Basic equations

\[ \nabla \times \mathbf{E} = 0 \quad \Rightarrow \quad \mathbf{E} = -\nabla \phi \]

\[ \nabla \cdot \mathbf{D} = \rho \quad \Rightarrow \quad \nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \]

(Poisson's eq.)

Coulomb's Law

\[ F_{12} = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2 r_1^2}{r_{12}^3} = \text{force on charge} \quad \text{due to} \quad \text{charge} \quad \text{due to} \]

Where \( r_{12} = r_1 - r_2 \)

Imposed electric field at \( \mathbf{x} \).

If charge density is known in all space,

\[ \phi(\mathbf{x}) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x' \]
In finite region of space, we have boundary conditions.

**Dirichlet problem:** \[ \nabla^2 \Phi = -\frac{\rho}{\varepsilon_0} \text{ in } V \]
\[ \Phi \text{ specified on } S \]

Potential \( \Phi \) is determined uniquely.

**Neumann problem**
\[ \nabla^2 \Phi = -\frac{\rho}{\varepsilon_0} \text{ in } V \]
\[ \frac{\partial \Phi}{\partial n} \text{ specified on } S. \]
\( \Phi \) is determined to within a constant.

**Conductors**
A conductor has free charge.
\( \Phi = \text{const.} \) on a conductor (so \( \vec{E} = 0 \))
charge is confined to outer surface of a conductor.
At boundary between two conductors,

\[ E_1^\perp = E_2^\perp \]

and \( (E_1^2 - E_2^2) \cdot \hat{n} = \frac{\sigma}{\varepsilon_0} \)

where \( \sigma = \text{surface charge density} \)

Special case: medium 2 = conductor

Then \( E_2 = 0 \Rightarrow E_1^\perp = \frac{\sigma}{\varepsilon_0} \); \( E_1^\parallel = 0 \),

Force on surface of conductor \( -F = \frac{1}{2} \sigma E^2 \) per unit area

Boundary value problems in electrostatics: how to solve them?

Say want to solve Dirichlet problem in vol V surrounded by surface S.

Some methods include:

1. Green's function method (I haven't used this much)

\[ \Phi(x) = -\frac{1}{4\pi\varepsilon_0} \int_V \Phi(x') \frac{\partial G(x', x)}{\partial n'} \, d^3x' \]

\[ + \frac{1}{4\pi\varepsilon_0} \int_V \rho(x') G(x, x') \, d^3x' \]
where $\nabla^2 G_D(x, x') = -4\pi \delta(x-x') \cdot x'$ in $V$

$G_D(x, x') = 0 \text{ if } x' \text{ on } S.$

2. Method of images: find a set of images which satisfy boundary conditions.

3. (for Laplace's equation)
   Expand in a suitable complete set of functions for which satisfy Laplace's eq. in the relevant coordinate system.

Energy in electrostatics:

$$W = \frac{1}{2} \int \rho(x) \phi(x) d^3 x$$

$$= \frac{\varepsilon_0}{2} \int |E(x)|^2 d^3 x$$

Capacitance: If we have $n$ conductors at voltage $V_1 \cdots V_n$

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j \quad C_{ij} = \text{capacitance matrix}$$
\[ W = \frac{1}{2} \sum Q_i V_i = \frac{1}{2} \sum E_i V_i V_i \]

For two conductors with charge \( Q_i \), \( -Q_i \):

\[ W = \frac{1}{2} CV^2 \quad \text{where} \quad V = \text{voltage difference} \]

\underline{Method of Images:}

Can often satisfy boundary conditions by images (several examples in class).

\underline{Separation of Variables:}

\[
\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0
\]

General solution is

\[
\Phi(x, y, z) = e^{ik_1 x} e^{ik_2 y} e^{-k_3 z}
\]

where \( k_3 = \sqrt{k_1^2 + k_2^2} \)

Values of \( k_1, k_2, k_3 \) determined by boundary conditions
2D: Rectangular Coordinates

\[ \Phi(x, y) = e^{\pm ik_1 x} e^{\pm ik_2 y} \]

if \( \nabla^2 \Phi = 0 \)

2D Polar coordinates

\[ \Phi(\rho, \phi) = R(\rho) \psi(\phi) \]

\[ R(\rho) = a \rho^2 + b \rho^{-2} \]

\[ \psi(\phi) = A \cos \phi + B \sin \phi \]

(\( \nu \neq 0 \))

\( \nu = \text{integer} \)

\( \psi(\phi) = a + b \ln \rho \)

\( \psi(\phi) \text{ allowed for all } \phi \)

\( \psi(\phi) = A + B \phi \) when \( \nu = 0 \)

So a typical \( \psi^{\nu} \) is

\[ (a \rho^2 + b \rho^{-2})(A \cos \phi + B \sin \phi) \]

3D Spherical coords, azimuthal symmetry

If all angles allowed,

\[ \psi(r, \theta, \phi) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l (\cos \theta) \]
Another example:

Complete sets of functions

The $P_n(x)$'s are complete and orthogonal on $-1 \leq x \leq 1$

i.e. $\int_{-1}^{1} P_l(x) P_{l'}(x) dx = 0 \quad l \neq l'$

$$\frac{2}{2l+1} \quad l = l'$$

Since boundary value problem is unique, can use any method to get it.