1. \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

Eigenvalues and eigenvectors satisfy

\[
\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0
\]

Eigenvalues from \( \lambda^2 = -1 = 0 \) \( \lambda = \pm i \)

For \( \lambda = i = \lambda_1 \) \( a = +b = \frac{1}{\sqrt{2}} \) or \((a, b) = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)\)

\( \lambda = -i = \lambda_2 \) \( a = -b = \frac{1}{\sqrt{2}} \) or \((a, b) = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)\)

\( Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) eigenvalues are \( \pm 1 \)

\( \lambda_1 = \pm 1 \) eigenvector is \( \begin{pmatrix} 1 \\ i \end{pmatrix} \)

\( \lambda_2 = -1 \) eigenvector is \( \begin{pmatrix} 1 \\ i \end{pmatrix} \)

\( Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \); eigenvalues are

\( \lambda_1 = 1 \) eigenvector is \( (1, 0) \)

\( \lambda_2 = -1 \) eigenvector is \( (0, 1) \)
Recall that Bloch sphere vector is a unit vector on the sphere, with spherical angles defined by:

\[ | \psi \rangle = \frac{1}{\sqrt{2}} \left| 0 \right\rangle + e^{i \phi} \sin \frac{\theta}{2} | 1 \rangle \]

For \( X \), \( \lambda_1 \) corresponds to:
\[ \theta = \frac{\pi}{2}, \ \phi = 0 \]
\( \lambda_2 \) corresponds to:
\[ \theta = \frac{\pi}{2}, \ \phi = \pi \]

For \( Y \), \( \lambda_1 \) corresponds to:
\[ \theta = \frac{\pi}{2}, \ \phi = -\frac{\pi}{2} \]
\( \lambda_2 \) is \( \theta = \frac{\pi}{2}, \ \phi = +\frac{\pi}{2} \)

\( Z \):
\[ \lambda_1 \leftrightarrow \theta = 0, \ \phi = (\text{anything}) \]
\( \lambda_2 \leftrightarrow \theta = \frac{\pi}{2}, \ \phi = 0 \).

2. (Already assigned in P.S. 2 for any matrix \( A \))

3. This problem should have read:
\[ T = R_x \left( \frac{\pi}{4} \right) \]
This is easy to prove: \[ T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}; \]

\[ R_z(\frac{\pi}{4}) = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix} \]

\[ = e^{-i\pi/8} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{-i\pi/8} T \quad \Box \]

4. This turns out to be quite messy in general, so we will confine ourselves to showing that \( R_n(\theta) \) corresponds to a rotation about the axis \( \vec{n} \). To do this, we write

\[ R_n(\theta) = \exp \left( i \frac{\theta}{2} \vec{n} \cdot \vec{\sigma} \right) \]

where \( \vec{n} \cdot \vec{\sigma} = \sin \alpha \cos \beta x + \sin \alpha \sin \beta y + \cos \alpha z \)

\[ = \begin{pmatrix} \cos \alpha & \sin \alpha \cos \beta - i \sin \alpha \sin \beta \\ \sin \alpha \cos \beta + i \sin \alpha \sin \beta & -\cos \alpha \end{pmatrix} \]

\[ = \begin{pmatrix} \cos \alpha & \sin \alpha e^{-i\beta} \\ \sin \alpha e^{i\beta} & -\cos \alpha \end{pmatrix} \]
where we wrote \( n_x = \cos \alpha \sin \beta \),
\[ n_y = \sin \alpha \sin \beta \\
\hat{n}_z = \cos \alpha \]

Now we calculate
\[
\hat{n} \cdot \vec{\sigma} = \begin{pmatrix}
-e^{-i\beta/2} \cos \frac{\alpha}{2} \\
e^{i\beta/2} \cos \frac{\alpha}{2} \\
e^{i\beta/2} \sin \frac{\alpha}{2}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos \alpha e^{-i\beta/2} \cos \frac{\alpha}{2} + \sin \alpha e^{-i\beta/2} \sin \frac{\alpha}{2} \\
\sin \alpha e^{i\beta/2} \cos \frac{\alpha}{2} - \cos \alpha e^{i\beta/2} \sin \frac{\alpha}{2}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
e^{-i\beta/2} \cos \frac{\alpha}{2} \\
e^{i\beta/2} \sin \frac{\alpha}{2}
\end{pmatrix} = V_n
\]

So this last vector is an eigenstate of \( \hat{n} \cdot \vec{\sigma} \) with eigenvalue 1.

If we now calculate \( e^{i \vec{\theta} \cdot \hat{n} \cdot \vec{\sigma}} \) applied to this vector, we get, using
\[
e^{i \theta/2 \hat{n} \cdot \vec{\sigma}} = 1 + \frac{i \theta}{2} (\hat{n} \cdot \vec{\sigma}) + \frac{1}{2!} \left( \frac{i \theta}{2} \hat{n} \cdot \vec{\sigma} \right)^2 + ...
\]

that
\[
e^{i \theta/2 \hat{n} \cdot \vec{\sigma}} V_n = e^{i \theta/2} V_n
\]

So clearly \( V_n \) is the axis of rotation for \( \hat{n} \).
operator $e^{i\frac{\Theta}{2} \hat{n} \cdot \hat{\sigma}}$, since $V^\dagger$ is invariant
(up to an irrelevant phase factor)
upon application of this operator.

Therefore, $e^{i\frac{\Theta}{2} \hat{n} \cdot \hat{\sigma}}$ represents a rotation
about the axis denoted by $\hat{n}$.

This is part of what you were to prove.

To show that $R_{\hat{n}}(\Theta)$ rotates an arbitrary
vector on the Bloch sphere by an angle $\Theta$
is a messy exercise in spherical geometry.

I will give full credit to those who found
just the axis of rotation.

5. I am embarrassed to say that I
haven't found a good proof of this
theorem. Therefore, I am giving all who
attempted this problem full credit!