Two-dimensional arrays of Josephson junctions in a magnetic field: A stability analysis of synchronized states

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We report the results of a Floquet analysis of two-dimensional arrays of resistively and capacitively shunted Josephson junctions in an external transverse magnetic field. The Floquet analysis indicates stable phase locking of the active junctions over a finite range of values of the bias current and junction capacitance, even in the absence of an external load. This stable phase locking is robust against critical current disorder, up to at least a $\pm 25\%$ rms spread in critical currents. [S0163-1829(99)02310-3]

I. INTRODUCTION

One interest in linear chains of Josephson junctions, and in two-dimensional (2D) arrays of such junctions, lies in their possible use as voltage controlled oscillators in the millimeter and submillimeter region. Experimental work has generally focused on fabricating such arrays and demonstrating appreciable power outputs (in the microwatt to milliwatt range).[1–8] Theoretical studies have begun to determine under what conditions and for what ranges of circuit parameters coherent emission should be possible from such arrays.[9–23] These arrays are also of interest as paradigms of nonlinear dynamical systems with many degrees of freedom.[24–27]

Of particular interest have been the so-called phase-locked states of such arrays. Generally, two junctions can be considered phase locked, or synchronized, if the difference between the Josephson phase drops across the two junctions is time independent. This condition results in identical voltages across the two junctions.[10] Much theoretical work has focused on understanding the conditions and junction parameters for which all (or many) of the junctions in an array are indeed stably phase locked.

In this paper, we report the results of a local stability analysis, based on Floquet theory, for 2D square arrays of up to 5×5 plaquettes of *underdamped* junctions in an external magnetic field. We demonstrate stable phase locking in the absence of a load for a finite range of bias currents and junction capacitances. We also show that for such bias currents and capacitances, the synchronized state is robust against critical current disorder of at least $\pm 25\%$.

Because some experiments have demonstrated power outputs of the order of microwatts or more,[2,7,8] much theoretical effort has already been expended to understand the dynamics of linear and 2D arrays. In linear arrays of N junctions coupled to an impedance-matched load, it is theoretically expected that the power delivered to the load will scale as N times the power available from a single junction. Similarly, for a 2D array with M columns of N junctions each, the power delivered to a matched load should scale as MN times that from a single junction.[9] These estimates assume, however, the existence of stable phase locking in the array—an assumption which is not necessarily valid in either geometry.

Early theoretical work focused on serial arrays of N identical resistively and capacitively shunted junctions (RCSJ's) globally coupled to a load. The so-called in-phase oscillations, in which the voltages across all junctions are identical, were generally found to be most stable for values of the McCumber parameter β_c in the range of $0-1$, and for bias currents slightly above the critical current of the individual junctions. (The McCumber parameter is defined below, $\beta_c \gg 1$ corresponds to highly underdamped junctions, $\beta_c \ll 1$, to overdamped junctions.) Remarkably, this result was found to hold true whether the loads were resistive, capacitive, or inductive loads.[16,17] More recently, several authors have considered so-called splay-phase oscillations in serial arrays. In such oscillations, the voltages across neighboring junctions all have the same wave form but are shifted in time by an amount T/N, where T is the period. In this case, the arrays were found to have a high degree of *neutral stability*, i.e., small perturbations to the solutions of the underlying Josephson equations neither grow nor decay with time. Furthermore, the number of degrees of freedom having neutral stability was found to depend on the load impedance as well as the junction capacitance.[18,24]

2D arrays of square plaquettes have been found to have additional impediments to stable phase locking. Wiesenfeld, Benz, and Booi[9] showed that dc-biased *overdamped* arrays would suffer from a high degree of neutral stability. They also argued that *underdamped* arrays would have the same problem. The physics of this result follows from the geometry of the array. (See Fig. 1.) If one assumes identical junctions and a bias current I_B greater than the critical current, only the horizontal junctions will be in the voltage state. Except for initial transients, the vertical junctions will be inactive and in essence behave similar to superconducting shorts. This guarantees that the active junctions in a given column will have identical, in-phase voltages. The absence of currents in the inactive junctions, however, means that the voltages across active junctions in two *different* columns need not be identical but can have a phase shift that will depend on the initial values of the voltages and Josephson phases across the junctions. Intriguingly, however, analytic

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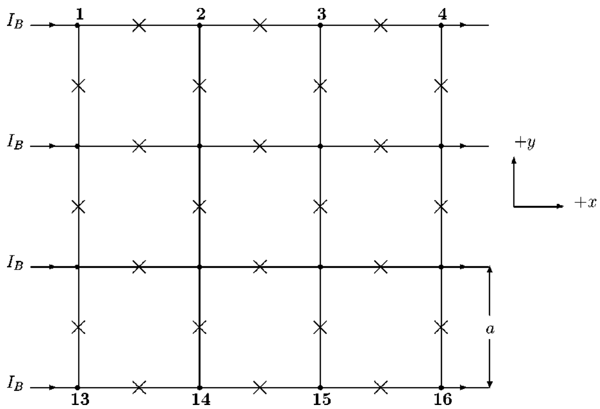

FIG. 1. Schematic of a square array with 3×3 plaquettes. An $N\times N$ array has $(N+1)\times(N+1)$ nodes, some of which are numbered in the figure. The dc bias current is applied and removed horizontally (x direction) and a constant magnetic field is applied in the $+z$ direction. Each junction has a resistance R, a capacitance C, and a critical current $I_{c,jk}$, where j and k index the nodes on either side of the junction.

work has shown that the presence of an external magnetic field can result in intercolumn coupling of the active junctions.[13]

Many workers have looked for ways to stabilize phase locking of a 2D array in the absence of a load. Whan, Cawthorne, and Lobb[10] studied *underdamped* junctions in zero field both with and without critical current disorder and for different biasing schemes. They used a simple criterion to determine which junctions in the array would be classified as phase locked. Namely, they required the absolute difference between the Josephson phases across two junctions to remain less than a chosen value for a sufficiently long time. They then defined a synchronization order parameter $r=\lim_{N_{\rm tot}\to\infty}(N_s/N_{\rm tot})$, where $N_{\rm tot}$ is the total number of active junctions in the array and N_s is the number of junctions in the largest phase-locked cluster. They concluded that, in order to maintain a phase-locked state in the presence of critical current disorder, the array must be biased both horizontally (as in Fig. 1) *and* vertically, i.e., all junctions in the array must be made to enter the voltage state.[28]

Similar thinking led Filatrella, Pedersen, and Wiesenfeld[14] to study a 2×2 array with identical junctions using what they described as a cross-type biasing. In contrast to the current injection scheme of Fig. 1, their scheme would be equivalent to having a bias current exiting the array in the $+y$ direction at nodes 1, 2, 3, and 4, as well as a bias current exiting node 4 in the $+x$ direction. No bias currents would exit from nodes 8, 12, or 16. The motivation for this biasing is to have both vertical and horizontal junctions active, thereby (it is hoped) to produce stable phase locking. Filatrella *et al.* also included loop inductance in their model. Their biasing scheme was found to remove the neutral stability for the 2×2 array but not for the larger 3×3 array.

Larsen and Benz[15] studied a two-cell ladder array of *underdamped* junctions, biasing the array along the legs of the ladder. An inductor was placed along the horizontal rung separating the two cells, and an additional, transverse bias current was injected into one end of the rung and extracted from the other (see the figure in Ref. 15). The transverse current was found to stabilize phase locking in the array for certain specific ranges of the inductance and the junction capacitance.

All these studies have been seeking a mechanism to couple together different rows of junctions oriented parallel to the direction of the bias current. It is also known that the desired coupling can be produced by adding an external load to the array. Kautz[11] studied an array, biased vertically, consisting of four columns, each with two vertical junctions. The horizontal junctions, which would not be in the voltage state in this geometry, were replaced by inductors, and a load resistor was connected parallel to the direction of the bias current. For *identical*, underdamped junctions, there existed a range of bias currents for which phase locking was stable. In the presence of critical current disorder, two striking results were found in this geometry. First, it was found that 2D arrays could experience a larger spread of critical currents than linear arrays and still remain phase locked. Secondly, when an external load was applied, it was found that the intercolumn coupling of junctions needed to lift the neutral stability came primarily from the load itself, rather than from any internal coupling mechanism within the array. However, as noted by Whan *et al.*,[10] it is not clear how the dynamics of the system are changed when the so-called inactive junctions are replaced by inductors.

In this work, we report on a formal stability analysis of square arrays in the presence of an external magnetic field and critical current disorder. We find that phase locking in such arrays is inherently *stable* for a range of bias currents and McCumber parameters. The parameter range is similar to that for which serial arrays are stable in the presence of an impedance load. We conjecture that the currents induced by the external field provide the mechanism for the stabilizing intercolumn couplings. (See Fig. 1.) The remainder of the paper is organized as follows. Section II describes our calculational method, including a brief description of Floquet theory, which provides the local stability analysis of the phase-locked solutions. Section III discusses our results, and Sec. IV presents our conclusions and some suggestions for future work.

II. FORMALISM AND CALCULATIONAL METHOD

The coupled Josephson equations describing the array of Fig. 1 are standard. Conservation of charge at the jth node yields

$$I_{B,j}+\sum_{\langle k\rangle}\left[I_{c,jk}\sin(\phi_j-\phi_k-A_{jk})+\frac{\hbar}{2eR}\frac{d}{dt}(\phi_j-\phi_k)+\frac{\hbar C}{2e}\frac{d^2}{dt^2}(\phi_j-\phi_k)\right]=0. \tag{1}$$

Here ϕ_j is the superconducting phase at node j, and $I_{B,j}$ is the bias current entering or leaving node j. The argument of the sine function is the gauge-invariant phase difference across the junction between nodes j and k. The sum runs over all nearest neighbor nodes to j. We allow for critical current disorder but assume uniform junction resistance R and junction capacitance C. As usual, A_{jk} is the line integral of the vector potential between nodes j and k,

$$A_{jk}=\frac{2\pi}{\Phi_0}\int_j^k \mathbf{A}\cdot d\mathbf{l}, \quad (2)$$

where $\Phi_0=hc/2e$ is the flux quantum. We assume a uniform magnetic field in the $+z$ direction, $\mathbf{B}=B\hat{z}$, and use the gauge $\mathbf{A}=Bx\hat{y}$. We can also define a *frustration parameter* for the array, given by $f=(Ba^2)/\Phi_0$, where a^2 is the area of a single plaquette. We have assumed the transverse penetration depth, $\lambda_\perp$, is much larger than the array size so that magnetic field induced by the screening currents may be neglected.[29]

It is useful to write Eq. (1) in dimensionless form. Let the characteristic time for the junctions be given by $t_c \equiv \hbar/(2e\langle I_c\rangle R)$, where $\langle I_c\rangle$ is the arithmetic average of the critical currents of the junctions in the array. We then define a dimensionless time, via $\tau\equiv t/t_c$. The dimensionless bias current at node j is $i_j\equiv I_{B,j}/\langle I_c\rangle$, and the dimensionless critical current for the junction connecting nodes j and k is $i_{c,jk}\equiv I_{c,jk}/\langle I_c\rangle$. Finally, the McCumber parameter, which is just the dimensionless capacitance of the junctions, is given by $\beta_c\equiv 2eC\langle I_c\rangle R^2/\hbar$. In terms of these new variables, Eq. (1) now takes the form

$$i_j+\sum_{\langle k\rangle}\left[i_{c,jk}\sin(\phi_j-\phi_k-A_{jk})+\frac{d}{d\tau}(\phi_j-\phi_k)+\beta_c\frac{d^2}{d\tau^2}(\phi_j-\phi_k)\right]=0. \quad (3)$$

We have solved these equations numerically using the fourth-order Runge-Kutta method, using time steps of size $\Delta\tau=0.001$. Typically, the code was run for a total time of $\tau_{\text{total}}=300$ before performing the stability analysis, which we now describe.

Suppose that $\phi_{0j}(\tau)$ is a solution to Eq. (3). We now perturb the phase at node j by a small amount, $\eta_j(\tau)$, so that the new phase is $\phi_j(\tau)=\phi_{0j}(\tau)+\eta_j(\tau)$. Linearizing Eq. (3) with respect to η_j, we arrive at the following:

$$\sum_{\langle k\rangle}\left[i_{c,jk}\cos(\phi_{0j}-\phi_{0k}-A_{jk})(\eta_j-\eta_k)+\frac{d}{d\tau}(\eta_j-\eta_k)+\beta_c\frac{d^2}{d\tau^2}(\eta_j-\eta_k)\right]=0. \quad (4)$$

Because the coefficients of the η_j are periodic, with period T/t_c in dimensionless units, we can apply Floquet's Theorem,[30] which tells us that there exist solutions to Eq. (4) of the form

$$\eta_j\left(\tau+\frac{T}{t_c}\right)=\mu\,\eta_j(\tau), \quad (5)$$

where μ is a (possibly complex) number called the Floquet multiplier. If the perturbations decay with time, the original solution $\phi_{0j}(\tau)$ is said to be linearly stable. From Eq. (5), we see that if $|\mu|>1$, the perturbation grows with time. If $|\mu|<1$, the perturbations diminish, and $|\mu|=1$ is the special case corresponding to neutral stability (perturbations neither grow nor decay).

An array with $N\times N$ plaquettes has $(N+1)^2$ nodes. We follow the usual convention and fix the phase on one of the nodes, leaving $(N+1)^2-1=N(N+2)$ coupled Josephson equations. Because we are solving a set of coupled second-order differential equations we actually have $2N(N+2)$ variables and the same number of resulting Floquet multipliers. We can think physically of the multipliers as describing the stability of the characteristic modes of the array. *At least* one of these multipliers must equal one; in the language of phase space, this corresponds to a perturbation tangent to the periodic orbit. As discussed in Ref. 18 we find the Floquet multipliers as follows. We perturb in one direction in phase space—that is, we start with $\eta_j=1$ and $\eta_k=0$ $(k\neq j)$. We use the Runge-Kutta algorithm to find the values of *all* the η_j one period later. These $2N(N+2)$ quantities then constitute one column of a $2N(N+2)\times 2N(N+2)$ matrix, the other columns coming from perturbing the solution for different values of j. The eigenvalues of the resulting matrix will be the Floquet multipliers of the system.[30] Excluding the multiplier of unity, we are most interested in the remaining largest multiplier, as that tells us by what factor the longest lived mode of the array decays (or grows) in one period.

We have calculated the multipliers for arrays of 3×3 and 5×5 plaquettes for bias currents and McCumber parameters in the approximate range $1.1\lesssim I_B/I_c\lesssim 5$ and $0.1\lesssim\beta_c\lesssim 5$, respectively, in the presence of a magnetic field corresponding to $f=1/3$, where f is the frustration parameter defined above. We now discuss the results of these calculations.

III. RESULTS

First, we offer numerical evidence that the underdamped array is indeed neutrally stable in the absence of a magnetic field.[9] Figure 2(a) shows the instantaneous voltages across each of the junctions in the top row of a 3×3 array as a function of time. In this case, the voltage at each of the nodes was initialized to zero, and we see that the voltages across the junctions are identical, as would seem to suggest stable synchronization. This behavior is misleading, however, since if we initially *randomize* the voltages at the nodes we see [in Fig. 2(b)] that the three junctions cannot lock in phase. In fact, the phase shift between $V_{1,2}$ and $V_{3,4}$ (for example) depends on the values of the initial voltages. Physically, this behavior is equivalent to the fact that if the voltages were initially in phase but then were perturbed, they would settle into some new configuration with phase shifts that are sensitive to the size of the perturbation.

We now consider some evidence that an applied magnetic field stabilizes the underdamped array. Figure 3(a) shows the calculated time-dependent difference between the phases across the first and third junctions in the top row of a 3×3 array, for $f=0$ and $\beta_c=1.0$. That is, we plot $\phi_{1,2}-\phi_{3,4}$ as a function of (dimensionless) time. At $\tau=60$, we simply perturb all the phases at all the nodes of the array by the same amount. We then watch to see if this phase difference returns to the same value, on average, it had before the perturbation. In zero field, before the perturbation, the junctions had synchronized in the sense that there was a roughly time-independent phase difference (with weak time-dependent oscillations superimposed on that phase difference). After the perturbation, the two junctions still synchronize, but with a

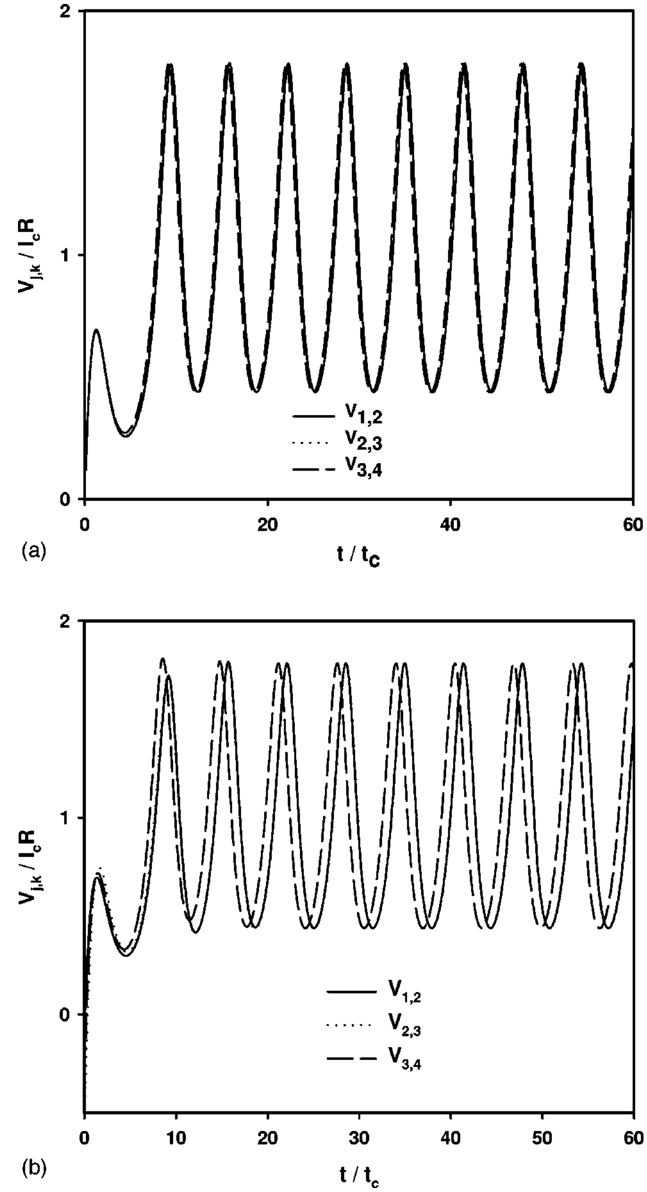

FIG. 2. Dimensionless voltage across each of the three junctions in the top row of a 3×3 array, plotted versus dimensionless time. (a) $I_B/I_c=1.2$, $\beta_c=1.0$, $f=0$. All voltages were initialized to zero. (b) $I_B/I_c=1.2$, $\beta_c=1.0$, $f=0$. All voltages were initialized randomly between zero and 1.

different value for the phase difference. This behavior is characteristic of neutral stability. Figure 3(b) shows that an applied magnetic field changes the behavior of the phase differences. We see that, for $f=1/3$, after the perturbation the average phase difference is clearly returning to its value before the perturbation, which appears to be approximately zero in this case. This simple plot indicates that the junctions are stably synchronizing. (We see similar behavior when comparing other pairs of junctions.)

Figure 3(c) shows the same kind of plot as Figs. 3(a) and 3(b), with the same scale on the vertical axis, but for $\beta_c=10$. In this case, for both $f=0$ and $f=1/3$, after the perturbation at $\tau=60$, there are rather long-lived transients that persists until $\tau\approx150$, after which the long-time behavior of the phase differences becomes clear. Indeed, for $\tau>150$, it is evident that the $f=0$ and $f=1/3$ behaviors are indeed different: the $f=1/3$ solution is weakly stable and the $f=0$ solution neutrally stable.[31]

Table I gives the three largest values of the Floquet multipliers for all the cases shown in Figs. 3(a)–3(c). It is important to remember that Figs. 3(a)–3(c) represent the solution to the fully nonlinear problem, as represented by Eq. (3), while the Floquet multipliers are obtained by linearizing Eq. (3) about the full solution. Nevertheless, as a simple check, the Floquet multiplier can be roughly estimated from the $f=1/3$ results in Fig. 3(c) and then compared with the value quoted in Table I. Using a standard plotting routine, an exponential curve can be fitted to the $f=1/3$ plot for $\tau>150$. The result is a fit of the form

$$\phi_{1,2}-\phi_{3,4}\sim e^{-0.002\tau}. \tag{6}$$

This form can be compared to the relationship between the Floquet multiplier μ and the corresponding Floquet exponent λ

$$\mu=e^{(\lambda t_c)(T/t_c)}, \tag{7}$$

where T/t_c is the period of the coefficients of the η_j in Eq. (4). From our numerical calculations, we find $T/t_c=3.170$. Thus we have, from Eqs. (6) and (7), and the definition of the Floquet multiplier [Eq. (5)]

$$\mu=e^{(-0.002)(3.170)}=0.994,$$

which agrees well with the result quoted in Table I.

It is also of interest to ask what happens in the overdamped limit, *i.e.*, as $\beta_c\to0$, in the presence of a magnetic field of $f=1/3$. Perhaps surprisingly, we find that a magnetic field is *not* sufficient to lift the neutral stability in this case. Evidently, the dynamics of an *underdamped* array in an external field, in which a perturbation created at one point in an array can travel a significant distance before diminishing, is conducive to stable synchronization. Similar behavior for small β_c was observed by Bhagavatula, Ebner, and Jayaprakash,[32] who studied 2D arrays of RCSJ's subjected to an ac electromagnetic field and horizontal dc bias currents. They found, numerically, that in the limit of small McCumber parameter, the horizontal rows of junctions decoupled, with no currents in the vertical junctions. This is just the recipe for neutral stability, as discussed in Sec. I: because of the inactive vertical junctions, two active junctions in the *same* column will have identical voltages, but two active junctions in *different* columns will have voltage waveforms separated by an arbitrary phase shift.

Our results for $\beta_c=0$, shown in Fig. 4, support this conclusion. The figure shows the same phase difference plotted in Fig. 3 (with a perturbation at $\tau=60$), but now graphed for $\beta_c=0$, at both $f=0$ and $f=1/3$. In both cases the phase difference eventually settles down to a time average that neither grows nor decays, indicative of neutral stability. For $f=1/3$ a long run time is required before the evidence of neutral stability is discernable, but this array is certainly only neutrally stable. The conclusion from these results is that *underdamped* arrays yield stable synchronization in a nonzero external field for a finite range of capacitances and bias currents. We now demonstrate this conclusion more comprehensively.

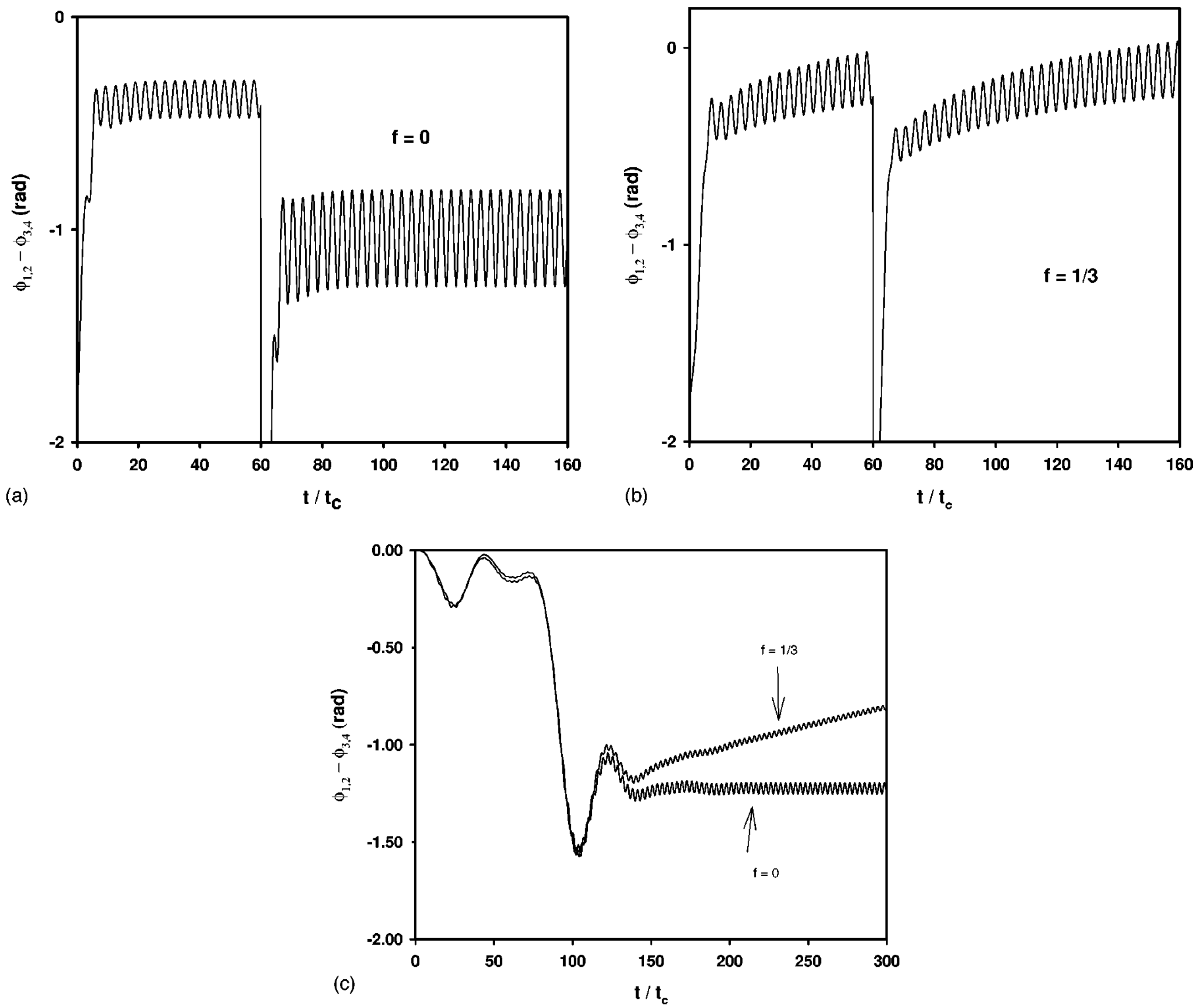

FIG. 3. Difference between the Josephson phase drops across two junctions in the top row of a 3×3 array, plotted versus dimensionless time. An identical perturbation is applied to the phase at each superconducting node at $t/t_c=60$. (a) $I_B/I_c=2.0$, $\beta_c=1$, $f=0$. The shift in the phase difference indicates neutral stability. (b) $I_B/I_c=2.0$, $\beta_c=1$, $f=1/3$. The return of the phase difference to its value before the perturbation indicates stable synchronization. (c) $I_B/I_c=2.0$, $\beta_c=10$. Both $f=1/3$ and $f=0$ are shown. The synchronization for $f=1/3$ is weakly stable; for $f=0$, it is neutrally stable.

The main results of this paper are contained in Figs. 5 and 6. These are contour plots of the largest Floquet multipliers as a function of the bias current and the McCumber parameter, for 3×3 and 5×5 arrays and $f=1/3$. To produce these plots, μ was calculated for hundreds of combinations of I_B and β_c. The contours were then created by interpolating between the calculated data. Looking at Fig. 5, we can think of the $\mu=0.99$ contour as roughly the boundary between stable ($\mu<0.99$) and unstable ($\mu>0.99$) synchronization. Clearly, there is a substantial region of the β_c-I_B parameter space for which synchronized solutions to the Josephson equations are stable. The most stable phase locking occurs for I_B near I_c and $\beta_c\approx1.25$. We have calculated the multipliers for bias currents as low as $I_B/I_c=1.1$. With smaller currents, the code that calculates the values of μ runs for a very long time, since the period of the junctions grows rapidly as $I_B\rightarrow I_c^+$.

Figure 6 shows the same contour plot for a 5×5 array. A quick comparison with Fig. 5 shows that the topography of the contour lines for the two plots is very similar, with the most stable region of the 5×5 array occurring for I_B near I_c and $\beta_c\approx2$. The stable region for the 5×5 array appears to encompass a slightly smaller portion of the plane than that for the 3×3 array. This dependence of the multipliers on array size is similar to that observed for ladder arrays,[33,34] in which the multipliers were found to approach unity from below as the array size was increased. More runs on larger square arrays would be needed, however, to determine the exact nature of the dependence of the largest μ on array size.

Note that we have used the same frustration parameter, $f=1/3$, for both square arrays studied here. For the 3×3 array, this means the array lattice contains an integer number of (3×3) unit cells of the vortex lattice, but for the 5×5 array does not contain an integer number of such cells. Since stable synchronization occurs in both cases, however, we conjecture that such a match is not required to produce it. As a test of this conjecture, we have calculated the multipliers for $f=1/2$ at several values of I_B and β_c for a 3×3 array.

TABLE I. Three largest Floquet multipliers for an ordered 3×3 array, as found in four different runs with differing values of β_c and f, always assuming $I_B/I_c=2$. Note that there are a total of 30 multipliers μ for a given set of values for I_B/I_c, β_c, and f. One of these always equals unity, corresponding to a perturbation in phase space which is tangential to the periodic orbit.

I_B/I_c	β_c	f	μ
2.0	1.0	0	1.001591 1.001590[a] 1.001556
2.0	1.0	$\frac{1}{3}$	0.9999847 0.938858[b] 0.868454
2.0	10.0	0	1.000426 1.000369[c] 1.000368
2.0	10.0	$\frac{1}{3}$	1.000084 0.992542[d] 0.984407

[a]Neutrally stable.
[b]Stable.
[c]Neutrally stable.
[d]Weakly stable.

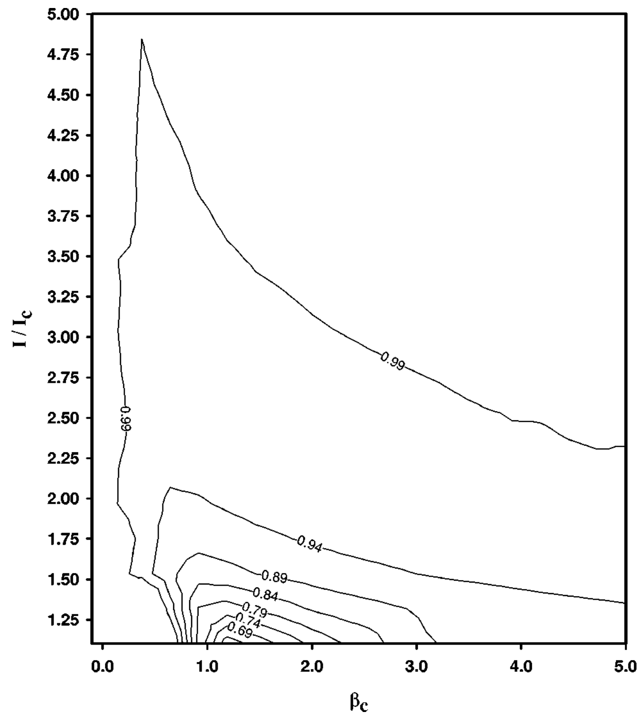

FIG. 5. Contour plot of largest Floquet multiplier as a function of bias current and junction capacitance for a 3×3 array, at $f=1/3$ with no disorder. The $\mu=0.99$ contour may be thought of as representing the boundary between stable ($\mu<0.99$) and unstable ($\mu>0.99$) synchronization.

We see behavior very similar to that of Fig. 5. It would be interesting, however, to see what happens to the phase locking at a small f, or an irrational value of f, for which the unit cell would be much larger than the array size.

We have qualitatively compared our results to calculations for *serial* arrays with a matched resistive load.[16,17] In such a geometry, the chain of junctions can be phase locked

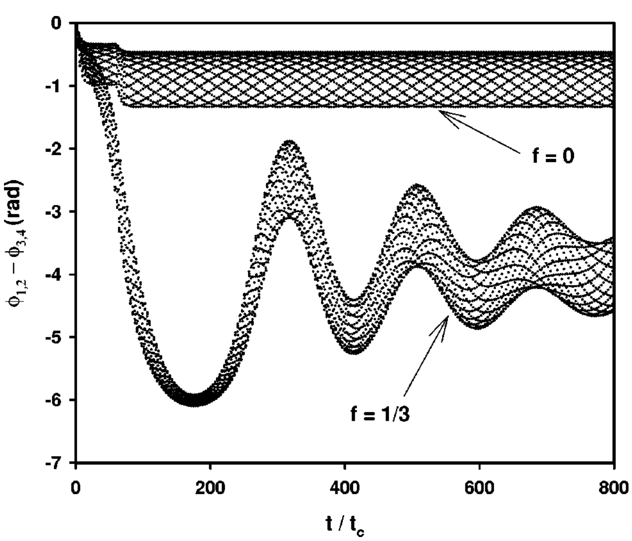

FIG. 4. Difference between the Josephson phase drops across two junctions on the top row of a 3×3 array as a function of dimensionless time. $I_B/I_c=2.0$, $\beta_c=0$. Both $f=1/3$ and $f=0$ are shown. An identical perturbation is applied to the phase at each superconducting node at $t/t_c=60$. This *overdamped* array is neutrally stable for both values of f.

(in the absence of an external magnetic field) for bias currents about twice the critical current and $\beta_c\approx1$. Because of the similarity to our results, we conjecture that, in the 2D array, the vertical junctions (i.e., those transverse to the bias current) act as loads on the horizontal junctions. It is crucial for stability, however, to have current flowing through those vertical junctions, which we accomplish via the external magnetic field. Not surprisingly, Kautz[11] has shown that one can stabilize the 2D array in the *absence* of an external field by adding a resistive load directly to the array. He concluded that the transverse junctions (inductors in his case), had little effect in stabilizing phase locking.

Finally, we have studied the sensitivity of the multipliers to critical current disorder in the 3×3 array, as shown in Fig. 7. To produce this plot, we chose values of I_B/I_c and β_c which would exhibit stable synchronization ($I_B/I_c=1.25$, $\beta_c=1.0$) without disorder, and then introduced a spread in the I_c's. The actual I_c's were chosen randomly from a Gaussian distribution of mean $\langle I_c\rangle$ and standard deviation σ.[35] Each point in Fig. 7 corresponding to a nonzero σ results from averaging over 10–15 different realizations of the I_c's. Clearly, even for critical current spreads of 25%, which is achievable with modern fabrication techniques,[12] the synchronization remains stable, at least for the sizes considered. To obtain these points, for each disorder realization, we calculated the oscillation period of each of the horizontal (i.e., active) junctions separately, then arithmetically averaged these periods to get a single number. We then used the *average* period of the junctions to calculate the Floquet multipliers. If we still obtained at least one multiplier equal to unity,[36] we would accept the array as synchronized and in-

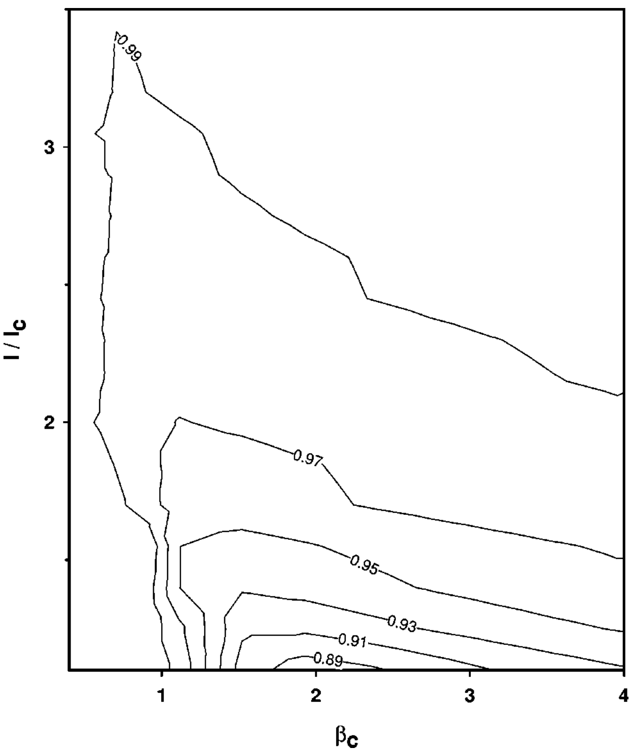

FIG. 6. Contour plot of largest Floquet multiplier as a function of bias current and junction capacitance for a 5×5 array, at $f=1/3$ with no disorder. The $\mu=0.99$ contour may be thought of as the boundary between stable ($\mu<0.99$) and unstable ($\mu>0.99$) synchronization. Note the similarity between the contour lines in this figure and those of Fig. 5.

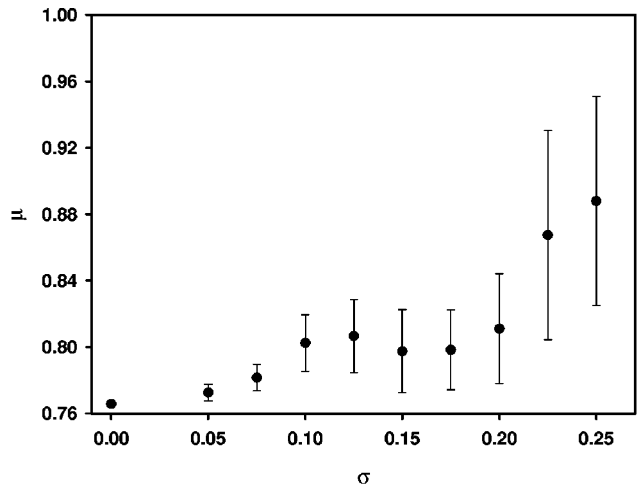

FIG. 7. Largest Floquet multiplier as a function of standard deviation of critical currents in a disordered 3×3 array, with $I_B/I_c=1.25$, $\beta_c=1.0$, $f=1/3$. These parameters resulted in stable phase locking in the uniform array [cf. Fig. 5].

cluded the resulting Floquet multiplier in the data of Fig. 7; otherwise, that realization of critical currents was rejected as leading to an unsynchronized array and not included in Fig. 7. We have also checked that, for each synchronized array, the periods of the active junctions are all equal to within a percent or so, i.e., these arrays really are oscillating synchronously.

We do find, as expected, that a sufficiently large critical current disorder can limit the ability of the junctions to lock frequencies. For example, we have examined the behavior of a few 3×3 arrays with $\sigma=0.5$ and $f=1/3$. In this case, we sometimes find that the individual active junctions oscillate at one of *two* different frequencies, differing by about 10%. In this case, a Floquet analysis seems premature. Clearly, additional work is desirable in order to fully understand the effects of disorder on array synchronization at larger values of disorder.[37]

IV. CONCLUSIONS

We have calculated the Floquet multipliers for a 3×3 and 5×5 array of *underdamped* Josephson junctions in the presence of an external magnetic field corresponding to $f=1/3$. We find a region of stable, synchronized solutions as a function of bias current and junction capacitance, even in the absence of a load, with the greatest stability occurring for I_B near I_c and $\beta_c\sim1-2$, where β_c is the McCumber parameter. No such stable solutions apparently exist for the *overdamped* 2D array (near $\beta_c=0$), nor do they exist in zero applied field in the absence of a load. Furthermore, these synchronized solutions are robust against moderate critical current disorder. The external magnetic field plays a crucial role in causing current to flow through the junctions transverse to the direction of the bias current, i.e., through the vertical junctions in our geometry. This current couples the horizontal junctions and lifts the neutral stability.

In the future, it would be useful to study stability as the frustration parameter is decreased, to see if there is a smooth return to neutral stability as $f\to0$. It would also be of interest to study the effects of including loop inductance. Such inductances would allow the current flowing around a given plaquette to induce a magnetic flux through the same (and neighboring) plaquettes, thereby possibly enhancing the stability of synchronous solutions. Finally, a more complete study of the effects of critical current disorder would be of value, especially in the regime of moderately large I_c disorder where not all the junctions may be locked to the same period.

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