The metric structure of compact rank-one ECS manifolds

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Abstract. Pseudo-Riemannian manifolds with nonzero parallel Weyl tensor which are not locally symmetric are known as ECS manifolds. Every ECS manifold carries a distinguished null parallel distribution $\mathcal{D}$, the rank $d \in \{1,2\}$ of which is referred to as the rank of the manifold itself. Under a natural genericity assumption on the Weyl tensor, we fully describe the universal coverings of compact rank-one ECS manifolds. We then show that any generic compact rank-one ECS manifold must be *translational*, in the sense that the holonomy group of the natural flat connection induced on $\mathcal{D}$ is either trivial or isomorphic to $\mathbb{Z}_2$. We also prove that all four-dimensional rank-one ECS manifolds are noncompact, this time without assuming genericity, as it is always the case in dimension four.

Introduction and main results

A pseudo-Riemannian manifold of dimension $n \geq 4$ whose Weyl tensor is parallel is referred to as *conformally symmetric* [**6**], and it is called *essentially conformally symmetric* (briefly, *ECS*) [**27**] if, in addition, it is neither conformally flat nor locally symmetric. It was shown by Roter that ECS manifolds exist in every dimension [**27**, Corollary 3] and that they all have indefinite metric signatures [**8**, Theorem 2]. The local structure of ECS manifolds is fully known [**10**].

Conformal symmetry of (M, g) is one of the *natural linear conditions* imposed on the covariant derivatives of the $\mathrm{SO}(p,q)$-irreducible components of its curvature tensor, in the sense of Besse [**2**, Chapter 16]. The interest in this subject is reflected in more work by other authors: Cahen and Kerbrat [**3**, Section 2], Hotloś [**21**], Mantica and Suh [**25**, Section 3], Schliebner [**28**], and Deszcz et al. in [**17**, Sect. 4], [**16**, Theorem 6.1]. The techniques used in the study of ECS manifolds are themselves also of interest, appearing in [**18**, Example 2.2], [**29**], [**1**, Theorem 3], [**4**], [**5**, Theorem 3.9], [**22**, Lemma 3], [**24**], [**31**, proofs of Theorems 1.1 and 4.5] and [**30**].

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As shown by Olszak [**26**], every ECS manifold (M, g) carries a distinguished null parallel distribution $\mathcal{D}$, whose sections are the vector fields corresponding under g to 1-forms ξ with $\xi \wedge [\mathrm{W}(v', v'', \cdot, \cdot)] = 0$ for all vector fields v', v''. The rank of $\mathcal{D}$ – always equal to 1 or 2 – is referred to as the *rank of* (M, g) [**15**]. In the rank-one case, the focus of this paper, we also call the ECS manifold in question

(∗) *translational* or *dilational*, depending on whether the holonomy group of the natural flat connection induced on $\mathcal{D}$ is finite or infinite.

In [**13**] and [**11**], compact rank-one ECS manifolds of every dimension $n \geq 5$ were constructed as suitable quotients of what we call *model ECS manifolds* (see Section 6). All such compact examples are geodesically complete and translational, but none of them is locally homogeneous. On the other hand, we show in [**14**] that, under a natural *genericity* assumption on the Weyl tensor, quotients of dilational model ECS manifolds cannot be compact unless they are locally homogeneous. More precisely, genericity refers to a certain self-adjoint endomorphism A of the vector space of parallel sections of $\mathcal{D}^\perp/\mathcal{D}$, and it means that only finitely many of its isometries commute with A (see also Section 4 and the end of Section 5). We have recently found [**12**] compact dilational examples in all odd dimensions $n \geq 5$, including locally homogeneous ones. They are all nongeneric and incomplete.

It is still not known whether a compact ECS manifold can be four-dimensional, or have rank two.

This paper provides partial answers to the above questions. We start with structure theorems, calling a rank-one ECS manifold $\mathcal{D}^\perp$-*complete* if all the leaves of $\mathcal{D}^\perp$ are complete relative to the induced connections (this definition makes sense for any foliation with totally geodesic leaves, such as $\mathcal{D}^\perp$ itself), and *maximally complete* if every non-complete maximal geodesic in its universal covering intersects all leaves of $\mathcal{D}^\perp$. Note that maximal completeness implies $\mathcal{D}^\perp$-completeness, while (due to (5.2) below) it follows from completeness.

Theorem A. *Any compact* $\mathcal{D}^\perp$*-complete rank-one ECS manifold is necessarily maximally complete.*

Theorem B. *Every generic compact rank-one ECS manifold is both maximally complete and* $\mathcal{D}^\perp$*-complete.*

Theorem C. *Any simply connected and maximally complete rank-one ECS manifold is isometric to a model ECS manifold.*

The next result is a trivial consequence of Theorems B and C.

Corollary D. *The universal covering of any generic compact rank-one ECS manifold is isometric to a model ECS manifold.*

For the Lorentzian signature, Schliebner [**28**] proved this last conclusion without assuming genericity. From Corollary D, we obtain the following strengthened version of [**14**, Theorem C], which refers to the dichotomy ($*$):

Theorem E. *Every generic compact rank-one ECS manifold is translational, as well as geodesically complete, and it cannot be locally homogeneous.*

Let us point out that Theorem E does not replace [**14**, Theorem C], but rather relies on it, since the latter is needed to prove the former. As we point out in Section 11, Corollary D combined with [**11**, Theorem 8.1] trivially leads to:

Corollary F. *Four-dimensional rank-one ECS manifolds are noncompact.*

In other words, if four-dimensional compact ECS manifolds do exist, they must necessarily be of rank two.

How the paper is organized

Unless stated otherwise, all manifolds, bundles, connections, mappings, and tensor fields are assumed to be smooth. The text is divided into two parts.

Part I. After Sections 1–3 dealing with preliminaries, in Section 4 we elaborate on the meaning of genericity. Sections 5 and 6 lay the groundwork for proving Theorems A and C, summarizing what is already known about the structure of the universal coverings of compact rank-one ECS manifolds, and describing our model ECS manifolds. In Section 7 we prove Theorems A–C, adapting to our situation the proofs of some weaker results from [**9**] (namely, Lemma 7.3 and Theorem 7.1 therein). Further details are provided in Appendices A and B.

Part II. Section 8 introduces the *transitive-commutation property* for a group-subgroup pair, crucial for understanding the structure of the isometry group of a locally homogeneous model ECS manifold. The focus on the locally homogeneous case is justified by [**14**, Theorem C], where we prove that a generic dilational compact rank-one ECS manifold must be locally homogeneous. Section 9 presents what we call *standard homogeneous rank-one ECS model manifolds*, used in Section 10 to prove Theorem E.

1. Completeness of connections

Lemma 1.1. *If ∇ is a connection on a manifold L, while $\boldsymbol{X}$ and $\boldsymbol{Z}$ are vector fields along a curve in L defined on an open interval $J \subseteq \mathbb{R}$ of the variable s containing* 0*, and*

$$\nabla_s \boldsymbol{Z} = \nabla_s \nabla_s \boldsymbol{X} = 0, \quad \boldsymbol{X}(0) = \boldsymbol{Z}(0), \quad [\nabla_s \boldsymbol{X}](0) = -\boldsymbol{Z}(0), \tag{1.1}$$

then $\boldsymbol{X}(s) = (1-s)\boldsymbol{Z}(s)$ for all $s \in J$.

In fact, $s \mapsto \boldsymbol{X}(s) = (1-s)\boldsymbol{Z}(s)$ satisfies (1.1), and we use the uniqueness of solutions for systems of second-order ordinary differential equations.

The next Lemma generalizes [**9**, Lemma 1.4] and is used in Section 7 to prove Theorem B. In its proof (and later in Appendix A) we adopt the notational convention of [**9**, end of Sect. 1]: given a *variation of curves* in a manifold M, that is, a C^∞ mapping $(t,s) \mapsto x(s,t)$ from an open set in $\mathbb{R}^2$ into M, and a connection on M, we denote by x_s, x_t (or, x_{ss}, x_{st}, x_{tss}, etc.) its partial (or, partial covariant) derivatives of orders $1, 2, 3$ etc., all of which are vector fields along the variation, meaning, as usual, sections of the corresponding pullback of TM. When the connections involved are flat and torsion-free,

(1.2) all such derivatives depend symmetrically on the subscripts.

Lemma 1.2. *Let $\mathcal{P}$ be a distribution on a manifold L, ∇ be a connection on L, and assume that:*

(a) *∇ is flat and torsionfree,*
(b) *$\mathcal{P}$ is trivialized by a vector space $\mathcal{X}$ of complete parallel vector fields,*
(c) *there is a vector space $\mathcal{Y}$ of complete vector fields on L which is isomorphically mapped via the quotient projection onto a vector space of sections trivializing the quotient bundle $TL/\mathcal{P}$ over L, and parallel relative to the connection induced on $TL/\mathcal{P}$.*

Then ∇ is complete.

Proof. The evaluation $\mathcal{X} \to \mathcal{P}_z$ at each $z \in L$ is an isomorphism, and so every ∇-geodesic of L starting tangent to $\mathcal{P}$ is complete, such a geodesic being an integral curve of a complete parallel vector field on L.

Given $z \in L$ and $\mathbf{v} \in \mathcal{Y}$, let $y \colon \mathbb{R} \to L$ be the integral curve of $\mathbf{v}$ with $y(0) = z$. As $\nabla_{\mathbf{v}}\mathbf{v}$ is always tangent to $\mathcal{P}$, we may choose $\zeta, \eta \colon \mathbb{R} \to \mathcal{X}$ with

$$[\zeta(t)]_{y(t)} = [\nabla_{\mathbf{v}}\mathbf{v}]_{y(t)} = \ddot{y}(t), \quad \ddot{\eta} = -\zeta. \tag{1.3}$$

Each $\eta(t)$ is complete, and the variation $\mathbb{R}^2 \ni (t,s) \mapsto x(t,s) = e^{s\eta(t)}y(t)$ has

$$x(t,0) = y(t), \quad x_s(t,0) = [\eta(t)]_{y(t)}, \quad x_{ss}(t,s) = 0. \tag{1.4}$$

Hence $x_{tt}(t,s) = (1-s)[\zeta(t)]_{x(t,s)}$ for all $(t,s) \in \mathbb{R}^2$, as one sees applying Lemma 1.1 to $\boldsymbol{Z}(s) = [\zeta(t)]_{x(t,s)}$ and $\boldsymbol{X}(s) = x_{tt}(t,s)$, with fixed t, the equalities (1.1) being immediate from (1.2)–(1.4). (Note that $\nabla[\zeta(t)] = 0$ and $x_{ttss} = x_{sstt} = 0$.) In particular, $t \mapsto x(t,1) = \mathrm{e}^{\eta(t)}y(t)$ is a complete geodesic whose initial velocity is, when $\eta(0) = 0$, equal to $\mathbf{v}_z + \dot{\eta}(0)$. However, every vector in T_zM is of this form for suitable $\mathbf{v}$ and η, as the values $\mathbf{v}_z$ realize all values at z in the complementary subbundle to $\mathcal{P}$ spanned by $\mathcal{Y}$, while the values $\dot{\eta}(0)$ realize all elements of $\mathcal{P}_z$. □

Lemma 1.3. *If ∇ is a complete connection on a manifold L and a non-constant function $f\colon L \to \mathbb{R}$ has $\nabla \mathrm{d}f = 0$, then f is surjective.*

Namely, along a maximal geodesic, f is an affine function of its parameter.

2. Properly discontinuous $\mathbb{R}^k$-subactions

Three well-known facts are phrased here as a remark for easy reference.

Remark 2.1. First, the composition of two fibrations (including covering projections) is clearly a fibration. Secondly, if a Lie group G acts on a manifold $\widehat{M}$ with a subgroup Γ acting on $\widehat{M}$ freely and properly discontinuously, then Γ is a discrete subset of G. Finally, whenever a compact topological manifold is contractible, it consists of a single point. (Otherwise, it would have a nontrivial top $\mathbb{Z}_2$ cohomology group.)

Lemma 2.2. *If $\mathbb{R}^k$ acts freely on a contractible manifold $\widehat{M}$ and a subgroup Γ of $\mathbb{R}^k$ acts on $\widehat{M}$ properly discontinuously with a compact quotient $\widehat{M}/\Gamma$, then $k = \dim \widehat{M}$, the action of $\mathbb{R}^k$ on $\widehat{M}$ is simply transitive, and Γ is a lattice in $\mathbb{R}^k$. Consequently, $\widehat{M}$ and $\widehat{M}/\Gamma$ are, respectively, an affine k-space and a k-dimensional torus.*

Proof. As Γ is a discrete subset of $\mathbb{R}^k$ (Remark 2.1), it forms a lattice in the subspace $Y \subseteq \mathbb{R}^k$ which it spans and, due to commutativity, the action of $\mathbb{R}^k$ on $\widehat{M}$ descends to a free action of the torus Y/Γ on $\widehat{M}/\Gamma$ which, according to [**19**, Corollary 4.2.11, p. 213], turns $\widehat{M}/\Gamma$ into the total space of a principal torus bundle over some compact base B. By Remark 2.1, the composition $\widehat{M} \to \widehat{M}/\Gamma \to B$ is a bundle projection with the fibre Y, and its homotopy long exact sequence [**20**, Theorem 4.49, p. 376] implies that B has trivial homotopy groups, being therefore contractible [**23**, Lemma 2.1]. Due to Remark 2.1, B consists of a single point and the resulting relations $\dim \widehat{M} = \dim Y \leq k \leq \dim \widehat{M}$, the last one immediate since the action of $\mathbb{R}^k$ is free, yield our assertion. □

3. Spectra of endomorphisms

Given an endomorphism B of a k-dimensional vector space $\mathcal{X}$, by the *spectrum* of B we mean the unordered system $\beta(1),\dots,\beta(k)$ formed by the complex characteristic roots of B listed with their multiplicities. If $B=[\mathrm{d}\sigma_q/\mathrm{d}q]_{q=1}$ is the infinitesimal generator of a Lie-group homomorphism $(0,\infty)\ni q\mapsto\sigma_q\in\mathrm{GL}(\mathcal{X})$ and the spectrum of each σ_q is $q^{\alpha(1)},\dots,q^{\alpha(k)}$, with $q^{\alpha(j)}=q^{\mathrm{Re}\,\alpha(j)}\mathrm{e}^{\mathrm{i}(\log q)\mathrm{Im}\,\alpha(j)}$, where $\alpha(j)\in\mathbb{C}$ *do not depend on* q, for $j=1,\dots,k$, then

$$B \text{ has the spectrum } \alpha(1),\dots,\alpha(k). \tag{3.1}$$

In fact, the complex-linear extension of B to $\mathcal{X}^{\mathbb{C}}$ has, in some basis, an upper triangular matrix with the diagonal entries $\beta(1),\dots,\beta(k)$ forming the spectrum of B. Thus $\sigma_q=\exp[(\log q)B]$ has the spectrum $q^{\beta(j)}$, $j=1,\dots,k$, and, up to a rearrangement, $q^{\beta(j)}=q^{\alpha(j)}$. Hence $[\beta(j)-\alpha(j)]\log q\in 2\pi\mathrm{i}\mathbb{Z}$ and so $\beta(j)=\alpha(j)$.

It is a trivial fact from linear algebra that, whether $\mathcal{X}$ is finite-dimensional or not, every family $\mathcal{F}$ of eigenvectors of an endomorphism $\Psi\in\mathrm{End}(\mathcal{X})$ corresponding to mutually distinct eigenvalues is linearly independent. As a consequence:

Lemma 3.1. *Given Ψ and $\mathcal{F}$ as above, let $(x_\alpha)_{\alpha\in A}$ be an* indexed *family of vectors such that $x_\alpha\in\mathcal{F}$ whenever $\alpha\in A$. If $A_0\subseteq A$ is a nonempty finite set with the property that $\sum_{\alpha\in A_0}x_\alpha\in\ker\Psi$, then $x_\alpha\in\ker\Psi$ for every $\alpha\in A_0$.*

Proof. There are positive integers n and $k_1,\dots,k_n$, as well as $\alpha_1,\dots,\alpha_n\in A$, such that $\sum_{\alpha\in A_0}x_\alpha=\sum_{i=1}^n k_i x_{\alpha_i}$, where $x_{\alpha_i}\neq x_{\alpha_j}$ whenever $i\neq j$. If λ_i is the eigenvalue of Ψ associated with x_{α_i}, it follows that $\sum_{i=1}^n k_i\lambda_i x_{\alpha_i}=0$, whence $\lambda_i=0$. Therefore $n=1$ and $\lambda_1=0$. ☐

The assumption and conclusion of Lemma 3.1 apply to $\Psi=q\,\mathrm{d}/\mathrm{d}q$ in the space of all complex-valued C^∞ functions of the variable $q\in(0,\infty)$ and the family $\mathcal{F}$ formed by all the power functions $(0,\infty)\ni q\mapsto q^{a+b\mathrm{i}}=q^a\mathrm{e}^{\mathrm{i}b\log q}$ with $a,b\in\mathbb{R}$. The proof of Theorem E, in Section 10, uses the following consequence:

(3.2) the sum of several terms of the form $q^{a+b\mathrm{i}}$ can be constant as a function of q only if each term in the sum is $q^0=1$.

4. Generic endomorphisms

Throughout this section, we let $(V,\langle\cdot,\cdot\rangle)$ be a pseudo-Euclidean vector space of dimension m, denote by $\mathcal{A}$ the space of all traceless self-adjoint endomorphisms of V, and say that $A\in\mathcal{A}$ is *generic* if only finitely many linear isometries of

$(V, \langle\cdot,\cdot\rangle)$ commute with A. For $m = 2$, unless $A = 0$, there are at most four linear isometries commuting with A, cf. [**9**, Remark 6.2], and so

$$\text{every } A \in \mathcal{A} \smallsetminus \{0\} \text{ is generic when } m = 2. \tag{4.1}$$

In all dimensions m, generic endomorphisms always exist. In fact, any $A \in \mathcal{A}$ with m distinct eigenvalues is generic, since its m eigenlines are mutually orthogonal and hence nondegenerate. Furthermore,

$$\text{the set of generic endomorphisms is an open and dense subset of } \mathcal{A}. \tag{4.2}$$

Indeed, note that an endomorphism $A \in \mathcal{A}$ is generic if and only if its isotropy group G_A under the action of $\mathrm{O}(V, \langle\cdot,\cdot\rangle)$ on $\mathcal{A}$ by conjugation is finite. However, finiteness of G_A amounts to its being countable, since G_A, given by the polynomial equation $CAC^* = A$, is an algebraic variety and so it has finitely many connected components [**32**, Theorem 3]. Consequently, genericity of A is equivalent to triviality of its isotropy algebra, which in turn means that the rank of $F(A)$ equals r, for $r = \dim \mathrm{O}(V, \langle\cdot,\cdot\rangle)$ and $F\colon \mathcal{A} \to \mathrm{Hom}(\mathfrak{so}(V, \langle\cdot,\cdot\rangle), \mathcal{A})$ given by $F(A)(B) = [A, B]$. We now fix some generic endomorphism $A_0 \in \mathcal{A}$, some bases of $\mathfrak{so}(V, \langle\cdot,\cdot\rangle)$ and $\mathcal{A}$, and a nonzero $r \times r$ subdeterminant of the matrix representing $F(A_0)$ in these bases. By analyticity, such a subdeterminant is nonzero on an open and dense subset of $\mathcal{A}$, thus proving (4.2).

In [**14**, Section 5] we show that a nilpotent endomorphism $A \in \mathcal{A}$ is generic if and only if $A^{m-1} \neq 0$, in which case there is a basis $(v_1, \dots, v_m)$ of V such that

(4.3) $Av_j = v_{j-1}$ and $\langle v_i, v_k \rangle = \varepsilon\delta_{ij}$ for all $i, j \in \{1, \dots, m\}$, where $\varepsilon = \pm 1$ is the semi-definiteness sign of $\langle A^{m-1}\cdot, \cdot\rangle$, $k = m + 1 - j$, and $v_0 = 0$. In addition, such a basis is also unique up to an overall sign change.

It follows [**14**, Corollary 5.3] that for every $q \in (0, \infty)$,

(4.4) there are only two linear isometries $C, -C$ of $(V, \langle\cdot,\cdot\rangle)$ with $CAC^{-1} = q^2 A$ and, in a basis satisfying (4.3), they are given by $Cv_j = \delta q^{m+1-2j} v_j$, for some sign factor $\delta = \pm 1$.

5. The universal coverings

In this section we fix a rank-one ECS manifold (M, g) of dimension $n \geq 4$ with arbitrary indefinite metric signature, and let Γ be the fundamental group of M. Consider the universal covering projection $\pi\colon \widetilde{M} \to M$, and set $\widetilde{\mathrm{g}} = \pi^*\mathrm{g}$, so that $(\widetilde{M}, \widetilde{\mathrm{g}})$ is a simply connected rank-one ECS manifold on which Γ acts freely and properly discontinuously by isometries, with quotient $M = \widetilde{M}/\Gamma$. We will also write $\widetilde{\mathcal{D}}$ for the Olszak distribution of $(\widetilde{M}, \widetilde{\mathrm{g}})$, defined in the Introduction.

As the Levi-Civita connection of $(\widetilde{M},\widetilde{\mathrm{g}})$ induces a connection on $\widetilde{\mathcal{D}}$, and the latter is flat [**10**, Lemma 2.2(f)], simple connectivity of $\widetilde{M}$ allows to us to fix

$$\text{(5.1)}\qquad \begin{array}{l}\text{a null parallel vector field } \mathbf{w} \text{ spanning } \widetilde{\mathcal{D}}, \text{ leading to a surjective}\\ \text{function } t\colon \widetilde{M}\to I \text{ onto an open interval } I\subseteq\mathbb{R}, \text{ with } \mathrm{d}t=\widetilde{\mathrm{g}}(\mathbf{w},\cdot).\end{array}$$

In addition, as shown in [**15**, end of Section 12],

$$\text{(5.2)}\qquad \text{the leaves of } \widetilde{\mathcal{D}}^{\perp} \text{ coincide with the level sets of } t\colon \widetilde{M}\to I.$$

By Lemma 1.3, (M,g) is incomplete when $I\neq\mathbb{R}$. Moreover, as *the Olszak distribution is a local geometric invariant of the given ECS metric*, t in (5.1) is unique up to affine substitutions, and so for every $\gamma\in\mathrm{Iso}(\widetilde{M},\widetilde{\mathrm{g}})$ there is $(q,p)\in\mathrm{Aff}(\mathbb{R})$ such that $t\circ\gamma=qt+p$, giving rise to two homomorphisms:

$$\text{(5.3)}\quad \text{a)}\ \mathrm{Iso}(\widetilde{M},\widetilde{\mathrm{g}})\ni\gamma\mapsto(q,p)\in\mathrm{Aff}(\mathbb{R}),\quad \text{b)}\ \mathrm{Iso}(\widetilde{M},\widetilde{\mathrm{g}})\ni\gamma\mapsto q\in\mathbb{R}\smallsetminus\{0\}.$$

The following principle will be repeatedly used:

$$\text{(5.4)}\qquad \begin{array}{l}\text{replacing } \Gamma \text{ with a finite index subgroup } \Gamma_0 \text{ amounts to replacing } M\\ \text{with the quotient } \widetilde{M}/\Gamma_0, \text{ which is also compact (as the total space of}\\ \text{a finite-sheeted covering of } M) \text{ and has } \widetilde{M} \text{ as its universal covering.}\end{array}$$

Using (5.4), we from now assume that

$$\text{(5.5)}\qquad \text{the image of } \Gamma \text{ under (5.3-b) is contained in } (0,\infty).$$

In [**15**, Section 12], we show that, if (5.5) holds and M is compact,

$$\text{(5.6)}\qquad \begin{array}{rl}\text{a)} & \text{there exists a smooth positive function } \psi \text{ on } \widetilde{M} \text{ such}\\ & \text{that the 1-form } \psi\,\mathrm{d}t \text{ is closed and } \Gamma\text{-invariant, and}\\ \text{b)} & \text{the vector field } \mathbf{w} \text{ in (5.1) is complete.}\end{array}$$

With q related to γ as in (5.3-b), these ψ and $\mathbf{w}$ satisfy the conditions

$$\text{(5.7)}\qquad \psi\circ\gamma=q^{-1}\psi \text{ and } \gamma_*\mathbf{w}=q^{-1}\mathbf{w},\ \text{for every } \gamma\in\Gamma,$$

due to the relation $\gamma^*(\mathrm{d}t)=q\,\mathrm{d}t$ and Γ-invariance of $\psi\,\mathrm{d}t$.

The Levi-Civita connection of $(\widetilde{M},\widetilde{\mathrm{g}})$ induces one on the quotient bundle $\widetilde{\mathcal{D}}^{\perp}/\widetilde{\mathcal{D}}$, which is flat by [**10**, Lemma 2.2(f)]. Thus, $\widetilde{M}$ being simply connected, the real vector space V of parallel sections of $\widetilde{\mathcal{D}}^{\perp}/\widetilde{\mathcal{D}}$ has the full dimension $m=n-2$. The space V also inherits a natural pseudo-Euclidean inner product $\langle\cdot,\cdot\rangle$ from $\widetilde{\mathrm{g}}$, and the Weyl tensor W of $(\widetilde{M},\widetilde{\mathrm{g}})$ induces, cf. [**9**, Section 4],

$$\text{(5.8)}\qquad \begin{array}{l}\text{a traceless self-adjoint operator } A\colon V\to V,\\ \text{given by } A(\mathbf{v}+\widetilde{\mathcal{D}})=\mathrm{W}(\mathbf{u},\mathbf{v})\mathbf{u}+\widetilde{\mathcal{D}}, \text{ where } \mathbf{u}\\ \text{is any vector field on } \widetilde{M} \text{ such that } \widetilde{\mathrm{g}}(\mathbf{u},\mathbf{w})=1.\end{array}$$

Clearly, $\mathbf{u}$ in (5.8) is unique modulo $\widetilde{\mathcal{D}}^\perp$, so that $A(X+\widetilde{\mathcal{D}})$ is well-defined and, by (5.7), $\gamma_*\mathbf{u}+\widetilde{\mathcal{D}} = q\mathbf{u}+\widetilde{\mathcal{D}}$ for every $\gamma\in\Gamma$. Every $\gamma\in\Gamma$ induces a linear isometry $C\colon V\to V$, acting via $C(\mathbf{v}+\widetilde{\mathcal{D}}) = \gamma_*\mathbf{v}+\widetilde{\mathcal{D}}$, which leads to

(5.9) a homomorphism $\Gamma\ni\gamma\mapsto C\in \mathrm{O}(V,\langle\cdot,\cdot\rangle)$ with $CAC^{-1}=q^2A$,

q being associated with γ as in in (5.3-b).

We will say that (M,g) itself is *generic* if A in (5.8) is generic in the sense of Section 4. By (4.1), (M,g) is always generic when $n=4$.

6. The rank-one ECS models and their isometry groups

Rank-one ECS models are built from the following data, cf. [**27**]:

(6.1) an integer $n\geq 4$, a pseudo-Euclidean vector space $(V,\langle\cdot,\cdot\rangle)$ of dimension $n-2$, a self-adjoint endomorphism $A\in\mathfrak{sl}(V)\smallsetminus\{0\}$, and a nonconstant smooth function $f\colon I\to\mathbb{R}$ defined on an open interval $I\subseteq\mathbb{R}$.

Then, defining $\kappa\colon I\times\mathbb{R}\times V\to\mathbb{R}$ by $\kappa(t,s,v) = f(t)\langle v,v\rangle+\langle Av,v\rangle$ and regarding $\langle\cdot,\cdot\rangle$ as a constant flat metric on V, we consider the simply connected n-dimensional pseudo-Riemannian manifold

$$(\widehat{M},\widehat{\mathrm{g}}) = \big(I\times\mathbb{R}\times V,\ \kappa\,\mathrm{d}t^2+\mathrm{d}t\,\mathrm{d}s+\langle\cdot,\cdot\rangle\big), \tag{6.2}$$

where we identify $\mathrm{d}t$, $\mathrm{d}s$ and $\langle\cdot,\cdot\rangle$ with their pull-backs to $\widehat{M}$.

By [**10**, Theorem 4.1] $(\widehat{M},\widehat{\mathrm{g}})$ is a rank-one ECS manifold. Calling the manifolds (6.2) *models* is justified by their being locally *universal*:

(6.3) every point of a rank-one ECS manifold of dimension n has a neighborhood isometric to an open subset of a manifold of type (6.2), with one possible exception in (6.1): f may be constant [**10**, Theorem 4.1].

Our two uses of the term 'generic' are mutually consistent:

(6.4) genericity of $(\widehat{M},\widehat{\mathrm{g}})$ – see the end of Section 5 – is equivalent to that of the endomorphism A in (6.1) as defined in Section 4.

Indeed, the Olszak distribution $\widehat{\mathcal{D}}$ of $(\widehat{M},\widehat{\mathrm{g}})$ – defined in the Introduction – is spanned by the null parallel coordinate vector field ∂_s [**27**, p. 93], so that the leaves of $\widehat{\mathcal{D}}^\perp$ are the $\mathbb{R}\times V$ factor submanifolds of $\widehat{M}$ in (6.2). This allows us to isometrically identify $(V,\langle\cdot,\cdot\rangle)$ with the space of parallel sections of $\widehat{\mathcal{D}}^\perp/\widehat{\mathcal{D}}$, which, as shown in [**15**, the lines following (7.3)], also identifies A in (6.1) with A in (5.8) (where one may set $\mathbf{u}=2\partial_t$).

Central to the discussion are: the $2(n-2)$-dimensional symplectic vector space

(6.5) $(\mathcal{E},\Omega)$ consisting of all solutions $u\colon I\to V$ of the second-order equation $\ddot u = fu + Au$, where Ω is defined by $\Omega(u,w) = \langle \dot u, w\rangle - \langle u, \dot w\rangle$, and its associated Heisenberg group H: the Cartesian product $\mathbb{R}\times\mathcal{E}$ with the operation defined by $(r,u)(\hat r,\hat u) = (r+\hat r - \Omega(u,\hat u)\,,\,u+\hat u)$.

We also need

(6.6) the subgroup S of $\mathrm{Aff}(\mathbb{R})\times \mathrm{O}(V,\langle\cdot,\cdot\rangle)$ formed by all (q,p,C) having $CAC^{-1} = q^2A$, with $qt+p\in I$ and $f(t) = q^2 f(qt+p)$ for all $t\in I$.

Each of q, (q,p), and C depends homomorphically on $\sigma = (q,p,C)$, so that S acts (from the left) on I, $\mathbb{R}$, and $\mathrm{C}^\infty(I,V)$ via, respectively,

$$\text{(6.7)}\qquad \text{i) } \sigma t = qt+p,\quad \text{ii) } \sigma s = q^{-1}s,\quad \text{iii) } (\sigma u)(t) = Cu(q^{-1}(t-p)).$$

As the notations in (6.7-i) and (6.7-ii) are in conflict, we will adopt only the former and explicitly write $q^{-1}s$ for the latter, always understanding that q is the first component of σ. The action of S on $\mathrm{C}^\infty(I,V)$, obviously leaving $\mathcal{E}$ in (6.5) invariant, restricts to an action on $\mathcal{E}$ with $\det\sigma = q^{2-n}$ on $\mathcal{E}$ for all $\sigma\in$ S, since

$$\text{(6.8)}\qquad \sigma^*\Omega = q^{-1}\Omega,\ \text{if } \sigma \text{ is regarded as an operator } \sigma\colon \mathcal{E}\to\mathcal{E}.$$

Rephrasing [**14**, Theorem 4.1], we have:

Theorem 6.1. *The isometry group of a model* $(\widehat M,\widehat{\mathrm{g}})$, *with* (6.1)–(6.2), *can be identified with the set* S × H, *cf.* (6.5)–(6.6), *so that* $\Phi = (\sigma,r,u)$ *acts on* $(\widehat M,\widehat{\mathrm{g}})$ *via*

$$\Phi(t,s,v) = \big(\sigma t\,,\,-\langle \dot u(\sigma t), 2\sigma v + u(\sigma t)\rangle + q^{-1}s + r\,,\,\sigma v + u(\sigma t)\big),$$

for every $(t,s,v)\in\widehat M$, *and the group operation in* S × H *becomes*

$$(\sigma,r,u)(\widehat\sigma,\widehat r,\widehat u) = \big(\sigma\widehat\sigma\,,\,r + q^{-1}\widehat r - \Omega(u,\sigma\widehat u)\,,\,u+\sigma\widehat u\big),$$

for $(\sigma,r,u),(\widehat\sigma,\widehat r,\widehat u)\in$ S × H. *Thus,* $\mathrm{Iso}(\widehat M,\widehat{\mathrm{g}})$ *is isomorphic to a semidirect product* S ⋉ H, *where the diagonal action of* S *on* H *is defined via* (6.7): $\sigma\cdot(r,u) = (q^{-1}r,\sigma u)$.

Remark 6.2. We identify H with the normal subgroup $\{(1,0,\mathrm{Id})\}\times$ H of $\mathrm{Iso}(\widehat M,\widehat{\mathrm{g}})$, the kernel of the homomorphism $\mathrm{Iso}(\widehat M,\widehat{\mathrm{g}})\ni(\sigma,r,u)\mapsto\sigma\in$ S.

Remark 6.3. In any rank-one ECS manifold, the leaves of $\mathcal{D}^\perp$ are totally geodesic, $\mathcal{D}^\perp$ being parallel. In addition, the resulting induced connection on each leaf is *flat*. Namely, (6.3) allows us to assume that the manifold has the form (6.1), with (6.2) except for nonconstancy of f. For i,j ranging over $2,\ldots,n-1$, any linear coordinates x^i on V form, along with $x^1 = t$ on I and $x^n = s/2$ on $\mathbb{R}$, a coordinate system on $I\times\mathbb{R}\times V$, and then – see [**15**, the lines following formula

(7.2)] – the coordinate vector fields ∂_n and ∂_i span $\mathcal{D}^\perp$, while, according to [**27**, p. 93], $\Gamma_{ij}^{\bullet} = \Gamma_{in}^{\bullet} = \Gamma_{nn}^{\bullet} = 0$, where $\bullet$ denotes any index.

Remark 6.4. In a rank-one ECS model manifold $(\widehat{M}, \widehat{\mathrm{g}})$, let there exist a subgroup Γ of $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ acting freely and properly discontinuously on $(\widehat{M}, \widehat{\mathrm{g}})$ with a compact isometric quotient $M = \widehat{M}/\Gamma$. Using (5.4) we also assume that $q > 0$ whenever $(q, p, C, r, u) \in \Gamma$. If the resulting ECS manifold (M, g) is translational, then all $(q, p, C, r, u) \in \Gamma$ have $q = 1$, due to [**14**, formula (3.5-a)]. Also, $I = \mathbb{R}$ in (6.1): otherwise, all $(1, p, C, r, u) \in \Gamma$ acting on $\widehat{M}$ (see Theorem 6.1) would have $p = 0$ (since $t \mapsto t + p$ sends I onto itself), and so

(6.9) t would descend to a function without critical points on the compact manifold M.

Finally, according to [**13**, formula (3.1)], such translational (M, g) is geodesically complete but not locally homogeneous.

7. Proofs of Theorems A, B, and C

In the first two proofs below, $(\widetilde{M}, \widetilde{\mathrm{g}})$ is the isometric universal covering manifold of the n-dimensional compact rank-one ECS manifold (M, g) in question. The objects π, Γ, $\widetilde{\mathcal{D}}$, $\mathbf{w}$, t, I, $(V, \langle \cdot, \cdot \rangle)$, and A are all defined as in Section 5.

Recall from the Introduction that $(\widetilde{M}, \widetilde{\mathrm{g}})$ is said to be $\widetilde{\mathcal{D}}^\perp$-complete if all the leaves of $\widetilde{\mathcal{D}}^\perp$ are complete, while its maximal completeness means that every non-complete maximal geodesic intersects all leaves of $\widetilde{\mathcal{D}}^\perp$. As $\widetilde{\mathcal{D}}^\perp$ and geodesics in $(\widetilde{M}, \widetilde{\mathrm{g}})$ are mapped under π onto their analogs in (M, g), it suffices to prove Theorem B for $(\widetilde{M}, \widetilde{\mathrm{g}})$ rather than (M, g).

Proof of Theorem B. We apply Lemma 1.2 to any given leaf L of $\widetilde{\mathcal{D}}^\perp$ with the flat connection induced on it (Remark 6.3), setting $\mathcal{X} = \mathbb{R}\mathbf{w}$ and $\mathcal{Y} = \bigoplus_{j=1}^{m} \mathbb{R}\widetilde{\boldsymbol{y}}_j$ for $m = n - 2$ vector fields $\widetilde{\boldsymbol{y}}_j$ on $\widetilde{M}$ defined below, which, due to their Γ-invariance, will descend to the compact manifold $M = \widetilde{M}/\Gamma$, making each $\mathbf{v} \in \mathcal{Y}$ complete.

We begin by assuming (5.5) and choosing a basis $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ of V. Let $K \subseteq (0, \infty)$ be the image of Γ under (5.3-b). By (5.5), K is either trivial, or infinite. In the former case, the image of (5.9) is finite due to genericity of A, and so its kernel has finite index in Γ. Thus, (5.4) allows us to further assume that Γ acts trivially on V, via (5.9), and $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ can be completely arbitrary. If K is infinite, (5.9) implies nilpotency of A, and we select a basis $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ of V for which (4.3)–(4.4) holds. As δ in (4.4) depends homomorphically on γ, using (5.4) we may require that $\delta = 1$ for every $\gamma \in \Gamma$.

In either case, we fix a Riemannian metric g° on M and lift $(\mathbf{v}_1,\ldots,\mathbf{v}_m)$ to vector fields $(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m)$ tangent to leaves of $\widetilde{\mathcal{D}}^\perp$ which are $\pi^*\mathrm{g}^\circ$-orthogonal to $\mathbf{w}$. For ψ as in (5.6-a), the vector fields $\widetilde{\boldsymbol{y}}_j = \psi^{2j-1-m}\boldsymbol{y}_j$, are Γ-invariant (cf. (4.4) and (5.7)), which completes the proof. (When $K=\{1\}$, we may set $\psi = 1$.) $\square$

Proof of Theorem A. In view of (5.2) and $\widetilde{\mathcal{D}}^\perp$-completeness of $(\widetilde{M},\widetilde{\mathrm{g}})$ (due to Theorem B), every maximal geodesic of $(\widetilde{M},\widetilde{\mathrm{g}})$ transverse to $\widetilde{\mathcal{D}}^\perp$ can be parametrized by t, as $t\colon \widetilde{M}\to I$ restricted to its image is a diffeomorphism onto a subinterval $I'\subseteq I$, cf. [**9**, Remark 7.2]. To show that $I'=I$, we invoke [**9**, Lemma 7.3] with minimal modifications in its proof, which never uses the assumption stated there that the metric under consideration is Lorentzian. See Appendix A for details. $\square$

Proof of Theorem C. Assume that $(\widetilde{M},\widetilde{\mathrm{g}})$ is a n-dimensional simply connected and maximally complete rank-one ECS manifold, and again choose π, Γ, $\widetilde{\mathcal{D}}$, $\mathbf{w}$, t, I, $(V,\langle\cdot,\cdot\rangle)$, and A as in Section 5. For the function $f\colon I\to\mathbb{R}$ such that $\widetilde{\mathrm{Ric}} = (2-n)f(t)\,\mathrm{d}t\otimes\mathrm{d}t$ (cf. [**15**, formula (6.6)]), we consider the model ECS manifold $(\widehat{M},\widehat{\mathrm{g}})$ built from these ingredients as in (6.2). An isometry between $(\widehat{M},\widehat{\mathrm{g}})$ and $(\widetilde{M},\widetilde{\mathrm{g}})$ is defined as in the proof of [**9**, Theorem 7.1], with no significant changes. Details are given in Appendix B. $\square$

8. The transitive-commutation property

For a group-subgroup pair (G,H), consider the *transitive-commutation property*: the commutation relation on $G\smallsetminus H$ is transitive, that is, the equalities $xy=yx$ and $yz=zy$ for $x,y,z\in G\smallsetminus H$ imply $xz=zx$.

Lemma 8.1. *Let (G,H) be a group-subgroup pair.*

(a) *(G,H) has the transitive-commutation property if and only if there exists a family $\mathcal{K}$ of Abelian subgroups of G such that $\{K\smallsetminus H\}_{K\in\mathcal{K}}$ is a partition of $G\smallsetminus H$, and any two elements of $G\smallsetminus H$ which commute lie in the same $K\in\mathcal{K}$.*

(b) *Whenever (G,H) has the transitive-commutation property and $\mathcal{K}$ is as in* (a)*, any Abelian subgroup of G is contained in H, or in a unique element of $\mathcal{K}$.*

Proof. The 'only if' part of (a) is obvious. Conversely, since commutation is now an equivalence relation on $G\smallsetminus H$, the subgroup K generated by any equivalence class $E\subseteq G\smallsetminus H$ of the commutation relation is Abelian and $K\smallsetminus H=E$, the right-to-left inclusion being immediate, the other one clear as elements in $K\smallsetminus H$ commute with all of E and hence lie in E. This yields (a).

To prove (b), let N be an Abelian subgroup of G. If there exists an element $x \in N \smallsetminus H$, we fix the unique subgroup $K \in \mathcal{K}$ with $N \smallsetminus H \subseteq K$. Then we must have $N \subseteq K$, for if $y \in (N \cap H) \smallsetminus K$, then $xy \in N \smallsetminus H$ and so $y = x^{-1}(xy) \in K$. □

Remark 8.2. One easily verifies the fact (not needed in our argument) that K associated with E in the above proof is both the unique Abelian subgroup of G containing E, and the centralizer of E.

9. Generic homogeneous models

By a *standard homogeneous rank-one ECS model manifold* (briefly, a *standard homogeneous model*), we mean $(\widehat{M}, \widehat{\mathrm{g}})$ as in (6.1)–(6.2) with

$$I = (0,\infty) \text{ and } f(t) = \frac{c^2 - 1/4}{t^2}, \text{ where } c \in [0,1/2) \cup (1/2,\infty) \cup \mathrm{i}(0,\infty). \tag{9.1}$$

All such $(\widehat{M}, \widehat{\mathrm{g}})$ are homogeneous, as pointed out in [**7**, Remark 1 on p. 172]. (This also follows from Theorem 6.1: one easily sees that the group of all elements $(q, p, C, r, u) \in \mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ having $p = 0$ acts on $\widehat{M}$ transitively.) The factor $c^2 - 1/4$ is just any real constant $h \neq 0$, written so for later convenience.

Lemma 9.1. *Every locally homogeneous rank-one ECS model manifold is isometric to an open submanifold* $(a, b) \times \mathbb{R} \times V$ *of a standard homogeneous model* (6.2).

Proof. By [**13**, formula (3.4)], in (6.1), $f \neq 0$ everywhere and $|f|^{-1/2}$ is a linear function of t, so that $f(t) = h(t - t_0)^{-2}$ for some real $h \neq 0$ and t_0. Under a suitable coordinate change $(t, s) \mapsto (qt + p, q^{-1}s)$, with $(q, p) \in \mathrm{Aff}(\mathbb{R})$, (6.1)-(6.2) remain valid, allowing us to assume that $t_0 = 0$ and $I \subseteq (0, \infty)$. □

As a trivial consequence of Lemma 9.1 and (6.3), we obtain:

Corollary 9.2. *All locally homogeneous rank-one ECS manifolds are locally isometric to standard homogeneous models.*

If a standard homogeneous model $(\widehat{M}, \widehat{\mathrm{g}})$ is also generic, cf. (6.4), then, with notations of (6.5) and Theorem 6.1, elements of

$$\text{the identity component } \mathrm{G}_0 \text{ of } \mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}}) \tag{9.2}$$

have the form $(q, 0, C_q, r, u)$, where $q \in (0, \infty)$, $(r, u) \in \mathrm{H}$, cf. (6.5) and Remark 6.2, while $C_q \colon V \to V$ is as in (4.4) with $\delta = 1$. We also consider

$$\sigma_q \colon \mathcal{E} \to \mathcal{E} \text{ defined as in (6.7-iii): } (\sigma_q u)(t) = C_q u(t/q). \tag{9.3}$$

Abbreviating $\Phi = (q, 0, C_q, r, u)$ simply to (q, r, u), we may now

$$\text{identify } \mathrm{G}_0 \text{ with } (0,\infty) \times \mathrm{H}, \tag{9.4}$$

and, for $\widehat{\Phi} \in \mathrm{G}_0$ and $(t,s,v) \in \widehat{M}$, we obtain, from Theorem 6.1 that

$$(9.5)\quad \begin{aligned} &\text{i)}\ \ \Phi\widehat{\Phi} = (q\widehat{q}, r + q^{-1}\widehat{r} - \Omega(u, \sigma_q\widehat{u}), u + \sigma_q\widehat{u}),\\ &\text{ii)}\ \ \Phi^{-1} = (q^{-1}, -qr, -\sigma_q^{-1}u),\\ &\text{iii)}\ \ \text{the } \mathcal{E}\text{-component of the commutator } [\Phi, \widehat{\Phi}] \text{ is } (\sigma_q - 1)\widehat{u} - (\sigma_{\widehat{q}} - 1)u,\\ &\text{iv)}\ \ \Phi(t,s,v) = \big(qt\,, -\langle \dot{u}(qt), 2C_qv + u(qt)\rangle + q^{-1}s + r\,, C_qv + u(qt)\big) \end{aligned}$$

With 1 denoting the identity operator $\mathcal{E} \to \mathcal{E}$, conjugation by Φ has the form

$$\Phi\widehat{\Phi}\Phi^{-1} = \Big(\widehat{q}\,, q^{-1}\widehat{r} + (1 - \widehat{q}^{-1})r - \Omega((1 + \sigma_{\widehat{q}})u, \sigma_q\widehat{u}) + \Omega(u, \sigma_{\widehat{q}}u)\,, (1 - \sigma_{\widehat{q}})u + \sigma_q\widehat{u}\Big).$$

Let us now fix a generic homogeneous model $(\widehat{M}, \widehat{\mathrm{g}})$.

By [**14**, Theorem 6.1], each $\sigma_q\colon \mathcal{E} \to \mathcal{E}$ has the spectrum $\lambda_j^{\pm} = q^{m+1-2j}\mu^{\pm}$, $j = 1, \dots, m$, for the eigenvalues $\mu^{\pm} \in \mathbb{C}$ of the operator T with $(Tu)(t) = u(t/q)$, on the space $\mathcal{W}$ of solutions $y\colon (0,\infty) \to \mathbb{C}$ to the ordinary differential equation $\ddot{y} = fy$. The expression for f in (9.1) gives $\mu^{\pm} = q^{-\frac{1}{2}\mp c}$, the corresponding T-diagonalizing (or, T-triangular) basis of $\mathcal{W}$ being $t \mapsto t^{\frac{1}{2}\pm c}$ if $c \neq 0$ (or, respectively, $t \mapsto t^{\frac{1}{2}}$ and $t \mapsto t^{\frac{1}{2}}\log t$ when $c = 0$). Hence

(9.6) the spectrum of σ_q becomes $\lambda_j^{\pm} = q^{m+\frac{1}{2}-2j\mp c}$, for $j = 1, \dots, m$.

The assignment $(0,\infty) \ni q \mapsto \sigma_q \in \mathrm{GL}(\mathcal{E})$, being a homomorphism, has an infinitesimal generator $B \in \mathfrak{gl}(\mathcal{E})$, with $\sigma_q = \exp[(\log q)B]$. By (3.1) and (9.6),

(9.7) the spectrum of B is $\kappa_j^{\pm} = m + \dfrac{1}{2} - 2j \mp c$, where $j = 1\dots, m$.

Lemma 9.3. *For $\mathcal{E}$ and B as above, let $\mathcal{E}_0 = \ker B$ and $\mathcal{E}_+ = B(\mathcal{E})$. Then:*

(a) *Either $\mathcal{E}_0$ is trivial, or $\dim \mathcal{E}_0 = 1$ and B is diagonalizable with $2m$ distinct real eigenvalues. In both cases, $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_+$ and $(\sigma_q - 1)(\mathcal{E}_0) = \{0\}$ if $q \in (0,\infty)$.*

(b) *For every $q \in (0,\infty) \setminus \{1\}$ the operator $\sigma_q - 1\colon \mathcal{E}_+ \to \mathcal{E}_+$ is an isomorphism.*

Proof. If $\det B = 0$, some $\kappa_j^{\pm}$ in (9.7) equals 0, so that $2c = \pm(2m - 4j + 1)$ is an odd integer. Hence, the eigenvalues of B are real and mutually distinct: $j \mapsto \kappa_j^{\pm}$ is injective for a fixed sign $\pm$, while $2c$ would be the *even* integer $2(i - j)$ if there existed i and j with $\kappa_j^{+} = \kappa_i^{-}$. Since $\sigma_q = \exp[(\log q)B]$, this yields (a).

To prove (b), we use (9.6) and consider two cases. If $c \in \mathbb{R}$, the resulting injectivity of $a \mapsto q^a$ on $\mathbb{R}$ for $q \neq 1$ gives $\lambda_j^{\pm} \neq 1$ except – see the last paragraph – when $\dim \mathcal{E}_0 = 1$ and $(j, \pm)$ is the unique pair with $\kappa_j^{\pm} = 0$. If $c \in \mathrm{i}(0,\infty)$, $|\lambda_j^{\pm}| = q^{m-2j+1/2}$, being a half-integer power of q, cannot equal 1 unless $q = 1$, and hence $\lambda_j^{\pm} \neq 1$ again. ☐

Lemma 9.4. *For a generic standard homogeneous model, the group-subgroup pair $(\mathrm{G}_0, \mathrm{H})$ given by (9.2) and (6.5) has the transitive-commutation property of Section 8,* *and its equivalence classes generate subgroups of* G_0 *acting freely on* $\widehat{M}$*, and isomorphic to* $\mathbb{R}^k$*, for* $k = \dim \mathcal{E}_0 + 1 \in \{1, 2\}$.

Proof. Define $J\colon \mathbb{R} \times \mathcal{E}_+ \times (0, \infty) \times \mathcal{E}_0 \to \mathrm{G}_0 \smallsetminus \mathrm{H}$ by

$$J(a, z, q, w) = \Big(q\,,\, a(1 - q^{-1}) + \Omega(z, \sigma_q z + (1 + q^{-1})w)\,,\, (\sigma_q - 1)z + w\Big)\,.$$

It is an obvious consequence of Lemma 9.3, (9.5-i) and (6.8) that

i) J maps $\mathbb{R} \times \mathcal{E}_+ \times [(0, \infty) \smallsetminus \{1\}] \times \mathcal{E}_0$ bijectively onto $\mathrm{G}_0 \smallsetminus \mathrm{H}$,
ii) $J(a, z, \cdot\,, \cdot\,)\colon (0, \infty) \times \mathcal{E}_0 \to \mathrm{G}_0$ is an injective group homomorphism whenever $(a, z) \in \mathbb{R} \times \mathcal{E}_+$.

By (ii), the images $K_{a,z}$ of $J(a, z, \cdot\,, \cdot\,)$, with $(a, u) \in \mathbb{R} \times \mathcal{E}_+$, are connected Abelian Lie subgroups of G_0, isomorphic to $\mathbb{R}$ (if $\mathcal{E}_0 = \{0\}$) or $\mathbb{R}^2$ (when $\dim \mathcal{E}_0 = 1$), while, due to (i), the family $\{K_{a,z} \smallsetminus \mathrm{H} : (a, z) \in \mathbb{R} \times \mathcal{E}_+\}$ is a partition of $\mathrm{G}_0 \smallsetminus \mathrm{H}$.

We now define $F : \mathrm{G}_0 \smallsetminus \mathrm{H} \to \mathbb{R} \times \mathcal{E}_+$ by $F(J(a, z, q, w)) = (a, z)$ when $q \neq 1$, which makes sense according to (i). Thus, F associates with $\Phi \in \mathrm{G}_0 \smallsetminus \mathrm{H}$ the unique (a, z) for which $\Phi \in K_{a,z}$. It follows that

iii) $\Phi, \widehat{\Phi} \in \mathrm{G}_0 \smallsetminus \mathrm{H}$ commute if and only if $F(\Phi) = F(\widehat{\Phi})$.

In fact, the 'if' part is obvious as Φ and $\widehat{\Phi}$ then lie in the same Abelian subgroup of G_0. By (9.5-i), two elements $(q, r, u), (\widehat{q}, \widehat{r}, \widehat{u}) \in \mathrm{G}_0$ commute if and only if

$$(9.8)\quad r + q^{-1}\widehat{r} - \Omega(u, \sigma_q \widehat{u}) = \widehat{r} + \widehat{q}^{-1} r - \Omega(\widehat{u}, \sigma_{\widehat{q}} u) \quad \text{and} \quad (\sigma_q - 1)\widehat{u} = (\sigma_{\widehat{q}} - 1)u.$$

We now prove the 'only if' part of (iii) assuming – see (i) – that $J(a, z, q, w)$ commutes with $J(\widehat{a}, \widehat{z}, \widehat{q}, \widehat{w})$ and $q \neq 1 \neq \widehat{q}$. The second equality of (9.8) with $(u, \widehat{u}) = ((\sigma_q - 1)z + w, (\sigma_{\widehat{q}} - 1)\widehat{z} + \widehat{w})$ yields $z = \widehat{z}$, since – by Lemma 9.3(b) – the operators $\sigma_q - 1$ annihilate $\mathcal{E}_0$, and form a family of mutually commuting automorphisms when restricted to $\mathcal{E}_+$ (the latter since $q \mapsto \sigma_q$ is a homomorphism). As $z = \widehat{z}$, the first equality of (9.8), for $(u, \widehat{u})$ chosen above, and $(r, \widehat{r})$ replaced by the $\mathbb{R}$-components of $J(a, z, q, w)$ and $J(\widehat{a}, \widehat{z}, \widehat{q}, \widehat{w})$, easily yields $a = \widehat{a}$, and (iii) follows. The conclusion is now immediate from Lemma 8.1(a). □

10. Proof of Theorem E

We fix a generic compact rank-one ECS manifold (M, g). By Corollary D, the universal covering of (M, g) is isometrically identified with a model $(\widehat{M}, \widehat{\mathrm{g}})$ as in (6.1)–(6.2), and then $\widehat{M}/\Gamma = M$, as at the beginning of Section 5, where (5.4) also allows us to assume that $\Gamma \subseteq \mathrm{G}_0$, cf. (9.2).

By Remark 6.4, the "translational" conclusion about (M, g), which we prove in the subsequent paragraphs, implies the other two assertions of Theorem E.

Next, suppose that, on the contrary, our (M,g) is not translational. The dichotomy $(*)$ in the Introduction and [**14**, Corollary D] imply that $(\widehat{M},\widehat{\mathrm{g}})$ is locally homogeneous. Lemma 9.1 now allows us to write $\widehat{M} = (a,b)\times\mathbb{R}\times V$. Then, $(a,b) = (0,\infty)$, or else all elements $\Phi = (q,r,u) \in \Gamma$, acting on $\widehat{M}$ via (9.5-iv), would have $q = 1$ (as $t \mapsto qt$ maps (a,b) onto itself), leading to (6.9).

The group $\Sigma = \Gamma\cap\mathrm{H}$, cf. Remark 6.2, is the kernel of the homomorphism

$$\Gamma \ni (q,r,u) \mapsto q \in (0,\infty). \tag{10.1}$$

As an immediate consequence of [**14**, Lemma 3.2(b) and (f) in Section 4] and [**14**, Theorems A, B, and Lemma 3.1],

$$\Sigma\cap(\{1\}\times\mathbb{R}\times\{0\}) \text{ is trivial and the image of (10.1) is dense in } (0,\infty). \tag{10.2}$$

The last line in (6.5) now has three consequences. First, $\Sigma \ni (1,r,u) \mapsto u \in \mathcal{E}$ is a homomorphism, and its injectivity due to (10.2) implies that Σ is Abelian. Secondly, the image Λ of this last homomorphism spans a subspace $\mathcal{L}$ of $(\mathcal{E},\Omega)$ which is isotropic, in the sense that $\Omega = 0$ on it. Finally, $\mathbb{R}\times\mathcal{L}$ is an Abelian subgroup of H (see Remark 6.2) containing Σ, and its group operation coincides with the addition in the vector space $\mathbb{R}\times\mathcal{L}$. Applying Remark 2.1 to Σ rather than Γ and the subspace $\mathcal{Z}$ of $\mathbb{R}\times\mathcal{L}$ spanned by Σ, we see that

$$\begin{array}{l}\Sigma \text{ is a lattice in } \mathcal{Z} \text{ and either } \mathcal{Z} = \mathbb{R}\times\mathcal{L}, \text{ or } \mathcal{Z} \text{ is}\\ \text{a hyperplane in } \mathbb{R}\times\mathcal{L} \text{ transverse to } \mathbb{R}\times\{0\}.\end{array} \tag{10.3}$$

Any $(q,r,u)\in\Gamma$ leads to the conjugation mapping $\mathrm{C}_{q,r,u}\colon \mathrm{H}\to\mathrm{H}$ given by

$$\mathrm{C}_{q,r,u}(\widehat{r},\widehat{u}) = \left(q^{-1}\widehat{r} - 2\Omega(u,\sigma_q\widehat{u}),\sigma_q\widehat{u}\right), \tag{10.4}$$

cf. Remark 6.2 and the lines after (9.5) with $\widehat{q} = 1$. This makes $\mathrm{C}_{q,r,u}$ a linear endomorphism of $\mathbb{R}\times\mathcal{E}$ and, by (9.6), for Λ and $\mathcal{Z}$ as in the lines before (10.3),

$$\begin{array}{l}\text{the spectrum of } \mathrm{C}_{q,r,u} \text{ consists of } q^{-1} \text{ and the spectrum of } \sigma_q,\\ \text{while } \mathrm{C}_{q,r,u}(\Sigma) = \Sigma, \text{ so that } \sigma_q(\Lambda) = \Lambda \text{ and } \mathrm{C}_{q,r,u}(\mathcal{Z}) = \mathcal{Z}.\end{array} \tag{10.5}$$

From (10.5), due to (9.4), (9.3), and the denseness conclusion in (10.2),

$$\sigma_q(\Lambda) = \Lambda \text{ and } \sigma_q(\mathcal{L}) = \mathcal{L} \text{ for every } q \in (0,\infty). \tag{10.6}$$

It follows that $\operatorname{rank}\Sigma \le 1$. Namely, by (10.5) and (10.4), each $\mathrm{C}_{q,r,u} \in \mathfrak{gl}(\mathbb{R}\times\mathcal{E})$ leaves invariant the subspaces $\mathcal{Z}$ and $\mathbb{R}\times\{0\}$, as well as $\mathcal{Z}' = \mathcal{Z}\cap(\mathbb{R}\times\{0\})$, so that it descends to a linear endomorphism of the quotient $\mathcal{Z}/\mathcal{Z}'$ and, in view of (10.4), the isomorphism $\mathcal{Z}/\mathcal{Z}' \to \mathcal{L}$ induced by the projection $(r,u) \mapsto u$ makes the latter endomorphism correspond to $\zeta_q : \mathcal{L} \to \mathcal{L}$ arising as the restriction of σ_q to $\mathcal{L}$. The spectrum of $\mathrm{C}_{q,r,u}$ acting on $\mathcal{Z}$ thus equals the spectrum of ζ_q in $\mathcal{L}$, augmented – only when $\mathcal{Z} = \mathbb{R}\times\mathcal{L}$ in (10.3) – by the eigenvalue q^{-1}.

At the same time, the infinitesimal generator of the Lie-group homomorphism $(0,\infty) \ni q \mapsto \zeta_q \in \mathrm{GL}(\mathcal{L})$, cf. (10.4), being the restriction to $\mathcal{L}$ of B appearing in (9.7), has the spectrum $\alpha(1),\dots,\alpha(k)$ which is a part of that in (9.7). For reasons stated three lines after (3.1), ζ_q then must have spectrum $q^{\alpha(1)},\dots,q^{\alpha(k)}$. Thus, the spectrum of $C_{q,r,u} : \mathcal{Z} \to \mathcal{Z}$ consists of

$$q^{\alpha(1)},\dots,q^{\alpha(k)} \text{ and, possibly, } q^{-1}, \tag{10.7}$$

which are complex powers of q with exponents not depending on q, so that, as $\mathrm{C}_{q,r,u}(\Sigma) = \Sigma$ in (10.5), the trace of $\mathrm{C}_{q,r,u} : \mathcal{Z} \to \mathcal{Z}$, being integer-valued and continuous in q, must be constant. By (3.2), the eigenvalues (10.7) are all equal to 1, which excludes q^{-1} and, due to continuity in q, gives $\alpha(1) = \ldots = \alpha(k) = 0$. From Lemma 9.3(a), it now follows that $k = \operatorname{rank}\Sigma \le 1$.

However, the conclusion that $\operatorname{rank}\Sigma \le 1$ further implies that Γ is Abelian. Indeed, when Σ is trivial, injectivity of (10.1) makes Γ isomorphic to a subgroup of $(0,\infty)$, while if $\operatorname{rank}\Sigma = 1$, we fix a generator $(1,b,w)$ of Σ, noting that (10.2) gives $w \ne 0$ and $\sigma_q w = w$ for every $q \in (0,\infty)$. Thus, the commutator $[\gamma,\widehat{\gamma}] \in \Sigma$ of any two elements $\gamma = (q,r,u)$ and $\widehat{\gamma} = (\widehat{q},\widehat{r},\widehat{u})$ in Γ equals $(1,\ell b,\ell w)$, for some $\ell \in \mathbb{Z}$. By (9.5-iii), $(\sigma_q - 1)\widehat{u} - (\sigma_{\widehat{q}} - 1)u = \ell w \in \mathcal{E}_0 \cap \mathcal{E}_+ = \{0\}$, cf. Lemma 9.3(a). Consequently, $\ell = 0$ and $[\gamma,\widehat{\gamma}] = (1,0,0)$ as required.

Now comes the contradiction: if Γ were Abelian, Lemmas 8.1(b) and 9.4 would give $\Gamma \subseteq \mathrm{H}$ or $\Gamma \subseteq K$ for some subgroup K of G_0 isomorphic to $\mathbb{R}^k$, $k \in \{1,2\}$, acting freely on $\widehat{M}$. The former case leads to (6.9), while the latter contradicts Lemma 2.2 as $\dim \widehat{M} > 2$.

11. The four-dimensional case: proof of Corollary F

Assume, on the contrary, that there exists a four-dimensional compact rank-one ECS manifold (M,g), and let Γ be its fundamental group. As (M,g) is generic – see the very end of Section 5 – it must be translational by Theorem E, while Corollary D allows us to identify its universal covering with a model $(\widehat{M},\widehat{\mathrm{g}})$ as in (6.1)–(6.2), so that $\Gamma \subseteq \mathrm{Iso}(\widehat{M},\widehat{\mathrm{g}})$ and $\widehat{M}/\Gamma = M$. Applying (5.4) if necessary, we use Remark 6.4 to conclude that $I = \mathbb{R}$ in (6.1) and that every $\gamma \in \Gamma$ has the form $\gamma = (1,p,\mathrm{Id},r,u)$, with $p \in \mathbb{R}$ and $(r,u) \in \mathrm{H}$ (cf. Theorem 6.1). Here, the $\mathrm{O}(V)$-component of γ is assumed to be trivial due to genericity combined with (5.4). The image P of the homomorphism $\Gamma \ni \gamma \mapsto p \in \mathbb{R}$ is infinite cyclic as its being dense (or, trivial) would imply constancy of f via the last condition in (6.6) (or, lead to (6.9)). While $\mathrm{G} = \{(1,p,\mathrm{Id},r,u) \in \mathrm{Iso}(\widehat{M},\widehat{\mathrm{g}}) : p \in \mathrm{P} \text{ and } (r,u) \in \mathrm{H}\}$ contains Γ as a subgroup, no subgroup of G can act on $\widehat{M}$ freely and properly discontinuously with a compact quotient, as shown in [**11**, Theorem 8.1].

Appendix A. How [9, Lemma 7.3] leads to Theorem A

With u in [**9**, Lemma 7.3] being $\mathbf{w}$ in (5.1), we fix $y\colon I \to \widetilde{M}$ parametrized by t (see [**9**, Remark 7.2]) and consider the differential equation

$$\nabla_{\dot y}\nabla_{\dot y}\mathbf{z} + R(\dot y, \mathbf{z})\dot y + \nabla_{\dot y}\dot y = -\frac{Q(\mathbf{z})\mathbf{w}}{4}, \tag{A.1}$$

where $Q(\mathbf{z}) = 3\langle A(\mathbf{z}+\widetilde{\mathcal{D}}), \mathbf{z}+\widetilde{\mathcal{D}}\rangle^{\boldsymbol{\cdot}} + 3f\widetilde{\mathrm{g}}(\mathbf{z},\mathbf{z})^{\boldsymbol{\cdot}} + 2\dot f\widetilde{\mathrm{g}}(\mathbf{z},\mathbf{z})$, imposed on sections $\mathbf{z}$ of $\widetilde{\mathcal{D}}^\perp$ along y. Here $\langle\cdot,\cdot\rangle$ stands for the inner product on the vector space V of parallel sections of $\widetilde{\mathcal{D}}^\perp/\widetilde{\mathcal{D}}$. For a maximal solution $\mathbf{z}$ of (A.1), we set $x(t,s) = \exp_{y(t)} s\mathbf{z}(t)$, and observe that $\mathbf{z}$ and x are now defined on I and $I\times\mathbb{R}$ instead of $\mathbb{R}$ and $\mathbb{R}^2$ as in [**9**]: namely, a solution $\mathbf{z}_0$ of $\nabla_{\dot y}\nabla_{\dot y}\mathbf{z} + R(\dot y,\mathbf{z})\dot y + \nabla_{\dot y}\dot y = 0$ is clearly defined on all of I, while for a function $\mu\colon I\to\mathbb{R}$ with $\ddot\mu = Q(\mathbf{z}_0)/4$, $\mathbf{z} = \mathbf{z}_0 - \mu\mathbf{w}$ is a solution to (A.1). With the subscript convention referred to in (1.2), the vector field $\mathbf{v}$ along x having $\mathbf{v}_s = 0$ for all (t,s) and $\mathbf{v} = \nabla_{\dot y}\dot y$ for $s=0$ satisfies the conditions $x_{tt} + (s-1)(\mathbf{v} - Q(x_s)\mathbf{w}/4) = 0$ and $[Q(x_s)]_s = 0$: they are precisely [**9**, formula (9)], being established here by the argument given there repeated *verbatim*. It follows that $I \ni t \mapsto x(t,1)\in\widetilde{M}$ is a maximal geodesic, which can be chosen to realize any t-normalized initial velocity.

Appendix B. From [9, Theorem 7.1] to Theorem C

Due to maximal completeness and (5.2), we may fix a maximal null geodesic $x\colon I\to\widetilde{M}$ parametrized by t (cf. [**9**, Remark 7.2]), and define a parallel field Π of Lorentzian planes along x by $\Pi_t = \mathbb{R}\dot x(t)\oplus\widetilde{\mathcal{D}}_{x(t)}$. As $\widetilde{\mathcal{D}}^\perp_{x(t)} = \Pi_t^\perp\oplus\widetilde{\mathcal{D}}_{x(t)}$ and V is the space of parallel sections of the quotient bundle $\widetilde{\mathcal{D}}^\perp/\widetilde{\mathcal{D}}$, for every $t\in I$ we have an isomorphism $V\to\widetilde{\mathcal{D}}^\perp_{x(t)}/\widetilde{\mathcal{D}}_{x(t)}\to\Pi_t^\perp$, the image of $v\in V$ under which will be denoted by $v(t)$. Therefore, for each $t\in I$ and $\mathbf{w}$ as in (5.1),

$$\mathbb{R}\times V \ni (s,v)\mapsto v(t) + \frac{s\mathbf{w}_{x(t)}}{2} \in \widetilde{\mathcal{D}}^\perp_{x(t)} \text{ is an obvious isomorphism.} \tag{B.1}$$

By [**10**, Lemma 5.1], $F\colon\widehat{M}\to\widetilde{M}$ given by $F(t,s,v) = \exp_{x(t)}(v(t)+s\mathbf{w}_{x(t)}/2)$ has $F^*\widetilde{\mathrm{g}} = \widehat{\mathrm{g}}$, and so, due to the italicized statement following (5.2), $F_*\widehat{\mathcal{D}} = \widetilde{\mathcal{D}}$. As the leaves of $\widetilde{\mathcal{D}}^\perp$ are simply connected – see [**15**, Theorem B] – and their induced connections are flat (Remark 6.3), each slice $\{t\}\times\mathbb{R}\times V$ is diffeomorphically mapped under F onto the leaf of $\widetilde{\mathcal{D}}^\perp$ passing through $x(t)$, so that injectivity of (B.1) yields the one of F. Finally, the definition of maximal completeness and surjectivity of (B.1) imply that F is surjective as well. Hence, F is a global isometry.

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