Physics 834: Prob. Set 6

Solutions

1. Jackson 4.1 (a) and (b)

\[ \frac{a}{a-x} - \frac{a}{a-y} \]

There are no quadrupole moments.

Why? All the qem are linear combinations of the second-order cubic harmonics discussed in class, which are of the form \( \int x y \rho(x) dx = Q_{12} \) etc. [see Jackson, eq. (4.9)].

All the \( Q_{ij} \)'s are of even powers of \( x, y, z \).

There. On the other hand, \( \rho(x) = -\rho(-x) \)

by inspection. Thus \( \rho(x) \) is an odd function of \( x \). Hence, the argument of the integral is an odd function of \( x \) and thus vanishes (for all the \( Q_{ij} \)).

Hence \( \text{qem} = 0 \) for \( l=2 \) (or for any even \( l \))
For the dipole moment, we can write

\[ \mathbf{p} = \int \mathbf{r} \rho(x) \, d^3x \]

\[ = q \int x [ \delta(x^2 - a^2) - \delta(x^2 + a^2) \]
\[ + \delta(x^2 - a^2) - \delta(x^2 + a^2) ] \, d^3x \]

\[ = q \left[ a \hat{x} - (a \hat{x}) + a \hat{y} - (a \hat{y}) \right] \]

\[ = 2qa \hat{(x+y)} \]

So

\[ q_{10} = \sqrt{\frac{3}{4\pi}} p_z = 0 \]

\[ q_{ll} = -\sqrt{\frac{3}{8\pi}} (p_x - i p_y) = -\sqrt{\frac{3}{8\pi}} \cdot 2qa (1-i) \]

\[ \Rightarrow \quad q_{1j-l} = -\sqrt{\frac{3}{8\pi}} \cdot 2qa (1+i) \]

Q. (b).

Here \( \rho(x) = \rho(-x) \), so all odd \( \sqrt{g_{2m}} = 0 \)

for \( m \) odd, or \( g_{2m} = 0 \)

\[ g_{00} = 0 \]

since no net charge

To get the \( g_{2m} \)'s, we first calculate the \( Q_{ij} \)'s [Jackson, eq. (4.9)].
\[ Q_{ij} = \int (3 x_i^1 - r_i^2 \delta_{ij}) q \left( \frac{x_j}{x^1} \right) d^3 x^1 \]

\[ = \int (3 x_i^1 x_j^1 - r_i^2 \delta_{ij}) q \left[ \delta \left( x^1 - a x_j^1 \right) - 2 \delta (x^1) + \delta \left( x^1 + a x_j^1 \right) \right] d^3 x^1 \]

\[ = 0 \quad i \neq j \]

\[ Q_{33} = \int (3 x_i^2 - r_i^2) q \left[ \delta \left( x^1 - a x_j^1 \right) - 2 \delta (x^1) + \delta \left( x^1 + a x_j^1 \right) \right] d^3 x^1 \]

\[ = 2 q a^2 + 2 q a^2 = 4 q a^2 \]

\[ Q_{11} = Q_{22} = -2 q a^2 \]

Thus \[ q_{22} = \frac{1}{12} \sqrt{\frac{15}{2 \pi}} (Q_{11} - 2 i Q_{12} - Q_{22}) = 0 = q_{21} \]

\[ q_{21} = 0 = q_{12} \]

\[ q_{20} = \frac{1}{2} \sqrt{\frac{5}{4 \pi}} Q_{33} = \sqrt{\frac{5}{4 \pi}} 2 q a^2 \]

2. Jackson 4.2:

Consider the potential of a dipole \( \vec{p} \) at the origin is \( \Phi(x) = \frac{1}{4 \pi \epsilon_0} \frac{\vec{p} \cdot \vec{x}}{|x|^3} \)
But the potential due to a charge density 
\[ \rho(x') = -\frac{\rho}{4\pi\epsilon_0} \nabla' \delta(x-x_0) \]  \hspace{1cm} (4)  

is 
\[ \Phi(x) = -\frac{1}{4\pi\epsilon_0} \int_V \left[ \frac{-\rho}{|x-x'|} \nabla' \delta(x-x_0) \right] d^3x' \]

\[ = -\frac{1}{4\pi\epsilon_0} \int_V \nabla' \cdot \left( \frac{\rho \delta(x-x_0)}{|x-x'|} \right) d^3x' \]

\[ + \frac{1}{4\pi\epsilon_0} \int_V \delta(x-x_0) \nabla' \left( \frac{\rho}{|x-x'|} \right) d^3x' \]

using the relation
\[ \nabla' \cdot (f \vec{G}) = \nabla f \cdot \vec{G} + f \nabla' \cdot \vec{G} \]

The first integral can be turned into a surface integral over any surface \( S \) enclosing the charge density, using the divergence theorem:
\[ \int_V \nabla' \cdot \left( \frac{\rho \delta(x-x_0)}{|x-x'|} \right) d^3x' = \int_S \nabla' \left( \frac{\rho}{|x-x'|} \right) d^3n' \]

\[ = \int_S \frac{\rho \cdot \vec{n}' \cdot \delta(x-x_0)}{|x-x'|} d^3x' = 0 \quad \text{so long as} \quad S \text{ is outside the pt} \quad x_0 \]

Also 
\[ \overline{\nabla'} \cdot \left( \frac{\rho}{|x-x'|} \right) = \frac{\rho \cdot \overline{\nabla'} \cdot 1}{|x-x'|} \]

\[ = \frac{\rho \cdot (x-x')}{|x-x'|^3} \quad \text{sine} \rho \text{ is} \]

\[ \frac{1}{|x-x'|} \text{ a const.} \]
Thus the potential is just

\[ \phi(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\delta(x - x_0) \cdot \hat{p} \cdot (\hat{x} - \hat{x}_0)}{|x - x_0|} \, d^3x \]

\[ = \frac{1}{4\pi\varepsilon_0} \frac{\hat{p} \cdot (\hat{x} - \hat{x}_0)}{|\hat{x} - \hat{x}_0|} \]

which is the potential at \( \hat{x} \) due to a dipole at \( \hat{x}_0 \). \( \square \)

The energy of a dipole in an external field, using eq. (i) at the top of p. 4, is

\[ U = \int \mathbf{p} \cdot \mathbf{\Phi}_{\text{ext}} \, d^3x \]

where \( \mathbf{\Phi}_{\text{ext}} \) is the potential due to the external field.

This is

\[ U = -\int \mathbf{p} \cdot \frac{\partial}{\partial x_0} \delta(x - x_0) \mathbf{\Phi}_{\text{ext}}(x_0) \, d^3x \]

\[ = -\int \left( \frac{\partial}{\partial x_0} \mathbf{\Phi}_{\text{ext}}(x_0) \right) \delta(x - x_0) \mathbf{p} \, d^3x \]

by divergence theorem

\[ + \int \delta(x - x_0) \frac{\partial}{\partial x_0} \mathbf{\Phi}_{\text{ext}}(x_0) \, d^3x \]
\[ \begin{align*}
\int & \delta(x' - x_0) \ p \cdot \nabla^1 \Phi_{\text{ext}}(x') \ d^3x' \\
= & \ - \int \delta(x' - x_0) \ p \cdot \vec{E}_{\text{ext}}(x') \ d^3x' \\
& \quad \text{using } \vec{E}_{\text{ext}} = -\nabla \Phi_{\text{ext}} \\
= & \ -p \cdot \vec{E}_{\text{ext}}(x_0)
\end{align*} \]

as derived in class

QED

3. Jackson 4.4(a)

We prove this theorem as follows.

Let the lowest non-vanishing moment be of order \( r^0 \). Then we suppose we calculate this moment with respect to the origin. Then the moment \( q_{em} \) will be some linear combination of the functions of the form

\[ \int x'^a y'^b z'^c \ \rho(x') \ d^3x' \quad (2) \]

where \( a + b + c = l \).

Now, calculate the moments with respect to a different origin \( x'_0 = (x_0, y_0, z_0) \).

Then the \( 0 \) moments \( q_{em} \) are linear combinations...
\[ \int (x'_0 - x^-)^a (y'_0 - y^-)^b (z'_0 - z^-)^c \rho (x^1) \, d^3 \chi \]

(actually, the same linear combinations as when the origins are taken as moments are calculated around the original origin.

Now the highest power of with the variable \(\chi\).

We can write this integral as

\[ \int x'^a \, y'^b \, z'^c \rho (x^1) \, d^3 \chi' \]

\[ 1 - \int a \chi_0 x'^{a-\frac{\alpha'_c}{\alpha_c}} \, y'^b \, z'^c \rho (x^1) \, d^3 \chi' \]

\[ + \text{ (various other integrals involving} \]

\[ x'^A \, y'^B \, z'^C \text{ with} \]

\[ A + B + C < \alpha_c \)

But all these other integrals could only enter as \( \text{terms with} \ l' < l \), and thus vanish, because we have assumed that the lowest nonvanishing moment is of order \( l \).

The first nonvanishing moment is thus the same, irrespective of choice of origin.
If we apply the same argument for \( l'' > l' \),
we get additional nonvanishing terms which depend on the choice of origin.
Thus, the lowest nonvanishing moments \( q_{l'm} \) are independent of the choice of origin, but the higher moments do depend on choice of origin. QED

4. Jackson 4.7

Let \( \rho(r) = \frac{1}{64\pi} r^2 e^{-r} \sin^2 \theta \)

We have \( q_{l'm} = \int Y_{l'm}^*(\theta, \phi) \rho(r, \theta, \phi) \, d^3r \)

Now \( \sin^2 \theta = 1 - \cos^2 \theta \)

\[
= \frac{1}{3} - \cos^2 \theta + \frac{2}{3}
\]

\[
= -\frac{2}{3} \sqrt{\frac{4\pi}{5}} \frac{2}{3} \left[ \sqrt{\frac{5}{4}} - \frac{3}{2} \cos^2 \theta \right] + \frac{2}{3} \sqrt{\frac{4\pi}{5}} \frac{1}{\sqrt{4\pi}}
\]

\[
= -\frac{2}{3} \sqrt{\frac{4\pi}{5}} Y_{20} + \frac{2}{3} \frac{2}{\sqrt{4\pi}} Y_{00}
\]
But we have the orthonormality relation

\[ \int_0^{2\pi} \int_0^\pi \left( Y_{em}^* (\theta, \phi) Y_{en} (\theta, \phi) \right) \sin \theta \, d\theta \, d\phi = \delta_{ll'} \delta_{mm'} \]

Hence, since \( \rho(r) \) is a linear combination of \( Y_{20} \) and \( Y_{00} \), only \( b_{20} \) and \( b_{22} \) are non-vanishing.

Thus,

\[ b_{00} = \int \left( Y_{00} \rho(r, \theta, \phi) \right) \, d^3r \]

\[ = \int \frac{1}{64\pi} \frac{1}{3} \frac{4\pi}{3} \int_0^\infty \frac{4}{r^2} \, dr \]

\[ = \frac{4^4}{64\pi} \frac{2}{3} \sqrt{4\pi} = \frac{1}{\sqrt{4\pi}} = b_{00} \]

Also,

\[ b_{20} = \int \left( Y_{20} \rho(r, \theta, \phi) \right) \, d^3r \]

\[ = -\int \left( \int \frac{1}{64\pi} \frac{1}{3} \frac{4\pi}{3} \, dr \right) \rho(r, \theta, \phi) \, \sin \theta \, d\theta \, d\phi \]

\[ = \frac{4^3}{64\pi} \frac{1}{3} \frac{4\pi}{3} \]
\[ = - \frac{1}{64\pi} \frac{2}{3} \sqrt{\frac{4\pi}{5}} \int_0^\infty \frac{\xi}{\xi} e^{-\xi} \, d\xi \]

\[ = - \frac{6! \sqrt{\frac{4\pi}{5}}}{64\pi} \frac{2}{3} \sqrt{\frac{4\pi}{5}} = -\sqrt{\frac{20}{\pi}} = 8.20 \]

The potential at large distances is therefore

\[ \Phi(x) = \frac{1}{\epsilon_0} \left( \frac{q_{oo}}{r} + \frac{q_{20} Y_0}{r^2} \right) \]

\[ = \frac{1}{\epsilon_0} \left( \frac{q_{oo}}{r} \frac{1}{\sqrt{4\pi}} P_0(\cos \theta) + \frac{q_{20}}{r^2} \sqrt{\frac{15}{4\pi}} P_2(\cos \theta) \right) \]

\[ = \frac{1}{4\pi \epsilon_0 r} - \frac{5}{\pi \epsilon_0} \frac{P_2(\cos \theta)}{r^3} \]

(6). The potential at any point in space is

\[ \Phi(x) = \frac{1}{4\pi \epsilon_0} \int \frac{\delta(x - x')}{|x - x'|} \, d^3x' \]

Use the \textit{expansion} (3.70) of Jackson to rewrite this as
or \( \Phi(x) = \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}(\theta, \phi)}{2l+1} \int_{r_1}^{r_2} \rho(r') Y^*_{lm}(\theta', \phi') \frac{r'}{r + 1} \, dr' \)

\[
= \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}(\theta, \phi)}{2l+1} \int_{r_1}^{r_2} \frac{r'^2}{r + 1} \left( \frac{1}{r^2 \sin \theta} e^{-r} \right) dr'
\]

\[
= \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}(\theta, \phi)}{2l+1} \int_{r_1}^{r_2} \frac{r'^2}{r + 1} \left( \frac{1}{r^2 \sin \theta} e^{-r} \right) dr'
\]

\[
= \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}(\theta, \phi)}{2l+1} \int_{r_1}^{r_2} \frac{r'^2}{r + 1} \left( \frac{1}{r^2 \sin \theta} e^{-r} \right) dr'
\]

\[
= \int d\Omega' \frac{1}{64\pi} \left( \frac{2}{3} \sqrt{4\pi} Y_{00}^{(1)} - \frac{2}{3} \sqrt{5} Y_{20}^{(20)} \right) \frac{1}{r_1^2 \sin \theta} e^{-r_1} dr_1
\]

where \( d\Omega' = \sin \theta' d\theta' d\phi' \) is an element of integration over solid angle.

Using the orthonormality of the \( Y_{lm} \) 's, this becomes

\[
\Phi(x) = \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \frac{Y_{00}(\theta, \phi)}{64\pi} \frac{2}{3} \sqrt{4\pi} \int_{r_1}^{r_2} \frac{r'}{r + 1} e^{-r'} dr'
\]

\[
= \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \frac{Y_{20}(\theta, \phi)}{64\pi} \frac{2}{3} \sqrt{5} \int_{r_1}^{r_2} \frac{r'}{r + 1} e^{-r'} dr'
\]

\[
= \int_{r_1}^{r_2} \frac{r'}{r + 1} e^{-r'} dr'
\]

\[
= \frac{1}{96\pi \varepsilon_0} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \int_{r_1}^{r_2} \frac{r'}{r + 1} e^{-r'} dr'
\]

\[
- \frac{1}{96\pi \varepsilon_0} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \int_{r_1}^{r_2} \frac{r'}{r + 1} e^{-r'} dr'
\]
The first integral is expressible as
\[ \int_0^\infty \frac{r^{1/6}}{r^3} e^{-r} \, dr' + \int_0^\infty \frac{r^{13/6}}{r^3} e^{-r} \, dr' \]

When \( r \to 0 \) the first term goes to zero and the second becomes \( \int_0^\infty r^{13/6} e^{-r} \, dr' = 6 \).

The second integral can be written as
\[ \int_0^\infty \frac{r^{16/3}}{r^3} e^{-r} \, dr' + \int_r^\infty \frac{r^{17/3}}{r^3} e^{-r} \, dr' \]

Again, these
We can rewrite this integral as
\[ \int_0^\infty \left( \frac{r^{16/3}}{r^3} - \frac{r^{17/3}}{r^3} \right) e^{-r} \, dr' + \int_0^\infty \frac{r^2}{r^{17/3}} e^{-r} \, dr' \]

The first integral, for
very small \( r \), goes like \( r^4 \),
while the second equals \( r^2 \). Hence
\[ \frac{3}{2} \cos^2 \theta - \frac{1}{2} = P_2 (\cos \theta) \]

the potential for small \( r \) is approximately
\[ \Phi(r, \theta) = \frac{1}{16 \pi \epsilon_0} - \frac{r^2}{480 \pi \epsilon_0} P_2 (\cos \theta) \]
\[ = \frac{1}{4 \pi \epsilon_0} \left( 8 \frac{1}{4} - \frac{r^2}{120} P_2 (\cos \theta) \right) \]
\[ \text{QED} \]
5. Jackson (4.10)

\[ -Q \]

(a) The key point is that \( \mathbf{E} \) is tangent to the surface between dielectric and air, so is continuous across the interface.

Hence \( V \) is constant.

We also expect \( \mathbf{E} = \frac{\mathbf{C}}{r^2} \) where \( \mathbf{C} \) is a constant to be determined.

To get \( \mathbf{C} \), we write

\[ \mathbf{D} = \frac{\mathbf{E} \cdot \mathbf{C}}{r^2} \quad \text{in region (1)} \]

\[ = \frac{\varepsilon_0 \mathbf{C}}{r^2} \quad \text{in region (2)} \]

Let \( \sigma_1 \) be surface charge density on the inner sphere in region (1) (assumed constant).

\[ \sigma_2 = \varepsilon_0 \sigma_1 \text{ surface charge density on inner sphere in region (2)}. \]
Then we have
\[ \sigma_1 = D_1 = \frac{e_c}{a^2} \]
\[ \sigma_2 = D_2 = \frac{\varepsilon_0 e_c}{a^2} \]

\[ \int d^2 x \; \sigma(x) \text{ over inner sphere} \]
\[ = Q = 2\pi (\varepsilon + \varepsilon_0) C \]

Thus \[ C = \frac{Q}{2\pi (\varepsilon + \varepsilon_0)} \]

So \[ E = \frac{Q \hat{r}}{2\pi (\varepsilon + \varepsilon_0) r^2} \text{ in both regions 1 and 2}. \]

(e). From (a)
\[
\begin{align*}
\sigma_1 &= \frac{e_c}{a^2} = \frac{e C}{2\pi (\varepsilon + \varepsilon_0) a^2} \\
\sigma_2 &= \frac{\varepsilon_0 e_c}{a^2} = \frac{\varepsilon_0 Q}{2\pi (\varepsilon + \varepsilon_0) a^2}
\end{align*}
\]

Check: \[ 2\pi a^2 (\sigma_1 + \sigma_2) = Q \]

(outward normal is \( \hat{r} \))

(c). We have \[ \bar{\sigma}_p = -p_{\text{out}} \hat{r} \]
\[ \bar{p} = (\bar{D} - \varepsilon_0 \bar{E}) = \frac{(\varepsilon - \varepsilon_0) Q \hat{r}}{2\pi (\varepsilon + \varepsilon_0) r^2} \]

Hence \[ \bar{\sigma}_p = -\frac{(\varepsilon - \varepsilon_0) Q}{2\pi (\varepsilon + \varepsilon_0) a^2} \text{ on inner sphere} \]