Mean-field theory for arrays of Josephson-coupled wires

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We describe a mean-field theory for phase transitions in a Josephson junction array consisting of two sets of parallel wire networks, arranged at right angles and coupled together by Josephson interactions. In contrast to earlier treatments, we include the variation of the superconducting phase along the individual wires; such variation is always present if the wires have finite thickness and are sufficiently long. The mean-field result is obtained by treating the individual wires exactly and the coupling between them within the mean-field approximation. For a perpendicular applied magnetic field of strength $f=p/q$ flux quanta per plaquette (where p and q are mutually prime integers), we find that the mean-field transition temperature $T_c(f)\approx T_c(0)q^{-b}$ with $b=1/4$. By contrast, a mean-field theory which neglects phase variation along the array predicts $b=1/2$, and gives a T_c which diverges in the thermodynamic limit. The model with phase variations agrees somewhat better with experiment on large arrays than does the approximation which neglects phase variations. [S0163-1829(98)07138-0]

I. INTRODUCTION

There has recently been considerable interest in so-called hybrid Josephson junction arrays.[1–5] Such arrays consist of extended superconducting objects, such as wires, which are coupled together by Josephson interactions. Because the objects are extended, each one may be Josephson-coupled to a large number of other such superconductors. In the limit when the number of these couplings becomes very large, it is believed[1] that mean-field theory[6] (MFT) should become exact, since in that limit a superconducting object would experience its average environment, and deviations from that average would become arbitrarily small.

Various experimental realizations of such arrays can readily be imagined. For example, Tinkham and collaborators[2,3,5] have studied networks of long superconducting wires, arranged into two parallel sets of N wires each, placed at right angles to one another. In such a geometry (see Fig. 1), a Josephson junction forms at each crossing point of two perpendicular wires. Moreover, each wire in one parallel set interacts with all the wires in the other parallel set, so that, in the large-N limit, there are very many such ''neighbors.'' These workers studied both the thermodynamics and the IV characteristics of these networks, and carried out mean-field calculations to account for their observations.

One problem with the standard mean-field theory is that, even for the arrays just described, it must break down in the thermodynamic limit. Whatever the wire thickness, there will always be a wire length Na (where a is the separation between junctions; see Fig. 1), beyond which it will become important to consider the variation of the phase of the superconducting order parameter along the wire. (This phase is assumed constant along the wire in the mean-field theory described by Vinokur *et al.*[1]) Indeed, effects of such phase variation were already observed in the original experiments by Sohn *et al.*[2,3] In these measurements, carried out on large ($N\geqslant 600$) networks, the network critical current was found to saturate at a finite value even for very large N. By contrast, in the absence of phase variation along the wire, the critical current should vary linearly with N, at zero magnetic field.

In an earlier paper,[4] we showed that the saturation behavior could be accounted for by a simple dynamical model, in which the wire segments were treated as overdamped Josephson junctions with a critical current much larger than that of the junctions connecting different wires. This gives the wires an effective inductance per unit length of $\hbar/(2eI_ca)$, where I_c is the critical current of the Josephson junctions used to model the wires. This model represents a simple way of including the phase variation along the wire numerically in studying the dynamical response of the array.

In this paper, we describe a simple mean-field theory for treating the static properties of a hybrid array, including the effects of phase variation along the individual wires. Following our earlier model, we accomplish this by treating each wire segment as a Josephson junction with a much larger

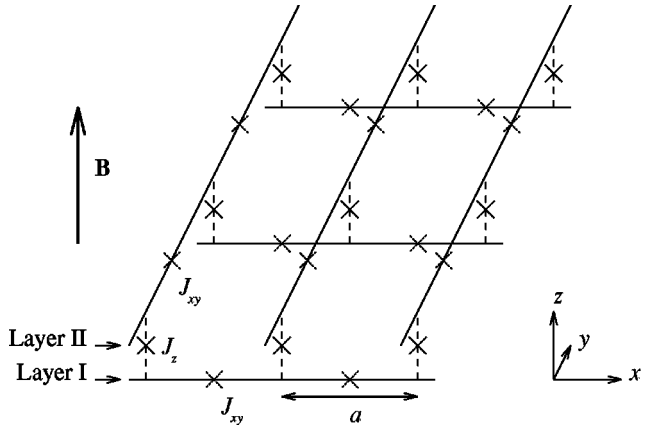

FIG. 1. Sketch of the model geometry. The network consists of two sets of parallel superconducting wires at right angles, and connected by Josephson junctions with coupling energy J_z. The wire segments between each junction are modeled as Josephson junctions with coupling energy $J_{xy}\geqslant J_z$. The thermodynamic variables are the phases of the superconducting order parameter at each wire intersection in the two planes, denoted $\theta_{k,l}$ and $\psi_{k,l}$.

critical current than that of the interwire junctions. We then treat the thermodynamic properties of the individual wires exactly, and treat the interactions between the wires within MFT.

The remainder of this paper is organized as follows. In the next section, we describe the MFT in the presence of such phase variation along the wires. The following section gives our results using this MFT, after which we discuss the results and compare them to experiment and to the predictions of other models.

II. MEAN-FIELD THEORY WITH PHASE VARIATION ALONG WIRES

A. Zero magnetic field

Our basic model for the wire network is shown in Fig. 1. There are two sets of parallel wires, equally spaced, and arranged at right angles. Wherever two wires cross, a Josephson junction is assumed to form, with coupling energy J_z. We model the wire segment between adjacent junctions as a Josephson junction with a larger coupling constant J_{xy}. The critical currents $I_{c;xy}$ and $I_{c;z}$ are related to the coupling constants by $I_{c;xy}=2eJ_{xy}/\hbar$; $I_{c;z}=2eJ_z/\hbar$.

We denote the two layers of wires as I and II. They are taken as parallel to the xy-plane, with wires in layer I and layer II parallel to the x-axis and y-axis respectively. The coordinates of points in plane I or plane II where the wires cross are $(x,y)=(ka,la)$, where a is the lattice constant, and k and l are (positive or negative) integers.

We describe the network by the classical Hamiltonian

$$\mathcal{H}=-J_{xy}\sum_{k,l}\cos(\theta_{k,l}-\theta_{k+1,l})-J_{xy}\sum_{k,l}\cos(\psi_{k,l}-\psi_{k,l+1})$$
$$-J_z\sum_{k,l}\cos(\theta_{k,l}-\psi_{k,l}). \tag{1}$$

Here $\theta_{k,l}$ and $\psi_{k,l}$ are the phases of the superconducting order parameters at the wire intersections. We write $J_{xy}=\lambda J_z$, where, in general, we expect $\lambda\gg 1$. The appropriate order parameters can then be introduced as

$$\eta_{\theta_{k,l}}\equiv\langle\exp(i\theta_{k,l})\rangle=\frac{\int d\Gamma\exp(i\theta_{k,l})\exp(-\beta\mathcal{H})}{\int d\Gamma\exp(-\beta\mathcal{H})},$$
$$\eta_{\psi_{k,l}}\equiv\langle\exp(i\psi_{k,l})\rangle, \tag{2}$$

where $\beta=1/k_BT$ and

$$\int d\Gamma\equiv\prod_{k,l}\int_{-\pi}^{+\pi}(d\theta_{k,l}/2\pi)\int_{-\pi}^{+\pi}(d\psi_{k,l}/2\pi).$$

In zero magnetic field, the order parameters will all be all real and identical, so that $\eta_{\theta_{k,l}}=\langle\cos\theta_{k,l}\rangle=\eta_{\psi_{k,l}}\equiv\eta$. We treat the interactions along each wire exactly, while treating those between wires within MFT. To obtain the MFT, $\mathcal{H}$ appearing in Eq. (2) is replaced by

$$\mathcal{H}_{mf}=\mathcal{H}_{xy}+\mathcal{H}_z, \tag{3}$$

where

$$\mathcal{H}_{xy}=-J_{xy}\sum_k\cos(\theta_{k,l}-\theta_{k+1,l}), \tag{4}$$

while $\mathcal{H}_z$ is obtained by making the replacement $\cos(\theta_{k',l}-\psi_{k',l})\to\eta\cos\theta_{k',l}$ where $\psi_{k',l}$ is the phase of any point in layer II which is connected to the lth wire in layer I by a Josephson junction:

$$\mathcal{H}_z=-J_z\eta\sum_{k'}\cos\theta_{k',l}. \tag{5}$$

The MFT is then specified by the equation

$$\eta\approx\frac{\int d\Gamma'\cos\theta_{k,l}\exp(-\beta\mathcal{H}_{mf})}{\int d\Gamma'\exp(-\beta\mathcal{H}_{mf})}, \tag{6}$$

where $\int d\Gamma'\equiv\Pi_{k'}\int_{-\pi}^{\pi}(d\theta_{k',l}/2\pi)$.

To find the transition temperature T_c, we assume that the order parameter $|\eta|\ll 1$, and expand the exponential $\exp(-\beta\mathcal{H}_z)$ to first order in η. The result is

$$\eta\approx\frac{\int d\Gamma'\cos\theta_{k,l}\exp(-\beta\mathcal{H}_{xy})\left(1+\beta J_z\eta\sum_{k'}\cos\theta_{k',l}\right)}{\int d\Gamma'\exp(-\beta\mathcal{H}_{xy})\left(1+\beta J_z\eta\sum_{k'}\cos\theta_{k',l}\right)}. \tag{7}$$

The term of order η^0 on the right-hand side vanishes. The surviving integral [i.e., the coefficient of η on the right-hand side of Eq. (7)] can be evaluated exactly by writing $\cos\theta_{k,l}\cos\theta_{k',l}=\frac{1}{2}[\cos(\theta_{k,l}-\theta_{k',l})+\cos(\theta_{k,l}+\theta_{k',l})]$, and noting that the average of the second of these terms vanishes exactly for the Hamiltonian $\mathcal{H}_{xy}$. After some algebra, we may finally reexpress the self-consistency condition as

$$\eta=\frac{\eta}{2}\beta J_z\mathrm{Re}\sum_{j=-\infty}^{+\infty}\left[\frac{\int_{-\pi}^{+\pi}d\varphi\,e^{i\varphi}e^{\beta J_{xy}\cos\varphi}}{\int_{-\pi}^{+\pi}d\varphi e^{\beta J_{xy}\cos\varphi}}\right]^{|j|}. \tag{8}$$

The quantity in brackets is simply a ratio of Bessel functions which we denote

$$\Lambda(\beta J_{xy})\equiv I_1(\beta J_{xy})/I_0(\beta J_{xy}). \tag{9}$$

The sum over j is the geometric series $\Sigma_{j=-\infty}^{+\infty}\Lambda^{|j|}=(1+\Lambda)/(1-\Lambda)$, and the self-consistency condition finally takes the form

$$1=\frac{1}{2}\beta J_z\left(\frac{1+\Lambda(\beta J_{xy})}{1-\Lambda(\beta J_{xy})}\right), \tag{10}$$

where we have used the fact that Λ is real if βJ_{xy} is real. The value of β which solves this equation gives T_c within this MFT at zero magnetic field.

As an illustration, we consider the limit $J_{xy}\gg J_z$. In this regime, we expect that $k_BT_c\ll J_{xy}$, so that the exponent $\beta J_{xy}\gg 1$ and $\Lambda(\beta J_{xy})\sim\exp(-1/2\beta J_{xy})$. k_BT_c is then the inverse of the value of β which satisfies Eq. (10), namely,

$$k_B T_c \sim \sqrt{2 J_z J_{xy}}. \tag{11}$$

If we do not require $J_{xy} \gg J_z$, the mean-field equation (10) cannot be solved in closed form, in general, but it reduces to an easily solved transcendental equation. The solution to this equation gives $\beta_c J_{xy}$ as a function of the dimensionless variable $\lambda \equiv J_{xy}/J_z$, and hence $k_B T_c \equiv 1/\beta_c$.

B. Finite magnetic field

At finite field, the order parameters (2) are all different, in general, and closed-form expression for T_c is not obtainable even when $J_{xy}/J_z \gg 1$. Instead, T_c is an eigenvalue of a certain matrix of large dimension which must be diagonalized numerically. In the presence of a magnetic field perpendicular to the layers, the Hamiltonian (1) may be expressed:

$$\mathcal{H} = -J_{xy} \sum_{k,l} \cos(\theta_{k,l} - \theta_{k+1,l}) - J_{xy} \sum_{k,l} \cos(\psi_{k,l} - \psi_{k,l+1})$$

$$- J_z \sum_{k,l} \cos(\theta_{k,l} - \psi_{k,l} - A_{k,l}), \tag{12}$$

where $A_{k,l} = 2\pi B a^2 k l/\Phi_0$. In the first two sums, the sums are carried out over pairs of bonds in the xy-plane, while in the third, it is evaluated over the weaker junctions in the z-direction, as before. B denotes the magnetic field and $\Phi_0 = hc/2e$ is the flux quantum. We have chosen a convenient gauge[7] to insure that the flux through any given plaquette perpendicular to the z-direction is Ba^2.

The order parameters of Eq. (2) are evaluated using a mean-field $\mathcal{H}_{mf;\theta}$ given by

$$\mathcal{H}_{mf;\theta} = \mathcal{H}_{mf;\theta;xy} + \mathcal{H}_{mf;\theta;z}, \tag{13}$$

$$\mathcal{H}_{mf;\theta;xy} = -J_{xy} \sum_{n=-\infty}^{+\infty} \cos(\theta_{k+n,l} - \theta_{k+n+1,l}),$$

$$\mathcal{H}_{mf;\theta;z} = \left\langle -J_z \sum_{n=-\infty}^{+\infty} \cos(\theta_{k+n,l} - \psi_{k+n,l} - A_{k+n,l}) \right\rangle_{\psi}.$$

Here, the notation $\langle \cdots \rangle_{\psi}$ means that the quantities in brackets are to be averaged over the variables $\psi_{i,j}$. Similarly, the variables $\eta_{\psi_{k,l}}$ are to be evaluated using, instead of $\mathcal{H}$, the corresponding mean-field $\mathcal{H}_{mf;\psi}$.

As in the zero-field case, the transition temperature T_c is obtained as a function of the frustration $f \equiv Ba^2/\Phi_0$ by linearizing the mean-field equations. This procedure results in the following set of coupled linear equations for the variables $\eta_{\theta_{k,l}}$ and $\eta_{\psi_{k,l}}$:

$$\eta_{\theta_{k,l}} = \frac{1}{2}\beta J_z \sum_{n=-\infty}^{+\infty} \eta_{\psi_{k+n,l}} \exp(iA_{k+n,l}) \left[\frac{I_1(\beta J_{xy})}{I_0(\beta J_{xy})}\right]^{|n|}, \tag{14}$$

$$\eta_{\psi_{k,l}} = \frac{1}{2}\beta J_z \sum_{m=-\infty}^{+\infty} \eta_{\theta_{k,l+m}} \exp(-iA_{k,l+m}) \left[\frac{I_1(\beta J_{xy})}{I_0(\beta J_{xy})}\right]^{|m|}.$$

Equivalently, these equations may be expressed as equations for the variables $\chi \equiv \beta J_{xy}$ and $\lambda = J_{xy}/J_z$. This system of equations can be separated into two sets involving the $\eta_{\theta_{k,l}}$'s and $\eta_{\psi_{k,l}}$'s separately. The result of this separation is

$$\eta_{\theta_{k,l}} = \left[\frac{\chi}{2\lambda}\right]^2 \sum_{n,m} \eta_{\theta_{k+n,l+m}} e^{-i2\pi f(k+n)m} \Lambda(\chi)^{|n|+|m|}, \tag{15}$$

$$\eta_{\psi_{k,l}} = \left[\frac{\chi}{2\lambda}\right]^2 \sum_{n,m} \eta_{\psi_{k+n,l+m}} e^{i2\pi f n(l+m)} \Lambda(\chi)^{|n|+|m|}. \tag{16}$$

In general, this is an infinite set of coupled equations. If, however, f is a rational fraction (i.e., $f = p/q$ where p and q are integers with no common factors), then it is reasonable to assume that $\eta_{\theta_{k,l}}$ and $\eta_{\psi_{k,l}}$ each have period q in both indices—that is, that the order parameter is periodic within a $q \times q$ unit cell as in more conventional Josephson junction arrays. This assumption is plausible in the present case because the exponential factor in the mean-field equations has the required periodicity. Then there are exactly q^2 mean-field equations for the set of variables $\eta_{\theta_{k,l}}$. This set of q^2 equations with q^2 unknowns can be represented as the linear homogeneous matrix equation $\vec{\eta}_\theta = \mathbf{C}(\chi)\vec{\eta}_\theta$, where $\mathbf{C}$ is a $q^2 \times q^2$ matrix. The critical value of β is determined by the value of χ such that the largest eigenvalue of $\mathbf{C}(\chi)$ is equal to unity. There is an equal number of equations for the $\eta_{\psi_{k,l}}$'s, which give the same information about $T_c(f)$. While these equations can be solved numerically for any value of J_{xy}/J_z, the approximation is best when $J_{xy} \gg J_z$, as at $f = 0$.

III. RESULTS

We have applied this MFT numerically to arrays at both finite and zero magnetic field. For an applied magnetic field $f = p/q$ flux quanta per plaquette, with p and q integers with no common divisors, one solves the set of q^2 equations involving only the variables $\eta_{\theta_{k,l}}$ [Eq. (15)].

For real χ, the eigenvalues $\{\alpha_{n,k}\}$ of $\mathbf{C}(\chi)$ have the form $\alpha_{n,k} = |\alpha_n| \exp(2\pi i k/q)$, where $n = 1,2,\ldots,q$ and $k = 0,1,\ldots,q-1$. The eigenvalues have q distinct magnitudes $|\alpha_n|$, each magnitude being q-fold degenerate. These eigenvalues are all transcendental functions of χ, but can be readily determined numerically. To find them, we vary parameter χ until the largest eigenvalue of $\mathbf{C}(\chi)$ is within 10^{-9} of unity. The χ_c so determined gives the critical temperature $k_B T_c = J_{xy}/\chi_c$. We calculate T_c in this way for a range of values of f and λ.

Our numerical results for $T_c(f)$ are shown in Figs. 2 and 3. Figure 2 shows the MFT critical temperature $k_B T_c(f)/J_{xy}$ for various frustration f plotted as a function of the coupling ratio λ. Numerically, to an excellent approximation, $T_c(p/q)$ is independent of p; it depends only on q. Presumably, this behavior is implied by the self-consistent equations (15), although we have not been able to show it analytically. The equations also imply that $T_c(f)$ is periodic in f with period unity.

For $\lambda \gg 1$, we find numerically to an excellent approximation that

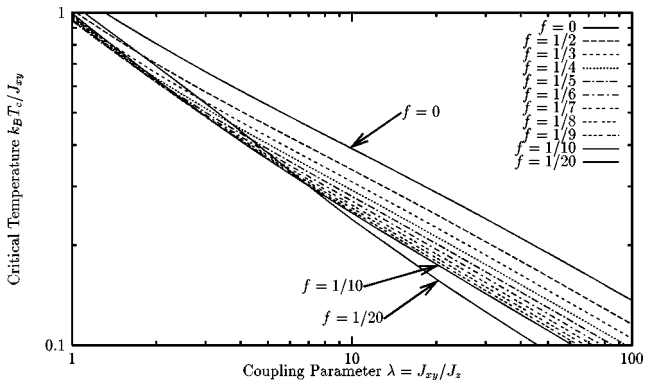

FIG. 2. Mean-field transition temperature $k_B T_c(f)$, in units of the coupling constant J_{xy}, plotted as a function of $\lambda = J_{xy}/J_z$ for various values of $f = p/q$, where p and q are mutually prime integers. $T_c(f)$ depends only on q and is independent of p.

$$k_B T_c(\lambda, f) \sim J_{xy}(\lambda/2)^{-1/2} q^{-1/4}. \tag{17}$$

This behavior is shown in Fig. 3, which shows $T_c(\lambda, f)$ versus f for $\lambda = 1000$ [similar behavior is found for any $\lambda \gg 1$; in this plot we interpret $f = 0$ as corresponding to $q = 1$]. A corollary to Eq. (17) is that $T_c(f)$ vanishes for irrational f. A similar result is believed to hold for conventional two-dimensional Josephson junction arrays on a square lattice,[9] but in these conventional arrays, the result emerges from Monte Carlo (i.e., numerically exact) solutions for $T_c(f)$ whereas in the present case, it appears in a version of MFT.

In the conventional mean-field theory for these wire arrays (which assumes that each wire is described by a single constant phase),[3] $T_c(f) \propto N J_z q^{-1/2}$, where N is the number of wires. The present MFT with phase variation along the wire differs from the conventional one in that it gives a finite $T_c(f)$ in the thermodynamic limit ($N \to \infty$). The conventional MFT is, however, well-behaved in the physically achievable finite-N regime, where it agrees well with suitably designed experiments.[5]

The variation $T_c(f = p/q) \sim q^{-1/4}$ implies that $T_c(f)$ is an everywhere discontinuous function of f. This behavior is, of course, a consequence of MFT and might disappear in a more exact Monte Carlo calculation. But it does have a possible interpretation. As noted, the unit cell of the ordered

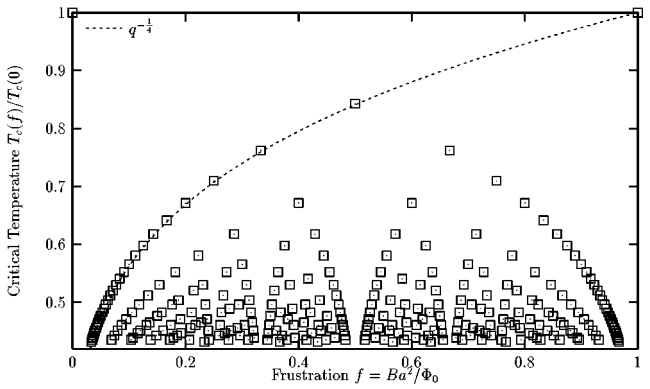

FIG. 3. Mean-field transition temperature $T_c(f)/T_c(0)$, as a function of f for $\lambda = 1000$. In this regime, an excellent approximation for $T_c(f)$ is $k_B T_c(f)/T_c(0) \sim (2/\lambda)^{1/2} q^{-1/4}$, where $f = p/q$. The dashed curve is a plot of $q^{-1/4}$.

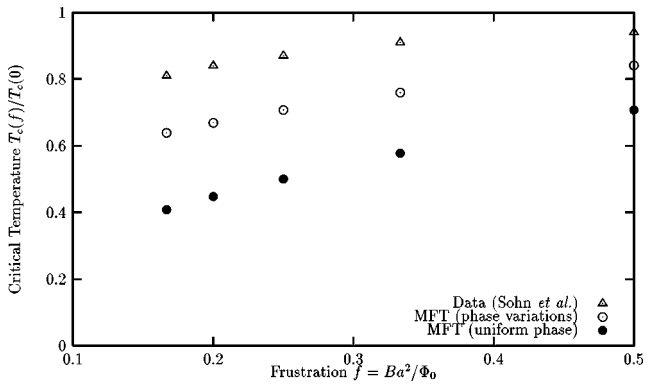

FIG. 4. Triangles: measured ratio of transition temperatures $T_c(f)/T_c(0)$ for a network of two sets of 600 parallel, equally spaced, superconducting wires, arranged at right angles, as reported by Sohn *et al.* (Ref. 3). Empty circles: predictions of MFT including phase variations along each wire for the same ratio (taken from Fig. 3). Filled circles: predictions of MFT with constant phase on each wire, i.e., $T_c(f)/T_c(0) = q^{-1/2}$.

state at $f = p/q$ should have dimensions $q \times q$. As q becomes larger, the energy required to disrupt the ordered state should become smaller, approaching zero in the large-q (irrational) limit, suggesting a very small $T_c(f)$ in this limit.

Figure 4 shows a plot of the calculated $T_c(f)$ for several values of $f = p/q$, as measured by Sohn *et al.*,[2] and as calculated within conventional MFT,[3] and within the present MFT. In all three cases, we normalize $T_c(f)$ by its zero-field value $T_c(0)$. Evidently, the inclusion of phase variation along the wires improves the agreement between calculation and experiment. This improvement is expected, since the experiments are carried out in wire networks containing 600 wires in each layer (i.e., 600^2 junctions) and in a regime where the phase variation along a wire should be significant within the dimensions of the array.

We have also carried out a few Monte Carlo (MC) calculations of the properties of an array at $f = 0$ described by the Hamiltonian (1) in our geometry. Specifically, we have evaluated the specific heat

$$C_V = \frac{1}{N^2 k_B T^2} (\langle \mathcal{H}^2 \rangle - \langle \mathcal{H} \rangle^2), \tag{18}$$

where $\langle \cdots \rangle$ denotes a canonical average, and the helicity modulus[11] γ_{xx}, which is a measure of the superfluid density of the array. It takes the form[12]

$$\gamma = \lambda \left\langle \sum_{\langle ij \rangle} \cos\theta_{ij} \right\rangle - \frac{\lambda J_{xy}}{k_B T} \left[\left\langle \left(\sum_{\langle ij \rangle} \sin\theta_{ij} \right)^2 \right\rangle - \left\langle \sum_{\langle ij \rangle} \sin\theta_{ij} \right\rangle^2 \right]. \tag{19}$$

Here $\Sigma_{\langle ij \rangle}$ denotes a sum over nearest-neighbor bonds in plane I, and θ_{ij} is a nearest-neighbor phase difference. The MC calculations are carried out using a conventional algorithm, with $2N^2$ phases for an $N \times N$ array and periodic boundary conditions. Typically, the arrays are allowed to evolve on the order of $> 10^8$ Metropolis steps after reaching

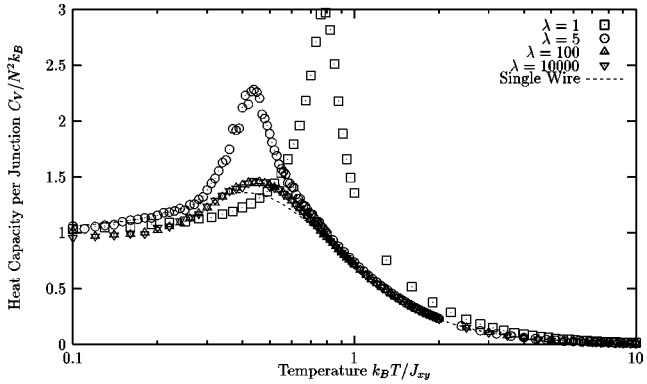

FIG. 5. Specific heat C_V of a network of two sets of 10 parallel, equally spaced, superconducting wires arranged at right angles, described by the Hamiltonian (1), and calculated by Monte Carlo simulations with periodic boundary conditions. $\lambda = J_{xy}/J_z$. Dashed line indicates the exact C_V for a single very long wire.

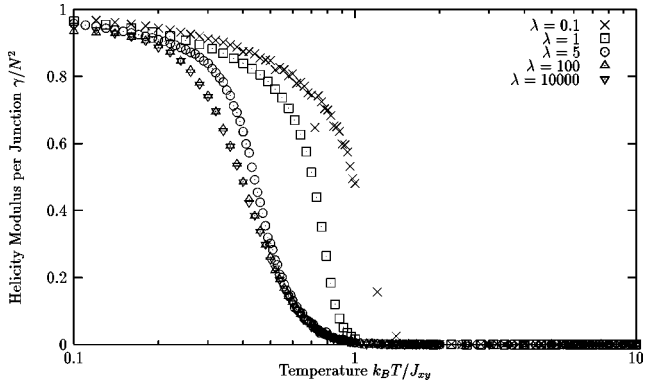

FIG. 6. The helicity modulus γ_{xx} for the model of Fig. 5, as obtained via Monte Carlo simulations, for various values of λ.

equilibrium from two different initial conditions. We consider 10×10 arrays and several values of λ.

Our MC results are shown in Figs. 5 and 6. In the limit of large λ, C_V approaches the independent wire limit, i.e., $C_V^{\rm wire} = \chi^2 - \chi[I_1(\chi)/I_0(\chi)] - [\chi[I_1(\chi)/I_0(\chi)]]^2$. For small λ, the wire network should behave like a conventional two-dimensional array, since $\theta_{i,j}$ and ψ_{ij} should lock together and behave as a single variable in this limit. Indeed, C_V in this limit is characteristic of a conventional Kosterlitz-Thouless (KT) transition,[8] with $k_BT_c \sim 0.90 J_{xy}$.[10] Our calculated C_V's differ from the independent-wire limit because of the finite wire length in our MC simulations.

The critical temperature T_c is signaled by the vanishing of the helicity modulus γ_{xx}. For large λ, γ_{xx} appears to vanish continuously with temperature, but in the opposite limit, there is a discontinuous jump from a finite value to zero at T_c, characteristic of a KT transition and of the expected universal magnitude. The behavior of γ_{xx} at large λ is due to the finite size of the sample. In the limit $N\to\infty$, γ_{xx} would, for $\lambda=\infty$, approach zero at all positive T, as expected for independent wires.

IV. DISCUSSION

We briefly compare the present results with our earlier work on dynamical properties of arrays.[4] In that work (carried out only at $f=0$), the array critical current $I_c^{\rm array}$ was also found to be affected by finite λ; specifically, $I_c^{\rm array}/I_{c;xy} \sim \lambda^{-1/2}$, where λ was defined as the ratio of junction critical currents $I_{c;xy}/I_{c;z}$ (equivalent to the coupling constant ratio λ in this paper). Thus, $I_c^{\rm array}$ behaves, for large λ, just like $T_c(f=0)$, and for the same reason. In the absence of phase variation along a wire, a given point on one wire interacts with all the wires in the other layer. With phase variation present, that point only interacts with a finite number of wires on the other array. The number of such wires is determined by a ''Josephson screening length,'' which increases as λ becomes larger.

We now estimate this screening length, which we denote ζ, and use it to estimate the validity limits of the $q^{-1/4}$ law as λ becomes smaller. The root-mean-square (rms) phase difference $\delta\theta_1(T)$ between adjacent nodes in the xy-plane at temperature T is of order $(2k_BT/J_{xy})^{1/2}$. The rms phase difference across n such nodes is of order $\delta\theta_n \sim \delta\theta_1\sqrt{n}$. ζ is the distance in the xy-plane such that $\delta\theta_n \sim 1$. At $T=T_c \sim (2J_{xy}J_z)^{1/2}$, this screening length is given by $n \sim J_{xy}/(J_{xy}J_z)^{1/2} = \sqrt{\lambda}$, or $\zeta \approx a\sqrt{\lambda}$.

The $q^{-1/4}$ rule should be valid if ζ is large compared to the linear dimension of the $(q\times q)$ flux lattice unit cell, i.e., $\sqrt{\lambda} \gg q$. Thus, the larger the q, the larger the value of λ which would be required for this rule to hold. Indeed, Fig. 2 shows that the log-log plot of $T_c(f)$ deviates from linearity at a larger value of λ for large q than for small q, consistent with the argument just given.

The present model is not restricted to the specific problem considered here. It could be applied to other problems in which strongly coupled objects interact weakly with other such objects. One such system might be systems of 1D spin chains weakly coupled with other such chains. If the 1D problem can be solved exactly, the exact solution could be used as an input to the mean-field problem in higher dimensions.

ACKNOWLEDGMENTS

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would involve **A**; its summand will read $\cos(\psi'_{k,l}-\psi'_{k,l+1}-2\pi Bka^2/\Phi_0)$. Form (12) is then obtained by making the change of variables $\psi'_{k,l}=\psi_{k,l}-2\pi Bkla^2/\Phi_0$.

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