1. Jackson, (4.13):

There are two forces acting on the liquid:

1. The electrostatic force

\[ F_{\text{Wes}} = \left( \frac{\partial W_{\text{Wes}}}{\partial h} \right) V \]

where \( W_{\text{Wes}} \) is the electrostatic energy and

The positive sign is valid because the force is acting at constant \( V \) = potential difference.

Now \( W_{\text{Wes}} = \frac{1}{2} \int \vec{D} \cdot \vec{E} \, d^3x \)

where the integral is taken over the length \( L \) as shown.
Now \( \vec{E} = \frac{C}{\rho} \hat{\rho} \) where \( C \) is a constant independent of height (since \( \vec{E} \) is continuous across the interface) while \( \hat{\rho} \) is a unit vector in the radial direction. \( C \) is determined by

\[
\int_{a}^{b} \frac{C}{\rho} \, d\rho = V = 2\pi r \ln \left( \frac{b}{a} \right)
\]

Therefore \( C = \frac{V}{\ln(b/a)} \)

and the energy is

\[
W_{es} = \frac{1}{2} \left[ \int_{0}^{h} \epsilon \vec{E}^2(r) \, \rho \, d\rho \, dz + \int_{h}^{L} \epsilon' \vec{E}^2(r) \, \rho \, d\rho \right] 2\pi
\]

\[
= \frac{1}{2} \left[ \epsilon h + \epsilon_0 (L-h) \right] (2\pi) \frac{V^2}{\ln^2(b/a)} \int_{a}^{b} \frac{1}{\rho} \, d\rho
\]

\[
= \pi \left[ \epsilon h + \epsilon_0 (L-h) \right] \frac{V^2}{\ln(b/a)}
\]

Hence

\[
F_{es} = \left( \frac{\partial W_{es}}{\partial h} \right)_{V} = \frac{\pi (\epsilon_0 - \epsilon) V^2}{\ln(b/a)} \quad \text{upwards}
\]

(2). The gravitational force is

\[
F_{grav} = - \frac{\partial W_{grav}}{\partial h}
\]
where \( W_{grav} = \pi (b^2-a^2) \int_0^h \rho g h' \, dh' \)

\[ = \frac{\pi}{2} (b^2-a^2) \rho g h^2 \]

where \( \rho = \text{mass density of liquid} \)

Thus \( F_{grav} = -\frac{\partial W_{grav}}{\partial h} = -\frac{\pi}{2} (b^2-a^2) \rho g h \) downward

The height is determined by

\[ F_{es} + F_{grav} = 0 \]

or \( \varepsilon_{\text{eff}} - \varepsilon_0 = \frac{\rho g h (b^2-a^2) \ln(b/a)}{\sqrt{2}} = \varepsilon_0 \chi_e \)

or

\[ \chi_e = \frac{\rho g h (b^2-a^2) \ln(b/a)}{\varepsilon_0 V} \]

where \( \chi_e = \frac{\varepsilon - \varepsilon_0}{\varepsilon_0} \) is the electrical susceptibility
Everywhere except at the origin, the potential satisfies Laplace's equation. Also, it is azimuthally symmetric. Therefore, both inside and outside the cavity, it can be expressed as a Legendre series as discussed in class.

Inside the cavity

\[ \Phi (r, \theta) = \Phi_{\text{in}} (r, \theta) = \]

\[ = \frac{1}{4\pi \epsilon_0} \frac{P \cos \theta}{r^2} + \sum_{l=0}^{\infty} A_l \, r^l \, P_l (\cos \theta) \]

The first term is the potential due to a dipole; the other terms are finite at the origin.

Outside, we expect only negative terms, so that \( \Phi_{\text{out}} \to 0 \) as \( r \to \infty \). Therefore

\[ \Phi_{\text{out}} (r, \theta) = \sum_{l=0}^{\infty} B_l \, r^{-(l+1)} \, P_l (\cos \theta) \]
Also, only expect \( l = 1 \) terms \((A_1\) and \(B_1\)) to survive, since must satisfy B.C. at \( r = a \) for all \( \theta \).

At \( r = a \),
\[ \Phi_{\text{in}}(a, \theta) = \Phi_{\text{out}}(a, \theta) \quad (i) \]

And
\[ -\varepsilon_0 \frac{\partial \Phi_{\text{in}}(a, \theta)}{\partial r} = -\varepsilon_0 \frac{\partial \Phi_{\text{out}}(r, \theta)}{\partial r} \quad (ii) \]

Or
\[ \frac{1}{4\pi\varepsilon_0} \frac{P}{a^2} + A_1 a = \frac{B_1}{a^2} \quad (i') \]

\[ \varepsilon_0 \left( -\frac{1}{2\pi\varepsilon_0} \frac{P}{a^3} + A_1 \right) = \varepsilon \left( -\frac{2B_1}{a^3} \right) \quad (ii') \]

Solving for \( A_1 \) and \( B_1 \), we find

\[ A_1 = -\frac{1}{2\pi a^3 \varepsilon_0} \frac{\varepsilon - \varepsilon_0}{2\varepsilon_0 + \varepsilon_0} P \]

And
\[ B_1 = \frac{3P}{4\pi (2\varepsilon + \varepsilon_0)} \]

Thus the potential outside for \( r > a \) is
\[ \Phi_{\text{out}} = \frac{3P \cos \theta}{4\pi (2\varepsilon + \varepsilon_0) r^2} \]
and for $r < a$ is

$$\Phi_{in} = \frac{\rho \cos \theta}{4\pi \varepsilon_0 r^2} - \frac{1}{2\pi a^2 \varepsilon_0} \frac{\varepsilon - \varepsilon_0}{2\varepsilon + \varepsilon_0} \rho r \cos \theta$$

Within the cavity

$$\vec{E}_{in} = -\nabla \left( \frac{\rho \cos \theta}{4\pi \varepsilon_0 r^2} \right) + \frac{1}{2\pi a^2 \varepsilon_0} \frac{\varepsilon - \varepsilon_0}{2\varepsilon + \varepsilon_0} \vec{P}$$

i.e., the field due to the dipole + a const. field

Outside the cavity

$$\vec{E}_{out} = -\nabla \left( \frac{3\rho \cos \theta}{4\pi (2\varepsilon + \varepsilon_0) r^2} \right)$$

i.e., the field due to a dipole

$$\vec{B}_{out} = \frac{3\varepsilon_0 \vec{P}}{2\varepsilon + \varepsilon_0}$$

$$\vec{D}_{out} = \varepsilon \vec{E}_{out} = \varepsilon_0 \vec{E}_{out} + \vec{P}_{out}$$

$$\Rightarrow \vec{P}_{out} = (\varepsilon - \varepsilon_0) \vec{E}_{out}$$

The surface polarization charge is

$$\sigma_{pol} = \vec{P}_{out} \cdot \hat{n}$$

where $\hat{n}$ is the unit normal out into the cavity

$$= -\vec{P}_{out} \cdot \hat{r}$$

Since $\nabla \cdot \vec{P}_{out} = 0$, there is no volume polarization charge.
Explicit evaluation of surface polarization charge

\[ \sigma_{\text{pol}} = -\vec{P}_{\text{out}} \cdot \hat{r} \]

\[ = \left[ \hat{r} \cdot \nabla \frac{3 \rho \cos \theta}{4\pi (2\varepsilon + \varepsilon_0) r^2} \right]_{r=a} \]

\[ = \frac{\partial}{\partial r} \left[ \frac{3 \rho \cos \theta}{4\pi (2\varepsilon + \varepsilon_0) r^2} \right]_{r=a} \]

\[ = -\frac{3 \rho \cos \theta}{4\pi (2\varepsilon + \varepsilon_0) a^3} \]

Note that \( \sigma_{\text{pol}} \) has a finite dipole moment, which is why the field outside the cavity still has a dipole form.
2. Jackson 5.3

Surface current density is
\[ \mathbf{J} = NI \mathbf{\phi} \]

We calculate the field due to a little ribbon of solenoid of width \( dz \) along the \( z \)-axis.

We can use Eq. (5.4) of Jackson (with \( k = \frac{\mu_0}{4\pi} \); see p. 176) to write
\[
\mathbf{dB} = \frac{\mu_0 NI}{4\pi} \oint_C \frac{d\ell \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}
\]

where the integral goes around the loop. Now the only non-vanishing component of \( \mathbf{dB} \) is the \( z \)-component, which is
\[ dB_z = \frac{\mu_0 NI}{4\pi} \int\frac{1}{(\vec{x} - \vec{x}')^2 + a^2} \sin \theta' \, d\vec{x}' \]

Note that \( dB_z \) is the contribution of the entire loop shown in the sketch to \( B_z \). Also, \( \partial \vec{E} = 0(\vec{x} - \vec{x}') \).

Since the integral around the loop gives \( 2\pi a \), we get

\[ B_z = \frac{\mu_0 NI}{4\pi} (2\pi a) \int_{z_1}^{z_2} \frac{\sin \theta' \, d\vec{x}'}{(z' - z)^2 + a^2} \]

\[ = \frac{\mu_0 NI a^2}{2} \int_{z_1}^{z_2} \frac{1}{2} \left[ \frac{z - z_1}{(z' - z)^2 + a^2} \right]^{3/2} \, d\vec{x}' \]

\[ = \frac{\mu_0 NI}{2} \left\{ \frac{z_2 - z}{\sqrt{(z' - z)^2 + a^2}} \right\}^{z_2}_{z_1} \]

\[ = \frac{\mu_0 NI}{2} \left( \frac{z_2 - z}{\sqrt{(z_2 - z)^2 + a^2}} + \frac{z - z_1}{\sqrt{(z_1 - z)^2 + a^2}} \right) \]

where \( z_1 \) and \( z_2 \) are the ends of the solenoid

\[ = \frac{\mu_0 NI}{2} \left( \cos \theta_2 + \cos \theta_1 \right) \]

where \( \theta_1 \) and \( \theta_2 \) are shown in the sketch.

\[ \text{Q.E.D.} \]
Problem Set 6: Solutions

(Cont'd)

Jackson 4.7(c):

The interaction energy between a quadrupole moment and an external electric field is

\[ W = -\frac{1}{b} \sum_{i,j} Q_{ij} \frac{\partial E_i}{\partial x_j} (x=0) \]

[Jackson eq. (4.24)].

For our case, \( Q_{11} = Q_{22} = -\frac{1}{2} Q_{33} = -\frac{Q}{2} \)

with \( Q_{ij} = 0 \) for \( i \neq j \). [see discussion above eq. (4.25)]

Hence,

\[ W = \frac{Q}{6} \left[ \frac{1}{2} \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} \right) - \frac{\partial E_z}{\partial z} \right] \]

\[ x=0 \]

\[ = -\frac{Q}{6} \left[ \frac{\partial^2 \Phi}{\partial z^2} - \frac{1}{2} \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) \right] \]

\[ x=0 \]

For prob. (4.7), we have already shown that

\[ \Phi(x) \sim \text{const.} - \frac{r^2}{480\pi\varepsilon_0} P_2 (\cos \theta) \]

\[ = \text{const.} - \frac{1}{480\pi\varepsilon_0} \left( \frac{3}{2} \frac{x^2}{2} - \frac{1}{2} (x^2 + y^2 + z^2) \right) \]
Carrying out the derivatives, we find

\[ W = -\frac{Q}{2880 \pi \varepsilon_0} \text{ in the "natural units" of the problem.} \]

We have to convert this back into SI units.

Since \( Q \) is in m\(^2\), we must multiply by \( e^2 \) and divide by \( a_0^3 \), so

\[ W = -\frac{Q \cdot e^2}{2880 \pi \varepsilon_0 a_0^3} \text{ joules} \]

\[ = -\frac{(10^{-28}) (1.6 \times 10^{-19})}{2880 \pi (8.85 \times 10^{-12}) (10.5 \times 10^{-10})^3} \]

\[ \approx 2.28 \times 10^{-24} \text{ joule} \]

The corresponding frequency is

\[ \omega_0 = \frac{2.28 \times 10^{-24}}{\hbar} = \frac{2.28 \times 10^{-24}}{1.05 \times 10^{-34}} \]

\[ \approx 2.17 \times 10^9 \text{ rad/sec} \]

\[ \approx 3.5 \text{ GHz} \]