\[ \int_{V} \nabla \cdot \vec{D} \, d^{3}x = \int_{\partial V} \vec{D} \cdot d\vec{s} = \frac{\sigma_{f}}{\varepsilon_{0}} \]

\[ = \oint_{S} \nabla \cdot \vec{D} \, d^{2}x = (\vec{D}_{1} - \vec{D}_{2}) \cdot \hat{n}_{1} \, ds \]

So

\[ D_{1n} - D_{2n} = \frac{\sigma_{f}}{\varepsilon_{0}} \]

Thus our overall eqns. are

\[ \nabla \cdot \vec{D} = \frac{\sigma_{f}}{\varepsilon_{0}} \quad \text{with} \quad \vec{E}_{1t} = \vec{E}_{2t} \]

\[ \nabla \times \vec{E} = 0 \]

\[ D_{1n} - D_{2n} = \frac{\sigma_{f}}{\varepsilon_{0}} \]

Surface "bound charge density" \( \sigma_{b} \):

We have \( D_{1n} - D_{2n} = \frac{\sigma_{f}}{\varepsilon_{0}} \)

Also, showed previously

\[ E_{1n} - E_{2n} = \frac{\sigma}{\varepsilon_{0}} \]

where \( \sigma = \text{total surface charge density} = \sigma_{f} + \sigma_{b} \).
But \( \vec{D} = \varepsilon_0 \vec{E} + \vec{p} \)

So \( \vec{E} = \frac{1}{\varepsilon_0} (\vec{D} - \vec{p}) \)

\[
E_{1n} - E_{2n} = \frac{1}{\varepsilon_0} (D_{1n} - D_{2n}) - \frac{1}{\varepsilon_0} (P_{1n} - P_{2n})
\]

\[
= \frac{\sigma_f + \sigma_e}{\varepsilon_0} = \frac{1}{\varepsilon_0} \sigma_f - \frac{1}{\varepsilon_0} (P_{1n} - P_{2n})
\]

Hence, \( \sigma_f = -(P_{1n} - P_{2n}) \)

Why is \( -\vec{V} \cdot \vec{P} \) the change density?

Pictorially:

\[
\begin{array}{cccc}
\text{+} & \text{+} & \text{+} & \text{+} \\
\text{-} & \text{+} & \text{+} & \text{+} \\
\text{+} & \text{+} & \text{+} & \text{+} \\
\text{+} & \text{+} & \text{+} & \text{+} \\
\end{array}
\]

\( \rightarrow x \)

Lower density \( \Rightarrow \) higher density of dipole moments

In this picture \( \frac{dP_x}{dx} > 0 \)

So \( \sigma_f < 0 \). Hard to see physically; easier mathematically
Boundary value problems involving dialectics

I. Simple image problem

\[ \varepsilon_2 \quad \varepsilon_1 \]

Region (2) \quad Region (1)

\[
\begin{array}{c}
\vec{d} \quad \vec{x}_0 \\
\vec{z} \quad \text{axis}
\end{array}
\]

\[ z = 0 \]

\[ \varepsilon_1, \varepsilon_2 \text{ are dielectric constants in two regions} \]

In region (1)

\[ \nabla \cdot \vec{D} = \rho_f = \varepsilon_1 \nabla \cdot \vec{E} \]

So \[ \nabla \cdot \vec{E} = \frac{\rho_f}{\varepsilon_1} = \frac{q}{\varepsilon_1} \delta(\vec{x} - \vec{x}_0) \]

\[ \vec{x}_0 = (0, 0, d) \]

Also, \[ \nabla \times \vec{E} = 0 \Rightarrow \vec{E} = -\nabla \Phi \]

Boundary conditions: \[ D_{1z} = D_{2z}, \quad E_{1x} = E_{2x}, \]

\[ E_{1y} = E_{2y} \]
In region \( \Omega \),

\[
\nabla \Phi = -\nabla^2 \Phi = \frac{q}{\varepsilon_1 (x-x_0)}
\]

\[
\Rightarrow \Phi (x) = \Phi_1 (x) = \frac{q/\varepsilon_1}{4\pi \varepsilon_0 |x-x_0|} + F(x) \quad z > 0
\]

where \( F(x) \) must satisfy \( \nabla^2 \Phi = 0 \)

in region \( \Omega \)

We assume \( F(x) \) arises from an image charge \( q' \)

located at \( \vec{x}_i = (0, 0, -d) \). Thus

\[
F(x) = \frac{q'/\varepsilon_1}{4\pi \varepsilon_0 |x-x_i|}
\]

In region \( \Omega_2 \), there is no real charge, and the potential is due to the image charge located at the position of the original charge:

\[
\Phi_2 (x) = \frac{q''/\varepsilon_2}{4\pi \varepsilon_0 |x-x_0|} \quad z < 0
\]

Now, the b.c. state that

\[
\varepsilon_1 \left( \frac{\partial \Phi_1}{\partial z} \right)_{z=0} = \varepsilon_2 \left( \frac{\partial \Phi_2}{\partial z} \right)_{z=0}
\]
\[
\left( \frac{\partial \Phi_1}{\partial x} \right)_{z=0} = \left( \frac{\partial \Phi_2}{\partial x} \right)_{z=0}; \quad \left( \frac{\partial \Phi_1}{\partial y} \right)_{z=0} = \left( \frac{\partial \Phi_2}{\partial y} \right)_{z=0}
\]

The last two are automatically satisfied if we require that

\[\Phi_1(x, y, z=0) = \Phi_2(x, y, z=0)\]  (2)

Writing these out, we get

\[\nabla^2 \Phi = \frac{\partial^2}{\partial z^2} \left( \frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} \right)_{z=0}
\]

\[+ \frac{q}{\epsilon_0} \frac{\partial}{\partial z} \left( \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}} \right)_{z=0} = \frac{q}{\epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + d^2}}
\]

and

\[q + q' = \frac{1}{\epsilon_1} \frac{1}{\sqrt{x^2 + y^2 + d^2}} = \frac{q''}{\epsilon_2} \frac{1}{\sqrt{x^2 + y^2 + d^2}}\]  (1')

(1') soon becomes

\[q - q' = q''\]  (8'1)

and (2') becomes

\[\frac{1}{\epsilon_1} (q + q') = \frac{1}{\epsilon_2} q''\]

or

\[
q' = -\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} q\
q'' = \frac{2\epsilon_2}{\epsilon_2 + \epsilon_1} q
\]
Special cases:

(a) $\varepsilon_1 = \varepsilon_2$. Then $q' = 0$, $q'' = q$.

   (no actual boundary)

(b) $\varepsilon_2 > \varepsilon_1$: $q' = -q$, $q'' = 2q$

But $\vec{E}$ field in this case in region 2 is like that outside a conductor.

While field in region 1 goes to zero.

There is, of course, no real image charge. The actual extra charge is the polarization charge at the interface between the two regions.

Dielectric sphere in an external electric field $\vec{E}_0$.

$\varepsilon_0'$
Outside sphere
\[ \nabla \cdot \mathbf{D}_{\text{out}} = 0 \]
\[ \nabla \times \mathbf{E}_{\text{out}} = 0 \]
\[ \mathbf{D}_{\text{out}} = \varepsilon_0 \mathbf{E}_{\text{out}} \]

Inside sphere
\[ \nabla \cdot \mathbf{D}_{\text{in}} = 0 \]
\[ \nabla \times \mathbf{E}_{\text{in}} = 0 \]
\[ \mathbf{D}_{\text{in}} = \varepsilon \mathbf{E} \]

In both regions
\[ \mathbf{E} = -\nabla \Phi_{\text{out}} \text{ in region } r > a \]
\[ \mathbf{E} = -\nabla \Phi_{\text{in}} \text{ for } r < a \]

Now also outside sphere
\[ \nabla \cdot \mathbf{D}_{\text{out}} = \varepsilon_0 \nabla \cdot \mathbf{E}_{\text{out}} = -\varepsilon_0 \nabla^2 \Phi_{\text{out}} = 0 \]

Inside sphere
\[ \nabla \cdot \mathbf{D}_{\text{in}} = \varepsilon \nabla \cdot \mathbf{E}_{\text{in}} = -\nabla^2 \Phi_{\text{in}} = 0 \]

Since \( \nabla^2 \Phi = 0 \) in both places, can do a Legendre expansion as previously:
\[ \Phi_{\text{in}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) \]

For \( r > a \)
\[ \Phi_{\text{out}} = \sum_{l=0}^{\infty} B_l r^{l+1} P_l(\cos \theta) \]

+ potential due to uniform field
The potential due to a uniform field is

\[ -E_0 r P_1(\cos \theta) \]

Boundary conditions:

\[ E_{\text{in}} = E_{\text{out}}, \quad D_{\text{in}} = D_{\text{out}}, \quad \text{continuous} \]

or

\[ \left( \frac{\partial \Phi_{\text{in}}}{\partial \theta} \right)_{r=a} = \left( \frac{\partial \Phi_{\text{out}}}{\partial \theta} \right)_{r=a} \]  \hspace{1cm} (1)

\[ \varepsilon \left( \frac{\partial \Phi_{\text{in}}}{\partial r} \right)_{r=a} = \varepsilon_0 \left( \frac{\partial \Phi_{\text{out}}}{\partial r} \right)_{r=a} \]  \hspace{1cm} (2)

Only \( l=1 \) term survives. This gives

(11): \[ A_1 a \frac{d}{d \theta} P_1(\cos \theta) = + B_1 a^{-2} \frac{d P_1}{d \theta} - E_0 a \frac{d P_1}{d \theta} \]

(21) \[ \varepsilon A_1 P_1(\cos \theta) = -2 B_1 \varepsilon_0 a^{-2} P_1(\cos \theta) - \varepsilon_0 E_0 P_1(\cos \theta) \]

or

\[ A_1 = B_1 a^{-3} - E_0 \]  \hspace{1cm} (1'')

\[ \varepsilon A_1 = -2 B_1 \varepsilon_0 a^{-3} - \varepsilon_0 E_0 \]  \hspace{1cm} (2'')

\[ 2 \varepsilon_0 A_1 = 2 B_1 \varepsilon_0 a^{-3} - 2 \varepsilon_0 E_0 \]  \hspace{1cm} (1'')

\[ \varepsilon \text{add} \ (2 \varepsilon_0 + \varepsilon) A_1 = -3 \varepsilon_0 E_0 \]

\[ A_1 = \frac{-3 \varepsilon_0 E_0}{\varepsilon + 2 \varepsilon_0} \]

\[ B_1 a^{-3} = A_1 + E_0 = E_0 \left(1 - \frac{3 \varepsilon_0}{\varepsilon + 2 \varepsilon_0} \right) = \left( \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2 \varepsilon_0} \right) E_0 \]
\[
\Phi_{in} = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 r \cos \theta
\]

Corresponding to the uniform electric field \( \vec{E}_0 \)\

\[
\vec{E}_{in} = \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} \vec{E}_0
\]

\[
\Phi_{out} = -E_0 r \cos \theta + \frac{a^3}{r^2} \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 \cos \theta
\]

Corresponding to the uniform field \( \vec{E}_0 \) plus a dipole \( \vec{p} = 4\pi \varepsilon_0 a^3 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \vec{E}_0 \)

Polarization inside is\

\[
\vec{P} = \vec{D}_{in} - \varepsilon_0 \vec{E}_{in} = 6\varepsilon_0 \varepsilon_0 (\varepsilon - \varepsilon_0) \vec{E}_{in}
\]

\[
= \frac{3\varepsilon_0 (\varepsilon - \varepsilon_0)}{(\varepsilon + 2\varepsilon_0)} \vec{E}_{in}
\]

Dipole moment is\

\[
\vec{p} = \frac{4\pi a^3}{3} \vec{P} = \frac{4\pi \varepsilon_0 (\varepsilon - \varepsilon_0) a^3 \vec{E}_0}{\varepsilon + 2\varepsilon_0}
\]

As above
Where is the charge of this dipole moment?

We already showed that the surface polarization charge is
\[ \sigma_s = -(P_{in} - P_{2in}) \]
where \( \mathbf{n} \) is a unit vector into medium (1).

In the present case, we have polarization only inside the sphere.

\[ \mathcal{E} \]

Thus, surface bound charge density is
\[ \sigma_s = -P_{in} = +P_r = P \cos \theta \]
where \( \theta \) is the polar angle.

The dipole moment \( \vec{p} = \rho \hat{\mathbf{z}} \), where
\[
\rho = \int \int d^2 \mathbf{r} \sigma_s \zeta \\
= \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^a a^2 P \cos \theta a \sin \theta \cos \theta \\
= 2\pi a^3 \int_0^\pi \cos \theta \sin^2 \theta d\theta P \\
= 2\pi a^3 \int_0^1 \left[ \frac{x^2}{2} \right]_0^1 P = 2\pi a^3 P \frac{2^2}{3} \\
= 2\pi a^3 P \frac{2^3}{3} \]
More general case:

\[ E_i \]

\[ \frac{\varepsilon_2}{a^2} \]

\[ E_{0z} = \text{applied electric field.} \]

Can do same calculation.

Inside sphere

\[ \Phi_{in} = Ar \cos \theta \]

\[ \Phi_{out} = -E_0 r \cos \theta + \frac{B}{r^2} \cos \theta \]

One finds, upon matching boundary conditions, that

\[ A = -\frac{3 \varepsilon_1 E_0}{\varepsilon_2 + 2 \varepsilon_1} \]

\[ B = a^3 \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + 2 \varepsilon_1} E_0 \]

(Details left as exercise to be worked out but not turned in.)
How about total energy?

Change notation to make consistent with Jackson: call $\mathbf{J}$ $\mathbf{P}$ from now on.

**Problem 2.5: Microscopic Media**

Since macroscopic media may not be linear, we have to be careful. Calculate the work to make a small change $\delta \mathbf{P}(\mathbf{x})$ in a charge density $\mathbf{P}(\mathbf{x})$, when there is an electrostatic potential $\Phi(\mathbf{x})$ already present. The work needed to make this change is

$$\delta W = \int d^3x \, \delta \mathbf{P}(\mathbf{x}) \cdot \nabla \Phi(\mathbf{x})$$

(basically we are moving charge against an existing potential).

But $\mathbf{P} = \nabla \cdot \mathbf{D}$

so $\delta \mathbf{P} = \nabla \cdot (\delta \mathbf{D})$

and so

$$\delta W = \int_V d^3x \, \left( \nabla \cdot (\delta \mathbf{D}) \right) \Phi(\mathbf{x})$$

$$= \int_V d^3x \, \nabla \cdot (\delta \mathbf{D} \Phi(\mathbf{x})) - \int_V d^3x \, \delta \mathbf{D}(\mathbf{x}) \cdot \nabla \Phi(\mathbf{x})$$

$$= + \int_V d^3x \, \delta \mathbf{D}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x})$$
Now the first integral is can be converted into a surface integral, using the divergence theorem:

\[ \int_V d^3x \nabla \cdot (\delta D(x) \Phi(x)) = \int_S d^2x (\vec{D}(x) \cdot \hat{n}) \]

But suppose \( \delta(x) \) is localized in space.

Then \( \Phi(x) \sim \frac{1}{R} \) at a dist. \( R \) far from source.

Similarly \( \delta D(x) \sim \frac{1}{R^2} \).

So \( \int_S d^2x (\vec{D}(x) \Phi(x) \cdot \hat{n}) \sim \int_S d^2x \frac{1}{R^3} \).

\[ \sim R^2 \cdot \frac{1}{R^3} \rightarrow 0 \text{ as if surface very far from charge density} \]

Thus we are just left with

\[ \delta W = \int d^3x \vec{D}(x) \cdot \vec{E}(x) \]

If medium is linear and isotropic, then \( \vec{D}(x) = \varepsilon(x) \vec{E}(x) \).

\[ \delta W = \int d^3x \varepsilon(x) \delta \left( \frac{1}{2} \vec{E} \cdot \vec{E} \right) \]

\[ = \int d^3x \varepsilon(x) \delta \left( \frac{1}{2} \varepsilon \vec{E}^2 \right) = \int d^3x \frac{1}{2} \varepsilon \vec{D} \cdot \vec{E} \]
and thus \[ W = \frac{1}{2} \int \vec{D} \cdot \vec{E} \, d^3x \]

= total work required to assemble a linear

total energy stored in a linear dielectric

medium.

(even true if dielectric is anisotropic)

Key necessary property is linearity.

E.g. energy stored in a parallel plate capacitor

with charges \( \pm Q \) on the plates

\[
\begin{align*}
\varepsilon \quad &\quad \left\{ \begin{array}{c}
Q \\
- Q
\end{array} \right. \\
\end{align*}
\]

Let area of plates be \( S \)

So surface charge density is \( \frac{Q}{S} = \sigma \)

\( \vec{D} \)-field between plates is just \( \sigma \) (since inside plates \( \vec{D} = 0 \))

Thus \( \vec{E} \)-field is \( \frac{\sigma}{\varepsilon} \)

and energy stored in capacitor is

\[
W = \frac{S}{2} \int \vec{D} \cdot \vec{E} = \frac{1}{2} S \frac{Q^2}{\varepsilon} = \frac{1}{2} \frac{Q^2}{\varepsilon S}
\]
The capacitance is

\[ C = \frac{Q}{\Delta V} = \frac{Q}{E \cdot x} = \frac{\varepsilon Q}{\sigma x} = \frac{\varepsilon S}{x}. \]

Note that \( W = \frac{Q^2}{2C} \).

Forces:

If we have a dielectric which can move, and sources of field are fixed, then

\[ F_x = -\frac{\partial W}{\partial x} \]

i.e. system tries to minimize its energy at fixed charge.

E.g. parallel plate capacitor

\[ -Q \]

\[ \left\lfloor \begin{array}{c} \hline \ \varepsilon \ \hline \end{array} \right\rfloor \]

\[ x \]

\[ \left\lfloor -Q \right\rfloor \]

E-field across interface continuous

\[ D = \varepsilon E \] to right of interface

\[ = \varepsilon E \] to left of interface

Surface (free charge density) is \( \sigma = \varepsilon E \) to left of interface; \( \sigma = \varepsilon E \) to right of interface.
Thus \( Q = [e_y d + \varepsilon_0 (d-y) d]E \)

and hence

\[
E = \frac{Q}{d [e_y + \varepsilon_0 (d-y)]}
\]

Total energy stored in capacitor is

\[
W = \frac{1}{2} x y d \frac{\varepsilon_0 Q^2}{\left[e_y + \varepsilon_0 (d-y)\right]^2} + x(d-y) d \frac{\varepsilon_0 Q^2}{2 \left[e_y + \varepsilon_0 (d-y)\right]^2}
\]

\[
= \frac{1}{2} x d Q^2 \frac{e_y + \varepsilon_0 (d-y)}{[e_y + \varepsilon_0 (d-y)]^2} d^2
\]

\[
= \frac{1}{2} \frac{x Q^2}{d [e_y + \varepsilon_0 (d-y)]}
\]

\[
F = -\left(\frac{\partial W}{\partial y}\right)_Q = + \frac{1}{2} \frac{x Q^2 (\varepsilon - \varepsilon_0)}{d [e_y + \varepsilon_0 (d-y)]^2}
\]

> 0 if \( y < \varepsilon_0 \)

[+ tends to pull slab into capacitor]
General expression for force on a dielectric

Total energy is $W$

Let $d\xi$ be a small displacement of the dielectric. $V\frac{dW}{d\xi}$ acts as the potential energy, and system is isolated (no external sources to feed energy into system).

Then force on dielectric is

$$F = -\frac{dW}{d\xi}.$$ 

Now suppose the dielectric is moved at fixed potentials. That is the two electrodes (in this illustration) are kept at fixed $V$ rather than fixed $Q$. Situation is as below:
The total energy of the system is

\[ W = \frac{1}{2} \int \mathbf{E} \cdot \mathbf{E} \, d^3x = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{D} \, d^3x \]

\[ = -\frac{1}{2} \int \nabla \cdot (\mathbf{D} \Phi) \, d^3x + \frac{1}{2} \int \nabla \cdot \mathbf{D} \Phi \, d^3x \]

\[ = \frac{1}{2} \int \rho \Phi \, d^3x \]

turn into surface integral which goes to zero.

Now imagine our system consists of two plates holding \( Q \) of charge.

\[ \begin{align*}
Q & \quad -Q \\
\epsilon & \quad \epsilon
\end{align*} \]

Total energy is just

\[ \frac{1}{2} Q \Phi (\epsilon_1 - \epsilon_2) \]

\[ = \frac{1}{2} \rho \Delta \Phi = \frac{1}{2} C (\Delta \Phi)^2 \]

where \( C = \text{capacitance} \).

Now move dielectric a little bit. At fixed potential, the change in energy is

\[ \delta W_{\Phi} = \frac{1}{2} \delta C (\Delta \Phi)^2 = \nabla \Phi \cdot \delta \mathbf{E} \]
$$= \frac{1}{2} \left( \frac{\partial W}{\partial \Phi} \right)_\Phi \delta \Phi$$

where \( \left( \frac{\partial W}{\partial \Phi} \right)_\Phi = \frac{1}{2} \left( \frac{\partial C}{\partial \Phi} \right) (\Delta \Phi)^2 \).

But \( W \) is no longer the potential energy, because the batteries do some work in holding the plates at fixed potential. We know that \( \Phi = C(\Delta \Phi) \).

So, \( \delta Q = \delta C(\Delta \Phi) \) = charge which has to be moved to hold potential difference fixed at \( \Delta \Phi \).

The corresponding work to move this charge is

$$\delta Q \cdot \Delta \Phi = \delta C(\Delta \Phi)^2 = \left( \frac{\partial C}{\partial \Phi} \right)_\Phi (\Delta \Phi)^2 \delta \Phi$$

Thus, the battery supplies twice the change in energy as is actually supplied to the system. The effective potential force is then \( W = 2W = -W \) and therefore, the effective potential force acting on the dielectric is actually

$$+ \left( \frac{\partial W}{\partial \Phi} \right)_\Phi \Phi$$

because, at fixed potential, the system tries to maximize \( W \) rather than minimizing it.
E.g., in our previous battery example:

\[ W = \frac{1}{2} x d \left[ y \in \mathbb{E}^2 + (d-y)\varepsilon_0 \mathbb{E}^2 \right] \]

\[ = \frac{1}{2} x \frac{(\Delta \Phi)^2}{d} \left[ y \in + (d-y)\varepsilon_0 \right] \]

(Using \( \Delta \Phi = \mathbb{E} \cdot x \)).

The force on the dielectric is therefore:

\[ F = \nabla \left( \frac{\partial W}{\partial y} \right) = \frac{\mu}{2d} \frac{(\Delta \Phi)^2 (\varepsilon - \varepsilon_0)}{\varepsilon_0} > 0 \]

if \( \varepsilon > \varepsilon_0 \).

---

**Comment on electrohydrological fluids** (ER fluids):

\[ \varepsilon_2 \quad \varepsilon_0 \quad \varepsilon_1 \]

\[ \varepsilon_2 \quad \varepsilon_0 \quad \varepsilon_2 \]

Apply electric field \( \mathbb{E}_0 \).
Induce dipoles $\vec{p}$ in the spheres. These dipoles have a dipole-dipole interaction which tends to make them line up parallel to the field $\vec{E}_0$.

The lined-up dipoles cause viscosity of fluid to increase.

$\Rightarrow$ epsilon and viscosity increases with an applied electric field.