Conformal flatness of compact three-dimensional Cotton-parallel manifolds

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ABSTRACT. A three-dimensional pseudo-Riemannian manifold is called essentially conformally symmetric (ECS) if its Cotton tensor is parallel but nowhere-vanishing. In this note we prove that three-dimensional ECS manifolds must be noncompact or, equivalently, that every compact three-dimensional Cotton-parallel pseudo-Riemannian manifold must be conformally flat.

1. Introduction and main result

Pseudo-Riemannian manifolds of dimensions $n \geq 4$ whose Weyl tensor is parallel are called *conformally symmetric* [**3**]. Those which are not locally symmetric or conformally flat are called *essentially conformally symmetric* (*ECS*, in short).

It has been shown by Roter in [**9**, Corollary 3] that ECS manifolds do exist in all dimensions $n \geq 4$, and in [**4**, Theorem 2] that they necessarily have indefinite metric signature. The local isometry types of ECS manifolds were described by Derdzinski and Roter in [**6**]. Compact ECS manifolds exist in all dimensions $n \geq 5$ and realize all indefinite metric signatures – see [**8**] and [**7**]. It is not currently known if compact four-dimensional ECS manifolds exist.

When the dimension of M is $n \leq 3$, the Weyl tensor vanishes and this discussion becomes meaningless. In dimension $n = 3$, however, conformal flatness is encoded in the Cotton tensor as opposed to the Weyl tensor, and so the following natural definition has been proposed in [**1**]: a three-dimensional pseudo-Riemannian manifold is called *conformally symmetric* if its Cotton tensor is parallel, and those which are not conformally flat are then called *ECS* (note that every three-dimensional locally symmetric manifold is conformally flat). There, it is also

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shown [**1**, Theorem 1] that, reversing the metric if needed, any point in a three-dimensional ECS manifold has a neighborhood isometric to an open subset of

$$(\widehat{M},\widehat{\mathrm{g}}) = \big(\mathbb{R}^3, (x^3+\mathfrak{a}(t)x)\,\mathrm{d}t^2+\mathrm{d}t\,\mathrm{d}s+\mathrm{d}x^2\big), \tag{1.1}$$

for some suitable smooth function $\mathfrak{a}\colon \mathbb{R}\to\mathbb{R}$. The coordinates t and s of $\widehat{M}$ are called y and t in [**1**], respectively, but have been renamed here as to make (1.1) directly resemble the corresponding local model given in [**6**, Section 4] for $n \geq 4$.

The pursuit of compact three-dimensional ECS manifolds quickly comes to an end in view of the following result, interesting on its own right without reference to ECS geometry:

THEOREM A. *A compact three-dimensional pseudo-Riemannian manifold with parallel Cotton tensor must be conformally flat.*

While the compactness assumption here is crucial, Theorem A may be seen as a close relative (in general signature) of [**2**, Theorem 1]: compact Riemannian *Cotton solitons* are conformally flat, but nontrivial compact Lorentzian ones do exist.

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2. Preliminaries

Throughout this paper, we work in the smooth category and all manifolds considered are connected.

2.1. Symmetries of the Cotton tensor. The *Cotton tensor* of a n-dimensional pseudo-Riemannian manifold (M,g) is the three-times covariant tensor field C on M defined by

$$\mathrm{C}(X,Y,Z) = (\nabla_X P)(Y,Z) - (\nabla_Y P)(X,Z), \quad \text{for } X,Y,Z\in\mathfrak{X}(M). \tag{2.1}$$

Here, P is the *Schouten tensor* of (M,g), given by

$$P = \mathrm{Ric} - \frac{\mathrm{s}}{2(n-1)}\mathrm{g}, \tag{2.2}$$

where Ric and s stand for the Ricci tensor and scalar curvature of (M,g), respectively. The Cotton tensor satisfies the following symmetries:

$$\begin{aligned} &\text{(i)}\;\; \mathrm{C}(X,Y,Z)+\mathrm{C}(Y,X,Z)=0\\ &\text{(ii)}\;\; \mathrm{C}(X,Y,Z)+\mathrm{C}(Y,Z,X)+\mathrm{C}(Z,X,Y)=0\\ &\text{(iii)}\;\; \mathrm{tr}_{\mathrm{g}}\big((X,Z)\mapsto \mathrm{C}(X,Y,Z)\big)=0 \end{aligned} \tag{2.3}$$

for all $X,Y,Z\in\mathfrak{X}(M)$. Symmetry (i) is obvious, while (ii) follows from a straightforward computation (six terms cancel in pairs), and (iii) from $\operatorname{div} P = \mathrm{d}(\mathrm{tr}_{\mathrm{g}} P)$

(which, in turn, is a consequence of the twice-contracted differential Bianchi identity $\operatorname{div}\mathrm{Ric} = \mathrm{d}s/2$).

2.2. Algebraic structure in dimension 3. A routine computation shows that

$$\text{the Ricci and Cotton tensors of (1.1) are given by } \mathrm{Ric} = -3x\,\mathrm{d}t \otimes \mathrm{d}t \text{ and } \mathrm{C} = 3\,(\mathrm{d}t \wedge \mathrm{d}x) \otimes \mathrm{d}t. \tag{2.4}$$

The expression for C motivates the following result, analogous to [**5**, Lemma 17.1]:

THEOREM 2.1. *Let $(V, \langle\cdot,\cdot\rangle)$ be a three-dimensional pseudo-Euclidean space, and* C *be a nonzero* Cotton-like tensor *on V, i.e., a three-times covariant tensor on V which formally satisfies* (2.3)*, and consider $\mathcal{D} = \{u \in V \mid \mathrm{C}(u,\cdot,\cdot) = 0\}$. Then:*

(a) *$\mathcal{D}$ consists only of null vectors, and hence* $\dim \mathcal{D} \leq 1$.

(b) $\dim \mathcal{D} = 1$ *if and only if* $\mathrm{C} = (u \wedge v) \otimes u$ *for some $u \in \mathcal{D} \smallsetminus \{0\}$ and unit $v \in \mathcal{D}^\perp$.*

(c) *In* (b)*, u is unique up to a sign, while v is unique modulo $\mathcal{D}$.*

Here, we identify $V \cong V^*$ with the aid of $\langle\cdot,\cdot\rangle$.

PROOF. For (a), assuming by contradiction the existence of a unit vector $e_1 \in \mathcal{D}$, we will show that $\mathrm{C} = 0$. Considering an orthonormal basis $\{e_1, e_2, e_3\}$ for $(V, \langle\cdot,\cdot\rangle)$ and using (2.3-i) and (2.3-ii), we see that

$$\mathrm{C}_{ijk} \text{ is only possibly nonzero when } \{i,j,k\} = \{2,3\} \text{ with } i \neq j. \tag{2.5}$$

Now $\mathrm{C}_{322} = -\mathrm{C}_{232}$ and $\mathrm{C}_{323} = -\mathrm{C}_{233}$, while $\mathrm{tr}_{\langle\cdot,\cdot\rangle}\big((w,w') \mapsto \mathrm{C}(e_j, w, w')\big) = 0$ for $j = 2$ and $j = 3$ readily yields $\mathrm{C}_{233} = 0$ and $\mathrm{C}_{322} = 0$, respectively. Hence $\mathrm{C} = 0$, as claimed. As for (b), assume that $\dim \mathcal{D} = 1$, fix a null vector $e_1 \in \mathcal{D} \smallsetminus \{0\}$, and complete it to a basis $\{e_1, e_2, e_3\}$ of V satisfying the relations

$$\langle e_1, e_2\rangle = \langle e_2, e_3\rangle = \langle e_3, e_3\rangle = 0 \quad \text{and} \quad \langle e_1, e_3\rangle = \langle e_2, e_2\rangle = (-1)^{q+1}, \tag{2.6}$$

where $q \in \{1,2\}$ is the index of $\langle\cdot,\cdot\rangle$. By the same argument as in (a), we again obtain (2.5), but this time $\mathrm{tr}_{\langle\cdot,\cdot\rangle}\big((w,w') \mapsto \mathrm{C}(e_3, w, w')\big) = 0$ reduces to $\mathrm{C}_{232} = 0$ in view of (2.6). Writing $a = \mathrm{C}_{323} \neq 0$ for the last essential component of C, it follows that $\mathrm{C} = a(e^3 \wedge e^2) \otimes e^3$, where $\{e^1, e^2, e^3\}$ is the basis of V^* dual to $\{e_1, e_2, e_3\}$. Applying the isomorphism $V \cong V^*$ and setting $u = |a|^{1/2} e_1$ and $v = \mathrm{sgn}(a) e_2$, we obtain the required expression $\mathrm{C} = (u \wedge v) \otimes u$. Conversely, it is straightforward to verify that the tensor $(u \wedge v) \otimes u$ with u null and v unit and orthogonal to u is Cotton-like with $\mathcal{D} = \mathbb{R}u$ and $\mathcal{D}^\perp = \mathbb{R}u \oplus \mathbb{R}v$. Finally, (c) is clear from (b). □

As a consequence, whenever (M, g) is a three-dimensional pseudo-Riemannian manifold, we may assign to each point $x \in M$ the kernel $\mathcal{D}_x$ of C_x in (T_xM, g_x). In the ECS case, we have that

$$\text{(2.7)} \qquad \begin{array}{l}\mathcal{D} \text{ is a smooth rank-one parallel distribution on } M\text{, which} \\ \text{contains the image of the Ricci endomorphism of } (M, \mathrm{g}).\end{array}$$

Indeed, we may note that (2.7) holds in the model (1.1) (as (2.4) gives us that $\mathcal{D}$ is spanned by the coordinate vector field ∂_s, $\widehat{\mathrm{g}}$-dual to $\mathrm{d}t$ up to a factor of 2), and invoke [**1**, Theorem 1].

3. Proof of Theorem A

In this section, we fix a compact three-dimensional ECS manifold (M, g) and its universal covering manifold $\pi\colon \widetilde{M} \to M$, which equipped with the natural pull-back metric $\widetilde{\mathrm{g}} = \pi^*\mathrm{g}$ becomes an ECS manifold. We will use the same symbols Ric, P, C, ∇, and $\mathcal{D}$ for the corresponding objects in both (M, g) and $(\widetilde{M}, \widetilde{\mathrm{g}})$. Observe that

$$\text{(3.1)} \qquad \begin{array}{l}\text{the fundamental group } \Gamma = \pi_1(M) \text{ acts properly discontinu-} \\ \text{ously on } (\widetilde{M}, \widetilde{\mathrm{g}}) \text{ by deck isometries, with quotient } \widetilde{M}/\Gamma \cong M.\end{array}$$

As $\widetilde{M}$ is simply connected, we may fix two globally defined smooth vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\mathrm{C} = (\boldsymbol{u} \wedge \boldsymbol{v}) \otimes \boldsymbol{u}$ on $\widetilde{M}$. Now, as $\mathcal{D}$ is parallel, item (c) of Theorem 2.1 gives us that

$$\text{(3.2)} \qquad \begin{array}{rl}\text{(i)} & \boldsymbol{u} \text{ is a null parallel vector field spanning } \mathcal{D}; \\ \text{(ii)} & \text{every } \gamma \in \Gamma \text{ either pushes } \boldsymbol{u} \text{ forward onto itself or onto its opposite.}\end{array}$$

Next, as the Ricci endomorphism of $(\widetilde{M}, \widetilde{\mathrm{g}})$ is self-adjoint, (2.7) allows us to write

$$\text{(3.3)} \qquad \mathrm{Ric} = -f\, \boldsymbol{u} \otimes \boldsymbol{u}, \text{ for some smooth function } f\colon \widetilde{M} \to \mathbb{R}.$$

By (3.3) and (3.2-i), $(\widetilde{M}, \widetilde{\mathrm{g}})$ is scalar-flat, and so $P = \mathrm{Ric}$. Combining this with (3.2-i) again to compute C via (2.1), we obtain that

$$\text{(3.4)} \qquad \mathrm{C} = (\boldsymbol{u} \wedge \nabla f) \otimes \boldsymbol{u}, \text{ where } \nabla f \text{ is the } \widetilde{\mathrm{g}}\text{-gradient of } f.$$

However, it follows from (3.2-ii) and (3.3) that f is Γ-invariant, and so it has a critical point due to (3.1) and compactness of M. Such a critical point is in fact a zero of C by (3.4), and therefore $\mathrm{C} = 0$. This is the desired contradiction: (M, g) must be either noncompact, or conformally flat.

References

[1] E. Calviño Louzao, E. García-Río, J. Seoane-Bascoy, and R. Vázquez-Lorenzo. Three-dimensional conformally symmetric manifolds. *Ann. Mat. Pura Appl. (4)*, 193(6):1661–1670, 2014.

[2] E. Calviño Louzao, E. García-Río, and R. Vázquez-Lorenzo. A note on compact Cotton solitons. *Classical Quantum Gravity*, 29(20):205014, 5, 2012.

[3] M. C. Chaki and B. Gupta. On conformally symmetric spaces. *Indian J. Math.*, 5:113–122, 1963.

[4] A. Derdziński and W. Roter. On conformally symmetric manifolds with metrics of indices 0 and 1. *Tensor (N.S.)*, 31(3):255–259, 1977.

[5] A. Derdzinski and W. Roter. Projectively flat surfaces, null parallel distributions, and conformally symmetric manifolds. *Tohoku Math. J. (2)*, 59(4):565–602, 2007.

[6] A. Derdzinski and W. Roter. The local structure of conformally symmetric manifolds. *Bull. Belg. Math. Soc. Simon Stevin*, 16(1):117–128, 2009.

[7] A. Derdzinski and W. Roter. Compact pseudo-Riemannian manifolds with parallel Weyl tensor. *Ann. Global Anal. Geom.*, 37(1):73–90, 2010.

[8] A. Derdzinski and I. Terek. New examples of compact Weyl-parallel manifolds. *Monatsh. Math.* (published online), DOI `https://doi.org/10.1007/s00605-023-01908-0`.

[9] W. Roter. On conformally symmetric Ricci-recurrent spaces. *Colloq. Math.*, 31:87–96, 1974.

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