_{Cj} (n_j - \bar{n})^2 - E_{Cj} \bar{n}^2 - \frac{2\hbar g_j}{\Omega V} \eta \sin \phi_j - E_{Jj} \cos \phi_j, \quad (13)$$

where we have written $\bar{n} = -\mu/q$. (Note that with our definition of μ , it does not have the dimension of energy.) Introducing the notation

$$E_{\alpha;j}^2 = E_{Jj}^2 + E_{\text{int};j}^2, \quad (14)$$

where we define $E_{\text{int};j} = 2\hbar \eta g_j / (\Omega V)$ and $\phi_{\alpha;j} = \tan^{-1}(E_{\text{int};j}/E_{Jj})$, we obtain

$$H_j = -E_{\alpha} \cos(\phi_j - \phi_{\alpha;j}) + E_{Cj} [(n_j - \bar{n})^2 - \bar{n}^2]. \quad (15)$$

The Schrödinger equation

$$H_j \psi_j(\phi_j) = E_j \psi_j(\phi_j), \quad (16)$$

where H_j is given by expression (15), can be transformed into Mathieu's equation by a suitable change of variables. Specifically, if we use the representation $n_j = -i \partial / (\partial \phi_j)$, and we also make the change of variables $\psi_j(\phi_j) = \exp(i\bar{n}\phi_j) u_j(\phi_j)$, then Eq. (16) takes the form

$$(E_j + \bar{n}^2) u_j = -E_{\alpha;j} \cos(\phi_j - \phi_{\alpha;j}) u_j - E_{Cj} \frac{\partial^2 u_j}{\partial \phi_j^2}. \quad (17)$$

Since ϕ_j and $\phi_j + 2\pi$ represent the same physical state, the physically significant eigenstate $\psi_j(\phi_j)$ should satisfy $\psi_j(\phi_j + 2\pi) = \psi_j(\phi_j)$ or, equivalently,

$$u_j(\phi_j + 2\pi) = \exp(-2\pi i \bar{n}) u_j(\phi_j). \quad (18)$$

Thus, the solutions to Eq. (17) are Mathieu functions satisfying the boundary condition (18).

The total ground-state energy of the coupled system takes the form

$$E_{\text{tot}} = \sum_{j=1}^N E_{j;0} + E_{\text{phot};0} + E_d, \quad (19)$$

where $E_{j;0}$ is the lowest eigenvalue of the Schrödinger equation (16). Note that the $E_{j;0}$'s are also functions of the $\tilde{\lambda}_j$'s, but only through the variable η . E_d is a ‘‘double-counting correction’’ which compensates for the fact that the interaction energy is included in both $E_{\text{phot};0}$ and the $E_{j;0}$'s; it is given by the negative of the expectation value of the last term on the right-hand side of Eq. (12), i.e.,

$$E_d = -i \frac{\hbar}{\sqrt{V}} \langle a - a^\dagger \rangle \sum_{j=1}^N g_j \langle \sin \phi_j \rangle = \frac{2\hbar \eta^2}{\Omega V}. \quad (20)$$

Hence, the total ground-state energy is

$$E_{\text{tot}}(\eta) = \sum_{j=1}^N E_{j;0} + E_{\text{phot};0} + E_d = \sum_{j=1}^N E_{j;0}(\eta) + \frac{1}{2} \hbar\Omega + \frac{\hbar}{\Omega V} \eta^2. \quad (21)$$

The actual ground-state energy is found from this expression by minimizing E_{tot} with respect to the variable η , holding μ (or \bar{n}) fixed.

C. Approximate minimization

We begin by considering the case $\bar{n} = 0$, for which an approximate minimization of $E_{\text{tot}}(\eta)$ can be done analytically as follows. First, one must evaluate the energies $E_{j;0}$, which are the ground-state eigenvalues of $H_j \psi_j(\phi_j) = E_j \psi_j(\phi_j)$. For $\bar{n} = 0$, $E_{j;0}$ has the approximate value²³

$$E_{j;0} \sim -\frac{E_{\alpha;j}^2}{2E_{Cj}} \quad (22)$$

for $E_{\alpha;j} \ll E_{Cj}$ and

$$E_{j;0} \sim -E_{\alpha;j} \quad (23)$$

for $E_{\alpha;j} \gg E_{Cj}$. A function which interpolates smoothly between these limits is

$$E_{j;0} \sim E_{Cj} - \sqrt{E_{Cj}^2 + E_{\alpha;j}^2}. \quad (24)$$

Substituting this expression into Eq. (21), we obtain

$$E_{\text{tot}}(\eta) = \frac{\hbar}{\Omega V} \eta^2 + \sum_{j=1}^N \left[E_{Cj} - \sqrt{E_{Cj}^2 + E_{Jj}^2 + \left(\frac{2\hbar g_j}{\Omega V} \right)^2 \eta^2} \right]. \quad (25)$$

Setting $dE_{\text{tot}}/d\eta=0$, we obtain the condition

$$\eta = \eta \frac{2\hbar}{\Omega V} \sum_j \frac{g_j^2}{\sqrt{E_{Cj}^2 + E_{Jj}^2 + [2\hbar g_j / (\Omega V)]^2 \eta^2}}. \quad (26)$$

This equation always has the solution $\eta=0$. If

$$\frac{2\hbar}{\Omega V} \sum_{j=1}^N \frac{g_j^2}{\sqrt{E_{Cj}^2 + E_{Jj}^2}} > 1, \quad (27)$$

then there is also a real, nonzero solution for η . Whenever this solution exists, it is a minimum in the energy, and the $\eta=0$ solution is a local maximum. Thus, Eq. (27) represents a *threshold for the onset of coherence*.

In the opposite limit, when $E_{\alpha;j} \gg E_{Cj}$ and $\tilde{\lambda}_j \rightarrow 1$,

$$\eta = \sum_{j=1}^N g_j \tilde{\lambda}_j \rightarrow \sum_{j=1}^N g_j. \quad (28)$$

If we define $\bar{g} = \sum_{j=1}^N g_j / N$ and $\bar{\eta} = \eta / N$, then we see that $\bar{\eta}$ rises from zero at a threshold determined by Eq. (27) and approaches unity when the parameters $|E_{\alpha;j}|$ are sufficiently large.

For $\bar{n} \neq 0$, the threshold can still be approximately found analytically. Since H' is periodic in \bar{n} with period unity, one need consider only $-1/2 < \bar{n} \leq 1/2$. In this regime, we write H_j as

$$\begin{aligned} H_j &= -E_{\alpha;j} \cos(\phi - \phi_\alpha) + E_{Cj}[(n - \bar{n})^2 - \bar{n}^2] \\ &= -E_{\alpha;j} \cos(\phi - \phi_\alpha) + H_j^0. \end{aligned} \quad (29)$$

The coherence threshold occurs in the small-coupling regime, $|E_{\alpha;j}| \ll E_{Cj}$. The desired ground-state solution can be obtained as a perturbation expansion about the solutions to the zeroth-order Schrödinger equation, $H_j^0 \psi_j^0 = E_j^0 \psi_j^0$. The (unnormalized) solutions to this equation are $\psi_j^0 = \exp(im\phi_j)$, corresponding to eigenvalues $E_j^0 = E_{Cj}[(m - \bar{n})^2 - \bar{n}^2]$, with m integer. For $|\bar{n}| < 1/2$, the ground state is $m=0$. The second-order perturbation correction to this energy due to the perturbation $H_j' = -E_{\alpha;j} \cos(\phi - \phi_\alpha)$ is

$$\Delta E_j = \sum_{m=\pm 1} \frac{|\langle 0 | H_j' | m \rangle|^2}{E_0 - E_m}, \quad (30)$$

where $|m\rangle$ denotes the ket corresponding to $\exp(im\phi_j)$. After a little algebra, it is found that $\Delta E_j = -E_{\alpha;j}^2 / [(2E_{Cj})(1 - 4\bar{n}^2)]$. If $|E_{\alpha;j}| \gg E_{Cj}$, then the ground-state eigenvalue

$E_{j;0}$ approaches $-E_{\alpha;j}$ as in the case $\bar{n}=0$. The generalization of the formula (25) to the case $\bar{n} \neq 0$ is readily shown to be

$$E_{\text{tot}}(\eta) = \frac{\hbar}{\Omega V} \eta^2 + \sum_{j=1}^N \left[\tilde{E}_{Cj} - \sqrt{\tilde{E}_{Cj}^2 + E_{Jj}^2 + \left(\frac{2\hbar g_j}{\Omega V} \right)^2 \eta^2} \right], \quad (31)$$

where $\tilde{E}_{Cj} = E_{Cj}(1 - 4\bar{n}^2)$. To determine η for a given value of \bar{n} , and of the E_{Cj} 's, E_{Jj} 's, and g_j 's, one minimizes this energy with respect to η , as described above.

In practice, at any value of \bar{n} , and for any given distribution of the parameters g_j, C_{Jj} , and E_{Jj} , one can easily evaluate the energy numerically, using the known properties of Mathieu functions, hence obtaining both the coherence threshold and the value of the order parameter η . Once η is known, the individual values of the $\tilde{\lambda}_j$'s can be obtained by numerically solving the Schrödinger equation (16), using the Hamiltonian (12) for the ground-state eigenvalue. Finally, the constant-voltage condition can be imposed by choosing μ so that $\sum_{j=1}^N q \langle n_j \rangle / C_j$ equals the time-averaged voltage across the array.

III. RESULTS

Although our formalism applies equally to ordered and disordered arrays, we will present numerical results for ordered arrays only, purely for numerical convenience. In the ordered case, the constants g_j, E_{Cj} , and E_{Jj} are independent of j . In this ordered case, we denote the parameters $g, E_C = q^2/(2C)$, and E_J , respectively. For a specified value of \bar{n} , we can find the ground-state eigenvalue $E_{j;0}$ numerically by solving Eq. (16), using the well-known properties of the Mathieu functions. We can then minimize the total energy E_{tot} with respect to η . In the ordered case, as noted, all the $\tilde{\lambda}_j$'s are equal, and $\eta = N\tilde{\lambda}$. Furthermore, in this case, \bar{n} is related to Φ by $\Phi = Nq\bar{n}/C$. Hereafter, for given values of g, E_C, E_J , and \bar{n} , we define $\tilde{\lambda}_0$ as the value of $\tilde{\lambda}$ which minimizes the total energy E_{tot} .

In Fig. 2, we plot $\tilde{\lambda}_0$ for this ordered array, as a function of N , assuming $\bar{n}=0$. Two curves are plotted. The solid curve shows $\tilde{\lambda}_0$ for the case $E_J=0$, i.e., no direct Josephson coupling. The dashed curve in Fig. 2 shows $\tilde{\lambda}_0$ but for a finite direct Josephson coupling. In both cases, there is clearly a threshold array size N_c , below which $\tilde{\lambda}_0=0$. For $N > N_c$, we find $\tilde{\lambda}_0 > 0$. Since $\tilde{\lambda}_0 = \langle \sin \phi_j \rangle_0$ (that is, the expectation value of $\sin \phi_j$ in this energy-minimizing state), the Josephson array has a net supercurrent in this configuration. As N increases, $\tilde{\lambda}_0$ approaches unity, which corresponds to complete phase locking. The value of N_c is larger for finite Josephson coupling than for zero direct coupling; thus, it appears, paradoxically, that the finite direct coupling actually impedes the transition to coherence. This point will be discussed further below.

For $E_J=0$ and $\bar{n}=0$, N_c can easily be found analytically from Eq. (27). The threshold is found to satisfy

$$N_c = E_C / (2E_{J0}), \quad (32)$$

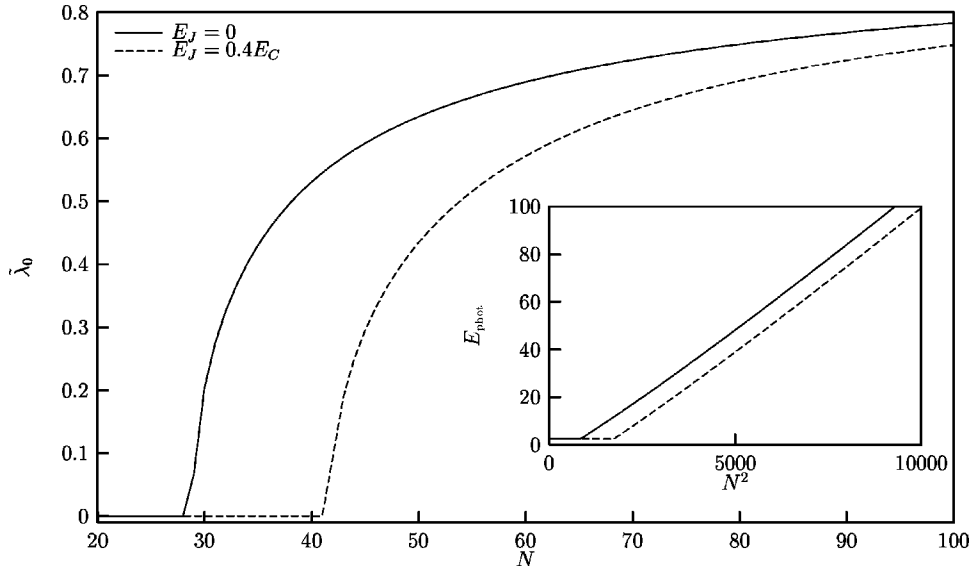


FIG. 2. Coherence order parameter $\tilde{\lambda}_0$ which minimizes $E_{\text{tot}}(\tilde{\lambda})$ for a one-dimensional array, plotted as a function of the number of junctions N , for two values of the direct Josephson coupling energy E_J . Other parameters are $\hbar g/\sqrt{V}=0.3E_C$, $\hbar\Omega/2=2.6E_C$, and $\bar{n}=0$. The coupling parameter $E_{J0}=\hbar g^2/(\Omega V)$ is given by $E_{J0}\approx 0.017E_C$. Inset: total energy in the photon field, E_{phot} , plotted as a function of N^2 , for the same parameters and the same two values of E_J .

where $E_{J0}=\hbar g^2/(\Omega V)$. This value agrees quite well with our numerical results (cf. Fig. 2). Note that for *any* nonzero value of the coupling E_{J0} , no matter how small, there always exists a threshold value of N , above which phase coherence becomes established.

If $E_J=0$ and $\bar{n}=0$, N_c can still be obtained as an implicit equation even in the disordered case, in terms of the distribution of the g_j 's and E_{Cj} 's. The result is readily shown to be

$$1 = \frac{2\hbar}{\Omega V} \sum_{j=1}^{N_c} \frac{g_j^2}{E_{Cj}}. \quad (33)$$

For a given distribution of the parameter g_j^2/E_{Cj} , there will always exist a threshold value of N such that this equation is satisfied, no matter how weak the coupling constants g_j . Thus, at least in this mean-field approximation, the disorder has no qualitative effect on the coherence transition discussed here. In particular, the critical number N_c does not necessarily either increase or decrease with increasing disorder; instead, N_c depends on the distribution of g_j , E_{Jj} , and E_{Cj} in the array.

The inset to Fig. 2 shows the total energy in the photon mode, $E_{\text{phot}}=\hbar\Omega(\langle a^\dagger a \rangle + 1/2)$, in the ordered case, plotted as a function of N for $\bar{n}=0$. From Eq. (11), we find that $E_{\text{phot}}=\hbar\Omega/2+N^2\tilde{\lambda}_0^2 E_{J0}$ for an ordered array; this is the quantity plotted in the inset. As is evident from the plot, E_{phot} varies approximately *linearly* with N^2 all the way from the coherence threshold to large values of N , where $\tilde{\lambda}_0 \rightarrow 1$. This N^2 dependence is a hallmark of phase coherence.

The voltage Φ across the array is determined by \bar{n} (or equivalently μ). In Fig. 3, we plot $\tilde{\lambda}_0$ as a function of \bar{n} for several array sizes (at fixed coupling constants E_{J0} , E_C , and E_J in an ordered array). Since, as already shown, $\tilde{\lambda}_0$ is periodic in \bar{n} with a period of unity, we plot $\tilde{\lambda}_0(\bar{n})$ only for a

single period, $0 \leq \bar{n} \leq 1$. Figure 3 shows that, for any given N and E_{J0} , the calculated $\tilde{\lambda}_0$ achieves its maximum value when \bar{n} has a half-integer value; i.e., the array is most easily made coherent at such values of \bar{n} . In particular, an array whose size is slightly below the threshold value at integer values of \bar{n} can be made to become coherent, with a nonzero $\tilde{\lambda}_0$, when \bar{n} is increased—that is, when a suitable voltage is applied. On the other hand, for values of N far above the threshold, $\tilde{\lambda}_0$ is little affected by a change in \bar{n} .

In Fig. 4, we show the quantity $\langle n_j \rangle$ as a function of \bar{n} , for several values of N and fixed value of the coupling constant ratios E_{J0}/E_C and E_J/E_C , for a single cycle ($0 \leq \bar{n} \leq 1$). This quantity is related to the voltage drop across one junction, in our model, by $\Phi/N=q\langle n_j \rangle/C$. For sufficiently large arrays, $\langle n_j \rangle \sim \bar{n}$ and the voltage drop is nearly linear in \bar{n} in this mean-field approximation. For arrays closer to the coherence threshold, $\langle n_j \rangle$ is a highly nonlinear function of \bar{n} . However, the deviation from linearity, $\langle n_j \rangle - \bar{n}$, is, once again, a periodic function of \bar{n} with period unity. The discontinuous jumps in \bar{n} as a function of $\langle n_j \rangle$ represent regions of incoherence ($\tilde{\lambda}_0=0$), whereas the regions in which $\langle n_j \rangle$ is a smooth function of \bar{n} are regimes of phase coherence ($\tilde{\lambda}_0 \neq 0$).

In Fig. 5, we again plot $\tilde{\lambda}_0(N)$ for two fixed ratios E_J/E_C , but this time for $\bar{n}=1/2$. From Fig. 3, we expect this choice of \bar{n} to maximize the tendency to phase coherence and thus to reduce the threshold array size for the onset of phase coherence. Indeed, in the absence of direct Josephson coupling, this threshold is reduced to below unity (that is, $\tilde{\lambda}_0$ remains nonzero, even at $N=1$, for our choice of E_{J0}). In fact, for this value of \bar{n} , only an infinitesimal coupling to the resonant mode is required to induce phase coherence in this

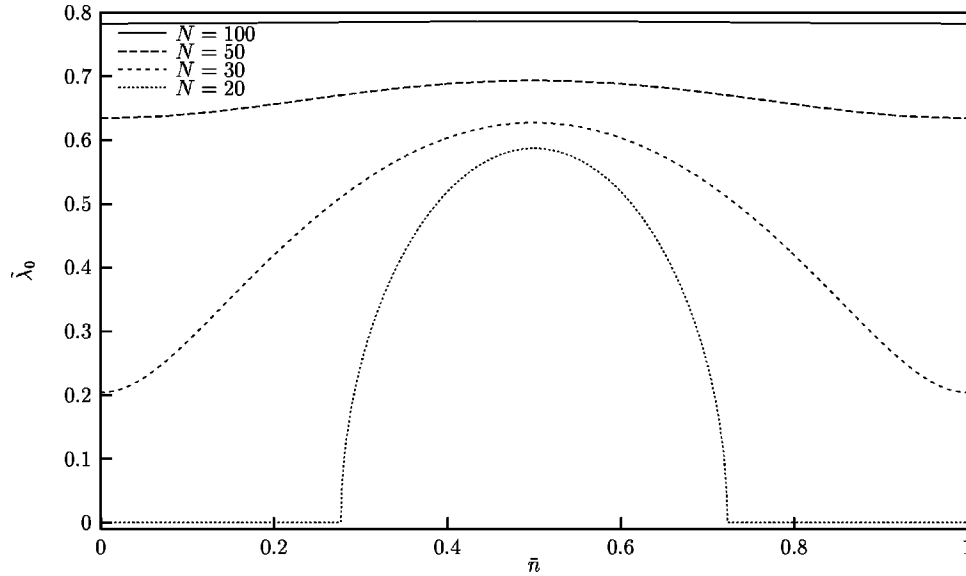


FIG. 3. Energy-minimizing value of the coherence order parameter $\tilde{\lambda}_0$, as a function of the parameter $\bar{n} = \mu/q$, for several values of the array size N . Other parameters are $\hbar g/\sqrt{V} = 0.3E_C$, $\hbar\Omega/2 = 2.6E_C$, $E_J = 0$, and $E_{J0} \approx 0.017E_C$.

model. Once again (cf. Fig. 5), the addition of a finite direct Josephson coupling actually increases the threshold number for phase coherence at $\bar{n} = 1/2$ as it does at $\bar{n} = 0$.

Although we have not carried out a similar series of calculations for a disordered array, our *analytical* results show that the essential features found in the ordered case will be preserved also in a disordered array. Most importantly, there remains a critical junction number for phase coherence in a disordered array, just as there does in the ordered case. The most important difference between the two cases is that the individual $\tilde{\lambda}_j$'s will be functions of j in the disordered case.

IV. DISCUSSION

Although the present work is only a mean-field approximation, we expect that it will be quite accurate for large N .

The reason is that, in this model, the one photonic degree of freedom is coupled to every phase difference and, thus, experiences an environment which is very close to the mean, whatever the state of the individual junctions. Such small fluctuations are necessary in order for a mean-field approach to be accurate. In fact, a similar approach has proved very successful in work on novel Josephson arrays in which each wire is coupled to a large number of other wires via Josephson tunneling.^{24,25}

It may appear surprising that a finite direct Josephson coupling actually increases the threshold array size for coherence. But in fact this behavior is reasonable. If there is no direct coupling ($E_J = 0$), the phase difference across each junction evolves independently, except for the global coupling to the resonant photon mode. When the array exceeds its critical size, this coupling produces coherence. If the same

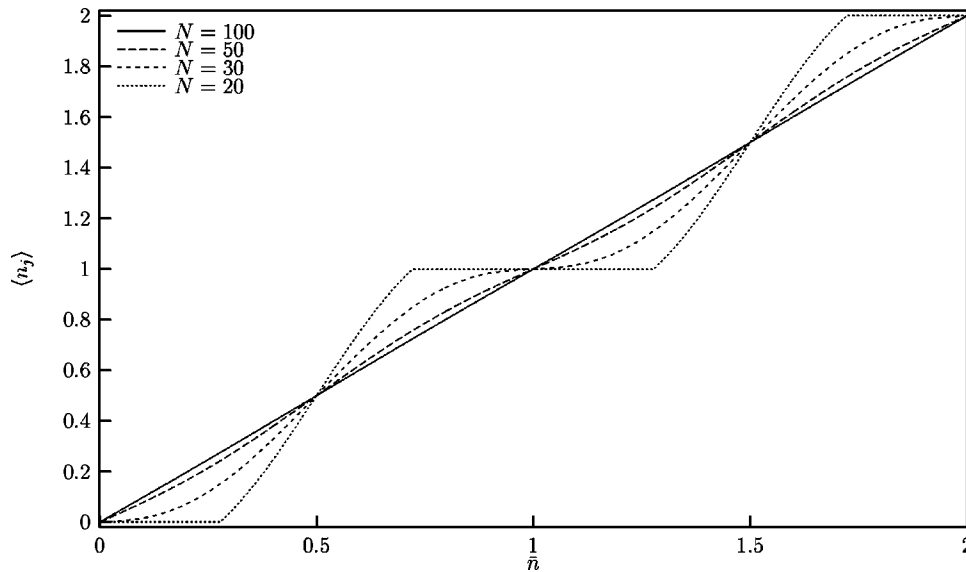
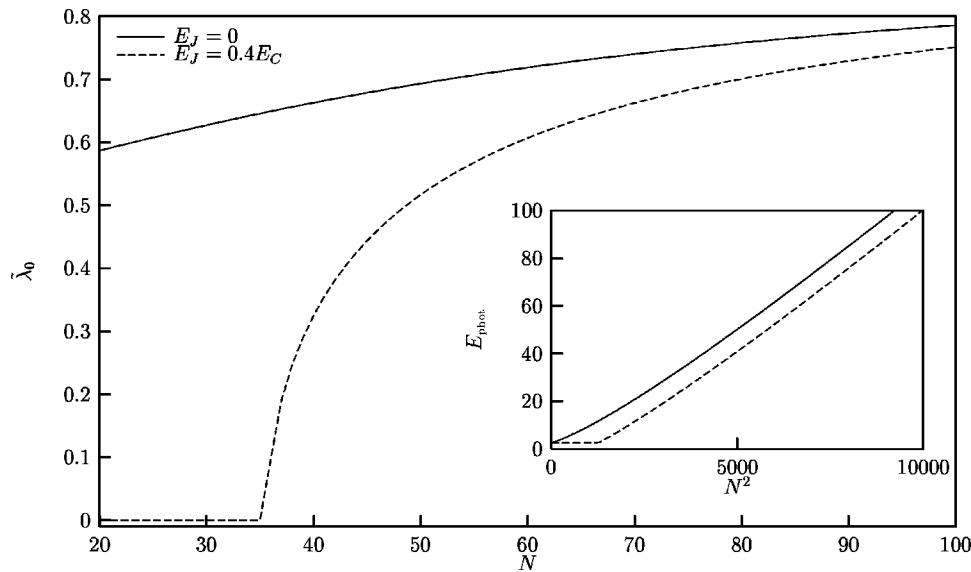


FIG. 4. The parameter $\langle n_j \rangle = \Phi C / (qN)$, where Φ/N is the voltage drop across one junction, plotted as a function of the parameter $\bar{n} = \mu/q$, for several values of the array size N . Other parameters are $\hbar g/\sqrt{V} = 0.3E_C$, $\hbar\Omega/2 = 2.6E_C$, $E_J = 0$, and $E_{J0} \approx 0.017E_C$.

FIG. 5. Same as Fig. 2, but for $\bar{n} = 1/2$.

array now has a finite E_J , there are two coupling terms. But these are not simply additive, but in fact are $\pi/2$ out of phase: the direct coupling favors $\phi_j = 0$, while the photonic one favors $\phi_j = \pi/2$. For a large enough array, the coupling to the photon field still predominates and produces global phase coherence, but this occurs at a higher threshold, at least in our model, than in the absence of direct coupling.

A striking feature of our results is the very low coherence threshold ($N=1$) when $\bar{n} = 1/2$. In fact, for any N and for $E_J = 0$, only an infinitesimal coupling to the cavity mode would be required to induce phase coherence at $\bar{n} = 1/2$. The reason for this low threshold is that, in the absence of coupling, junction states with $\langle n_j \rangle = 0$ and $\langle n_j \rangle = 1$ are degenerate. Any coupling is therefore sufficient to break the degeneracy and produce phase coherence. A related effect has been noted previously in studies of more conventional Josephson junction arrays in the presence of an offset voltage.²⁶

Finally, we comment on what is not included in the present work. This paper really considers only the minimum energy state of the coupled photon/junction array system under the assumption that a particular voltage is applied across the array. It would be of equal or greater interest to consider

the dynamical response of such an array. Specifically, it would be valuable to develop and solve a set of coupled dynamical equations which incorporate both the junction and the photonic degrees of freedom. Such a set of equations has already been proposed by Bonifacio under a particular set of simplifying assumptions. A more accurate set of equations is needed, which would include not only a driving current, but also the damping arising from both resistive losses in the junctions and losses due to the finite Q of the cavity. In the absence of damping, such equations can be written down from the Heisenberg equations of motion. The inclusion of damping may be more difficult. We hope to discuss some of these effects in a future publication.

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- ²¹The specific form of α_j is readily obtained from standard expressions for \mathbf{A} in terms of a and a^\dagger . If we choose a normalization such that $\int_V |\mathbf{E}_a|^2 d^3x = 1$, where V is the volume of the cavity and $\mathbf{E}_a(\mathbf{x})$ is the electric field of the normal mode, then we find that $\alpha_j = q/(\hbar c) \sqrt{4\pi} \int_j^{j+1} \mathbf{E}_a \cdot d\mathbf{s}$.
- ²²This approximation is equivalent to retaining terms only through first order in fluctuations about the mean. Specifically, if \mathcal{O}_1 and \mathcal{O}_2 are operators depending, respectively, on the photon and phase variables, and if $\delta\mathcal{O}_j = \mathcal{O}_j - \langle \mathcal{O}_j \rangle$ for $j=1,2$, then the approximation retains all terms in the product $\mathcal{O}_1\mathcal{O}_2$ through *first* order in the $\delta\mathcal{O}_j$'s.
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