Codazzi tensor fields in reductive homogeneous spaces

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Abstract. We extend the results about left-invariant Codazzi tensor fields on Lie groups equipped with left-invariant Riemannian metrics obtained by d'Atri in 1985 to the setting of reductive homogeneous spaces G/H, where the curvature of the canonical connection of second kind associated with the fixed reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ enters the picture. In particular, we show that invariant Codazzi tensor fields on a naturally reductive homogeneous space are parallel.

Introduction

Whenever M is a smooth manifold equipped with a connection ∇, a twice-covariant symmetric tensor field A on M is called a *Codazzi tensor field* if $\mathrm{d}^\nabla A = 0$, where d^∇ is the exterior derivative operator (defined with the aid of ∇) acting on tensor bundles over M, and we regard A as a T^*M-valued 1-form. When ∇ is torsionfree, A is a Codazzi tensor field if and only if

$$(\dagger) \qquad (\nabla_{\boldsymbol{X}} A)(\boldsymbol{Y}, \boldsymbol{Z}) = (\nabla_{\boldsymbol{Y}} A)(\boldsymbol{X}, \boldsymbol{Z}), \quad \text{for all } \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathfrak{X}(M),$$

which is to say that the covariant differential ∇A, a three-times covariant tensor field on M, is totally symmetric.

Codazzi tensors are ubiquitous in geometry, with the most prominent examples being the second fundamental form of a non-degenerate hypersurface in a pseudo-Riemannian manifold with constant sectional curvature (due to the Codazzi-Mainardi compatibility equation), and the Ricci or Schouten tensors of a pseudo-Riemannian manifold with harmonic curvature or harmonic Weyl curvature (due to the relations $\operatorname{div} R = \mathrm{d}^\nabla \mathrm{Ric}$ and $\operatorname{div} \mathrm{W} = \mathrm{d}^\nabla \mathrm{Sch}$). Whenever a Riemannian manifold (M, g) has constant sectional curvature K, every Codazzi tensor field locally has the form $\operatorname{Hess} f + Kf\mathrm{g}$ for some smooth function f, cf. [**4**].

2010 *Mathematics Subject Classification.* 53C30.

Key words and phrases. Homogeneous spaces · Codazzi tensors · Non-associative algebras.

Both topological and geometric consequences of the existence of a nontrivial Codazzi tensor field on a Riemannian manifold have been studied in [**4**, **7**], and the local structure of a Riemannian manifold carrying a Codazzi tensor field satisfying additional multiplicity assumptions on its spectra and eigendistributions is obtained in [**9**]. Many such results are compiled in [**3**, §16.6–§16.22], which then led to further work [**5**, **14**].

In a different and more specific direction, left-invariant Codazzi tensor fields on Lie groups equipped with left-invariant Riemannian metrics have been discussed in [**6**], with the goal of better understanding the harmonic curvature condition in this setting. New results have been recently obtained in [**1**], where it is shown that solvable Lie groups equipped with left-invariant Riemannian metrics having harmonic curvature must necessarily be Ricci-parallel.

In this paper, we extend the results in [**6**] to the more general class of invariant Codazzi tensor fields on reductive homogeneous spaces equipped with invariant Riemannian metrics. Our approach to achieve this is straightforward: once a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ for the homogeneous space G/H is fixed, we run the computations done in [**6**] in the reductive complement $\mathfrak{m}$ (a non-associative algebra) instead of in the Lie algebra $\mathfrak{g}$. However, unlike in some results in [**6**] which involve positivity and negativity of sectional and scalar curvatures, the curvatures of $(G/H, \langle \cdot, \cdot \rangle)$ are now compared with curvatures of the canonical connection of second kind associated with the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ — with its flatness when $\mathfrak{h} = \{0\}$ and $\mathfrak{m} = \mathfrak{g}$ explaining its absence in [**6**]. Full proofs are included for the sake of completeness.

Organization of the text

We work in the smooth category and all manifolds considered are connected.

In Section 1, we gather some well-known standard facts regarding reductive homogeneous spaces needed for the rest of the text, the most important ones being Nomizu's Theorem [**15**] on invariant connections and Lemma 1.1. Section 2 generalizes [**6**, Proposition 1] to Proposition 2.1: the same compatibility condition (2.3) ensures that a symmetric bilinear form on $\mathfrak{m}$ reconstructed from prescribed eigenspaces gives rise to a Codazzi tensor field on G/H.

Section 3 explores the effects of the existence of an invariant Codazzi tensor field on curvature, generalizing [**6**, Propositions 3 and 4] and expressing the new conclusions, Propositions 3.1 and 3.3, with the aid of the *difference curvature tensor* introduced in (3.1). In particular, we conclude that every invariant Codazzi tensor field on a naturally reductive homogeneous space is parallel.

Acknowledgements. We would like to thank Andrzej Derdzinski for bringing [**6**] to our attention, which ultimately motivated this work. We are also grateful to the anonymous referee, for the comments which allowed us to improve the exposition of the text.

1. Preliminaries

The material in this section is standard and it is included for the convenience of the reader. We refer to [**13**, Ch. X], [**11**, Ch. II], and [**2**, Ch. II–III] for more details.

Let G be a Lie group and H be a closed Lie subgroup of G, so that the quotient space G/H admits a unique smooth structure for which the natural projection $\pi\colon G \to G/H$ is a principal H-bundle. The group G acts transitively on G/H via the "left translations" $\tau_g\colon G/H \to G/H$ given by $\tau_g(aH) = (ga)H$. Writing $\mathfrak{g}$ and $\mathfrak{h}$ for the Lie algebras of G and H, we assume that G/H is *reductive*: there is a *vector space* direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $\mathfrak{m}$ is $\mathrm{Ad}(H)$-invariant. We write $(\cdot)_{\mathfrak{h}}\colon \mathfrak{g} \to \mathfrak{h}$ and $(\cdot)_{\mathfrak{m}}\colon \mathfrak{g} \to \mathfrak{m}$ for the direct sum projections, and so $(\mathfrak{m}, [\cdot,\cdot]_{\mathfrak{m}})$ becomes a non-associative algebra. The derivative $\mathrm{d}\pi_e$ restricts to an isomorphism $\mathfrak{m} \cong T_{eH}(G/H)$ and, in addition,

(1.1) for each $h \in H$, the derivative of $\tau_h\colon G/H \to G/H$ at the fixed point eH is nothing more than $\mathrm{Ad}(h)\colon \mathfrak{m} \to \mathfrak{m}$.

Our guiding principle is that for any G-equivariant smooth fiber bundle $E \to G/H$,

(1.2) G-equivariant sections of E are in one-to-one correspondence with points of E_{eH} fixed by H.

Indeed, any point $\phi \in E_{eH}$ which is fixed by H defines a G-equivariant section ψ of E via $\psi_{gH} = g \cdot \phi$. For example, taking E to be tensor powers of $T^*(G/H)$ gives us that G-invariant covariant tensor fields on G/H are in one-to-one correspondence with $\mathrm{Ad}(H)$-invariant covariant tensors on $\mathfrak{m}$, cf. [**2**, Proposition 5.1], while taking E to be Grassmannian bundles over G/H yields that G-invariant distributions on G/H are in one-to-one correspondence with $\mathrm{Ad}(H)$-invariant vector subspaces of $\mathfrak{m}$. In addition, it has been proved in [**17**] that

(1.3) a G-invariant distribution $\mathcal{P}$ on G/H is involutive if and only if the subspace $\mathcal{P}_{eH}$ is closed under $[\cdot,\cdot]_{\mathfrak{m}}$.

We will also need Nomizu's theorem [**15**, Theorem 8.1]:

(1.4) G-invariant affine connections on G/H are in one-to-one correspondence with $\mathrm{Ad}(H)$-equivariant multiplications $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$.

Following [**8**, Section 5.2], a G-invariant connection ∇ on G/H and an $\mathrm{Ad}(H)$-equivariant multiplication α in $\mathfrak{m}$ related via (1.4) determine each other by the relation

$$\alpha(X,Y) = (\nabla_{X^\#}Y^\#)|_{eH} + [X,Y]_{\mathfrak{m}}, \quad \text{for all } X,Y \in \mathfrak{m}. \tag{1.5}$$

Here, we are using that every $X \in \mathfrak{g}$ determines its corresponding action field $X^\# \in \mathfrak{X}(G/H)$, with $X^\#_{eH} = X_{\mathfrak{m}}$ and whose complete flow is explicitly given by $(t, aH) \mapsto \tau_{\exp(tX)}(aH)$. Note that the right-invariant vector field on G generated by X is π-related to $X^\#$. For future reference, we also observe that this implies that

$$\mathcal{L}_{X^\#}\Theta = 0 \text{ for every } X \in \mathfrak{g} \text{ and } G\text{-invariant tensor field } \Theta \text{ on } G/H, \tag{1.6}$$

as the flow of $X^\#$ leaves Θ invariant. The torsion and curvature of ∇ are given in $\mathfrak{m}$ in terms of α by

$$\begin{array}{rl} \text{i)} & T(X,Y) = \alpha(X,Y) - \alpha(Y,X) - [X,Y]_{\mathfrak{m}}, \\ \text{ii)} & R(X,Y)Z = \alpha(X,\alpha(Y,Z)) - \alpha(Y,\alpha(X,Z)) - \alpha([X,Y]_{\mathfrak{m}},Z) - [[X,Y]_{\mathfrak{h}},Z], \end{array} \tag{1.7}$$

for all $X,Y,Z \in \mathfrak{m}$, cf. [**15**, formulas (9.1) and (9.6)] or [**8**, formula (22)].

LEMMA 1.1. *For a G-invariant connection ∇ and a G-invariant k-times covariant tensor field Θ on G/H, corresponding to α and θ on $\mathfrak{m}$ under* (1.4)–(1.5) *and* (1.2)*, the covariant differential $\nabla\Theta$ is also G-invariant and corresponds under* (1.2) *to $\alpha(\cdot,\theta)$ on $\mathfrak{m}$ given by*

$$\alpha(X,\theta)(Y_1,\dots,Y_k) = -\sum_{i=1}^{k}\theta(Y_1,\dots,\alpha(X,Y_i),\dots,Y_k) \tag{1.8}$$

for all $X, Y_1, \dots, Y_k \in \mathfrak{m}$.

PROOF. We will establish (1.8) when $k = 1$, with the general case being an exercise in notation. The identity $(\nabla_{\boldsymbol{X}}\Theta)(\boldsymbol{Y}) = (\mathcal{L}_{\boldsymbol{X}}\Theta)(\boldsymbol{Y}) - \Theta(\nabla_{\boldsymbol{X}}\boldsymbol{Y} - [\boldsymbol{X},\boldsymbol{Y}])$ evaluated at the vector fields $\boldsymbol{X} = X^\#$ and $\boldsymbol{Y} = Y^\#$, with $X,Y \in \mathfrak{m}$, reads as $(\nabla_{X^\#}\Theta)(Y^\#) = -\Theta(\nabla_{X^\#}Y^\# - [X^\#,Y^\#])$ due to (1.6). As evaluating the relation $[X^\#,Y^\#] = -[X,Y]^\#$ at eH yields $[X^\#,Y^\#]_{eH} = -[X,Y]_{\mathfrak{m}}$, (1.8) follows from (1.5). □

Lastly, whenever G/H is equipped with a G-invariant pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$, α corresponding to the Levi-Civita connection under (1.4)–(1.5) is called the *Levi-Civita product* of $\langle\cdot,\cdot\rangle$. The Koszul formula for α becomes

$$2\langle\alpha(X,Y),Z\rangle = \langle[X,Y]_{\mathfrak{m}},Z\rangle - \langle X,[Y,Z]_{\mathfrak{m}}\rangle - \langle[X,Z]_{\mathfrak{m}},Y\rangle, \tag{1.9}$$

for all $X,Y,Z \in \mathfrak{m}$, cf. [**8**, Exercise 10].

2. The Codazzi compatibility condition in $\mathfrak{m}$

In this section, let G/H be a homogeneous space admitting a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be equipped with a G-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ and its Levi-Civita product α. By Lemma 1.1 and (†) in the Introduction, a twice-covariant G-invariant symmetric tensor field A on G/H is Codazzi if and only if

$$\alpha(X, A)(Y, Z) = \alpha(Y, A)(X, Z) \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{m}$. As A is symmetric and g is positive-definite, the spectral theorem allows us to write an orthogonal direct sum decomposition

(2.2) $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$, where $r \geq 1$ and each $\mathfrak{m}_i$ is the eigenspace of A associated with the eigenvalue λ_i, ordered so that $\lambda_1 < \cdots < \lambda_r$.

We will also write $(\cdot)_i \colon \mathfrak{m} \to \mathfrak{m}_i$ for the corresponding direct sum projections.

A subalgebra of $(\mathfrak{m}, [\cdot, \cdot]_{\mathfrak{m}})$ is called *totally geodesic* if it is closed under α. By (1.3) and (1.5), an $\mathrm{Ad}(H)$-invariant totally geodesic subalgebra of $(\mathfrak{m}, [\cdot, \cdot]_{\mathfrak{m}})$ determines a foliation of G/H by totally geodesic submanifolds. The next result generalizes [**6**, Proposition 1].

PROPOSITION 2.1. *Whenever A is a G-invariant Codazzi tensor field on G/H, all the factors in decomposition* (2.2) *are* $\mathrm{Ad}(H)$*-invariant totally geodesic subalgebras of* $(\mathfrak{m}, [\cdot, \cdot]_{\mathfrak{m}})$*, and the compatibility condition*

$$(\lambda_i - \lambda_k)^2 \big\langle [X_i, Y_j]_{\mathfrak{m}}, Z_k \big\rangle + (\lambda_j - \lambda_i)^2 \big\langle [X_i, Z_k]_{\mathfrak{m}}, Y_j \big\rangle = 0 \tag{2.3}$$

holds for all $X, Y, Z \in \mathfrak{m}$ and $i, j, k \in \{1, \ldots, r\}$. Conversely, if a direct sum decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$ into mutually orthogonal $\mathrm{Ad}(H)$*-invariant vector subspaces is given and* (2.3) *holds, any choice of mutually distinct real constants $\lambda_1, \ldots, \lambda_r$ gives rise to a G-invariant Codazzi tensor field on G/H via $A = \bigoplus_{i=1}^{n} \lambda_i \langle \cdot, \cdot \rangle|_{\mathfrak{m}_i \times \mathfrak{m}_i}$. In addition, $\nabla A \neq 0$ if and only if there exists a triple (i, j, k) of mutually distinct indices with $\langle X_i, [Y_j, Z_k]_{\mathfrak{m}} \rangle \neq 0$, in which case A has at least three distinct eigenvalues.*

PROOF. That each $\mathfrak{m}_i$ is $\mathrm{Ad}(H)$-invariant follows from $\mathrm{Ad}(H)$-invariance of both A and $\langle \cdot, \cdot \rangle$. Namely, if $X \in \mathfrak{m}_i$, $h \in H$, and $Y \in \mathfrak{m}$, we have

$$A(\mathrm{Ad}(h)X, Y) = A(X, \mathrm{Ad}(h^{-1})Y) = \lambda_i \langle X, \mathrm{Ad}(h^{-1})Y \rangle = \lambda_i \langle \mathrm{Ad}(h)X, Y \rangle,$$

so that $\mathrm{Ad}(h)X \in \mathfrak{m}_i$. Next, as (1.9) is manifestly skew-symmetric in the pair (Y, Z), we see that $\alpha(X, \cdot) \in \mathfrak{so}(\mathfrak{m}, \langle \cdot, \cdot \rangle)$ for every $X \in \mathfrak{m}$, from which the relation

$$-\alpha(Z_k, A)(X_i, Y_j) = (\lambda_i - \lambda_j) \big\langle X_i, \alpha(Z_k, Y_j) \big\rangle \tag{2.4}$$

follows for all $X, Y, Z \in \mathfrak{m}$. The Codazzi condition (2.1) now reads

$$(\lambda_i - \lambda_j) \big\langle X_i, \alpha(Z_k, Y_j) \big\rangle = (\lambda_k - \lambda_j) \big\langle Z_k, \alpha(X_i, Y_j) \big\rangle. \tag{2.5}$$

Using (1.9) twice and rearranging terms, (2.5) becomes

$$\begin{aligned}(\lambda_i-\lambda_k)\big\langle[X_i,Y_j]_{\mathfrak m},Z_k\big\rangle+&\\ +(\lambda_i-\lambda_k)\big\langle[Z_k,Y_j]_{\mathfrak m},X_i\big\rangle&+(\lambda_i+\lambda_k-2\lambda_j)\big\langle[X_i,Z_k]_{\mathfrak m},Y_j\big\rangle=0.\end{aligned}\tag{2.6}$$

Permuting elements, we also have

$$\begin{aligned}(\lambda_j-\lambda_i)\big\langle[Y_j,Z_k]_{\mathfrak m},X_i\big\rangle+&\\ +(\lambda_j-\lambda_i)\big\langle[X_i,Z_k]_{\mathfrak m},Y_j\big\rangle&+(\lambda_j+\lambda_i-2\lambda_k)\big\langle[Y_j,X_i]_{\mathfrak m},Z_k\big\rangle=0,\end{aligned}\tag{2.7}$$

and so $(\lambda_j-\lambda_i)(2.6)+(\lambda_i-\lambda_k)(2.7)=0$ becomes precisely (2.3). Making $i=j\neq k$ on (2.3) leads to $[X_i,Y_i]_{\mathfrak m}\in\mathfrak m_k^\perp$ for all $k\neq i$, so that $[X_i,Y_i]_{\mathfrak m}\in\mathfrak m_i$. Then, making $j=k\neq i$ on (2.3) gives us that $\big\langle[X_i,Y_j]_{\mathfrak m},Z_j\big\rangle+\big\langle[X_i,Z_j]_{\mathfrak m},Y_j\big\rangle=0$, which combined with (1.9) implies that each $\mathfrak m_i$ is closed under α.

Conversely, to verify that $A=\bigoplus_{i=1}^r\lambda_i\langle\cdot,\cdot\rangle|_{\mathfrak m_i\times\mathfrak m_i}$ defines a Codazzi tensor field whenever (2.3) holds, it suffices to note that it implies (2.6) (and hence (2.5), due to (1.9)). Indeed: (2.3) becomes (2.6) when $i=k\neq j$ while, if $i\neq j$, adding to (2.3) the expression obtained from it after permuting $(i,j,k)\mapsto(j,k,i)$ yields (2.7) (and hence (2.6)).

Finally, (2.3) also implies

$$\begin{aligned}&\text{i)}\ \big\langle[X_i,Z_k]_{\mathfrak m},Y_j\big\rangle=-\frac{(\lambda_i-\lambda_k)^2}{(\lambda_j-\lambda_i)^2}\big\langle[X_i,Y_j]_{\mathfrak m},Z_k\big\rangle,\\ &\text{ii)}\ \big\langle X_i,[Y_j,Z_k]_{\mathfrak m}\big\rangle=\frac{(\lambda_j-\lambda_k)^2}{(\lambda_j-\lambda_i)^2}\big\langle[X_i,Y_j]_{\mathfrak m},Z_k\big\rangle,\end{aligned}\tag{2.8}$$

whenever $i\neq j$. Substituting (2.8) into (1.9) and simplifying it with the aid of (2.4), we obtain

$$\big\langle\alpha(X_i,Y_j),Z_k\big\rangle=\frac{\lambda_i-\lambda_k}{\lambda_i-\lambda_j}\big\langle[X_i,Y_j]_{\mathfrak m},Z_k\big\rangle,\qquad i\neq j,\tag{2.9}$$

which directly implies the last assertions regarding ∇A. □

REMARK 2.2. The use of the spectral theorem to obtain (2.2) relies crucially on positive-definiteness of the Riemannian metric $\langle\cdot,\cdot\rangle$. When $\langle\cdot,\cdot\rangle$ has indefinite metric signature, we have *Milnor's indefinite spectral theorem* [**10**, p. 256]:

(2.10) if a self-adjoint endomorphism T of a pseudo-Euclidean space $(V,\langle\cdot,\cdot\rangle)$ with $\dim V\geq 3$ satisfies that $\langle Tv,v\rangle\neq 0$ for every null $v\in V\smallsetminus\{0\}$, then T is diagonalizable in an orthonormal basis of V.

To justify (2.10), it suffices to choose $\Phi=\langle T\cdot,\cdot\rangle$ and $\Psi=\langle\cdot,\cdot\rangle$ in the notation of [**10**, p. 256]. With (2.10) in place, we see that A gives rise to (2.2) and satisfies (2.3) even when $\langle\cdot,\cdot\rangle$ has indefinite metric signature, provided that $\dim\mathfrak m\geq 3$ and $A(X,X)\neq 0$ whenever $X\in\mathfrak m\smallsetminus\{0\}$ is null. On the other hand, that (2.2) and

(2.3) together give rise to G-invariant Codazzi tensor fields on G/H remains true without any additional assumptions.

As pointed out in [**6**], there is a simple interpretation for the compatibility relation (2.3). For each $k \in \{1, \dots, r\}$, considering the inner product $\langle\langle \cdot, \cdot \rangle\rangle_k$ on $\mathfrak{m}$ defined by[1]

$$\langle\langle X, Y \rangle\rangle_k = \sum_{j=1}^{r} (\delta_{jk} + (\lambda_j - \lambda_k)^2) \langle X_j, Y_j \rangle, \qquad X, Y \in \mathfrak{m},$$

it follows that $\langle\langle [Z_k, X]_{\mathfrak{m}}, Y \rangle\rangle_k + \langle\langle X, [Z_k, Y]_{\mathfrak{m}} \rangle\rangle_k = 0$ for all $Z \in \mathfrak{m}_k$ and $X, Y \in \mathfrak{m}_k^{\perp}$. Indeed, it suffices to apply (2.3), assuming that $X \in \mathfrak{m}_i$ and $Y \in \mathfrak{m}_j$ with $i, j \neq k$. This means that, writing $\mathrm{ad}_{\mathfrak{m}}(X)(Y) = [X, Y]_{\mathfrak{m}}$ for every $X, Y \in \mathfrak{m}$ and denoting by $\pi_k^{\perp}$ the projection of $\mathfrak{m}$ onto $\mathfrak{m}_k^{\perp}$, the composition $(\pi_k^{\perp} \circ \mathrm{ad}_{\mathfrak{m}})|_{\mathfrak{m}_k}$ is a representation of $\mathfrak{m}_k$ on $(\mathfrak{m}_k^{\perp}, \langle\langle \cdot, \cdot \rangle\rangle_k)$ by skew-adjoint operators. Here, the representation is a representation of the vector space $\mathfrak{m}_k$, not of the non-associative algebra $(\mathfrak{m}_k, [\cdot, \cdot]_k)$. As a consequence:

(2.11) for each $Z_k \in \mathfrak{m}_k$, the chararacteristic roots of the operator $\pi_k^{\perp} \circ \mathrm{ad}_{\mathfrak{m}}(Z_k)|_{\mathfrak{m}_k^{\perp}}$ are all purely imaginary.

Recall that a non-associative algebra $\mathfrak{a}$ is:

(a) *nilpotent* [**16**, p. 18] if there is a positive integer t such that the product of t elements in $\mathfrak{a}$, no matter how associated, equals zero.

(b) *split-solvable* (cf. [**12**, p. 21]) if there is a sequence $\mathfrak{a} = \mathfrak{a}_0 \supseteq \cdots \supseteq \mathfrak{a}_p = 0$ of ideals of $\mathfrak{a}$ with $\dim(\mathfrak{a}_i/\mathfrak{a}_{i+1}) = 1$ for every $i = 0, \dots, p-1$.

Following [**6**], we call a G-invariant Codazzi tensor field A on G/H *essential* if $\nabla A \neq 0$ and none of the eigenspaces $\mathfrak{m}_i$ is an ideal of $(\mathfrak{m}, [\cdot, \cdot]_{\mathfrak{m}})$. Note that $\mathfrak{m}_k$ is an ideal of $(\mathfrak{m}, [\cdot, \cdot]_{\mathfrak{m}})$ if and only if $\pi_k^{\perp} \circ \mathrm{ad}_{\mathfrak{m}}(Z_k)|_{\mathfrak{m}_k^{\perp}} = 0$ for every $Z_k \in \mathfrak{m}_k$. Using the above, we obtain:

PROPOSITION 2.3. *If G/H has a G-invariant essential Codazzi tensor field A, then $(\mathfrak{m}, [\cdot, \cdot]_{\mathfrak{m}})$ cannot be nilpotent or split-solvable.*

PROOF. As in the Lie category, one may define a 'Killing form' β for $(\mathfrak{m}, [\cdot, \cdot]_{\mathfrak{m}})$ via $\beta(X, Y) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{m}}(X) \circ \mathrm{ad}_{\mathfrak{m}}(Y))$ for all $X, Y \in \mathfrak{m}$. A direct computation shows that, for every $Z_k \in \mathfrak{m}_k$, the relation

$$\beta(Z_k, Z_k) = \beta_k(Z_k, Z_k) + \mathrm{tr}\left[(\pi_k^{\perp} \circ \mathrm{ad}_{\mathfrak{m}}(Z_k)|_{\mathfrak{m}_k^{\perp}})^2\right] \tag{2.12}$$

holds, where β_k stands for the Killing form of $(\mathfrak{m}_k, [\cdot, \cdot]_k)$.

[1]Beware of the typo in [**6**, formula (7)]: the formula there has $\langle X, Y \rangle$ instead of $\langle X_j, Y_j \rangle$.

Let $Z_k \in \mathfrak{m}_k$ be arbitrary, and assume that $(\mathfrak{m}, [\cdot,\cdot]_{\mathfrak{m}})$ is nilpotent. It follows that both operators $\mathrm{ad}_{\mathfrak{m}}(Z_k)$ and $\mathrm{ad}_{\mathfrak{m}}(Z_k)|_{\mathfrak{m}_k}$ are nilpotent, and so both $\beta(Z_k, Z_k)$ and $\beta_k(Z_k, Z_k)$ vanish. In particular, (2.12) leads to $\mathrm{tr}\left[(\pi_k^{\perp} \circ \mathrm{ad}_{\mathfrak{m}}(Z_k)|_{\mathfrak{m}_k^{\perp}})^2\right] = 0$. Together with (2.11), this implies that $\pi_k^{\perp} \circ \mathrm{ad}_{\mathfrak{m}}(Z_k)|_{\mathfrak{m}_k^{\perp}} = 0$.

Now, assume instead that $(\mathfrak{m}, [\cdot,\cdot]_{\mathfrak{m}})$ is split-solvable. By [**12**, Corollary 1.30], whose 'necessity' implication does not rely on the Jacobi identity, the characteristic roots of each $\mathrm{ad}_{\mathfrak{m}}(Z_k)$, for $Z_k \in \mathfrak{m}_k$, are real. Combined with (2.11), it follows that $\pi_k^{\perp} \circ \mathrm{ad}_{\mathfrak{m}}(Z_k)|_{\mathfrak{m}_k^{\perp}} = 0$ yet again. □

3. Codazzi tensors versus difference curvatures

In this section, we continue to work with a homogeneous space G/H equipped with a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, G-invariant Riemannian metric $\langle \cdot,\cdot \rangle$, and Levi-Civita product α.

We will also need the *canonical connection of second kind* induced by given reductive decomposition, that is, the affine connection ∇^0 on G/H corresponding under (1.4)–(1.5) to the zero product in $\mathfrak{m}$. By (1.7-ii), the curvature tensor R^0 of ∇^0 is given simply by $R^0(X,Y)Z = -[[X,Y]_{\mathfrak{h}}, Z]$, for all $X, Y, Z \in \mathfrak{m}$. It follows from the Jacobi identity

$$\sum_{\mathrm{cyc}} [[X,Y]_{\mathfrak{h}}, Z] + \sum_{\mathrm{cyc}} [[X,Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} = 0, \qquad X, Y, Z \in \mathfrak{m},$$

and $\mathrm{Ad}(H)$-invariance of $\langle \cdot,\cdot \rangle$ that:

i) $(\mathfrak{m}, [\cdot,\cdot]_{\mathfrak{m}})$ is a Lie algebra if and only if R^0 satisfies the Bianchi identity,
ii) the expression $\langle R^0(X,Y)Z, W\rangle$ is skew-symmetric in the pair (Z, W).

The *Ricci tensor* Ric^0 of ∇^0 is defined by $\mathrm{Ric}^0(Y,Z) = \mathrm{tr}(X \mapsto R^0(X,Y)Z)$, with no reference to the metric $\langle \cdot,\cdot \rangle$, and it is only guaranteed to be symmetric if R^0 satisfies the Bianchi identity. We also consider the *sectional* and *scalar curvature functions* K^0 and s^0 associated with ∇^0 and $\langle \cdot,\cdot \rangle$: for any plane $\Pi \subseteq \mathfrak{m}$ we let $K^0(\Pi) = \langle R^0(X,Y)Y, X\rangle$, where $\{X, Y\}$ is any orthonormal basis for Π (with its choice being immaterial due to (ii) above), and $\mathrm{s}^0 = \mathrm{tr}_{\langle \cdot,\cdot \rangle} \mathrm{Ric}^0$.

The results in this section are most conveniently stated and proved in terms of

(3.1) the *difference curvature tensor* $R^d = R - R^0$ and the corresponding notions of sectional, Ricci, and scalar curvatures: they are respectively defined by $K^d = K - K^0$, $\mathrm{Ric}^d = \mathrm{Ric} - \mathrm{Ric}^0$, and $\mathrm{s}^d = \mathrm{s} - \mathrm{s}^0$.

As setup for the next result, observe that whenever A is a G-invariant Codazzi tensor field on G/H and $\mathfrak{m}$ is decomposed as in (2.2), an equivalent formulation to

(2.9) is

$$\alpha(X_i, Y_j) = \sum_{k=1}^{r} \frac{\lambda_i - \lambda_k}{\lambda_i - \lambda_j}[X_i, Y_j]_k, \qquad i \neq j. \tag{3.2}$$

Applying (3.2) to separately compute each term in the curvature relation (1.7-ii) for $(X, Y, Z) = (X_i, Y_j, Y_j)$, with $i \neq j$, we obtain $\langle \alpha(X_i, \alpha(Y_j, Y_j)), X_i \rangle = 0$ and

$$\langle \alpha(Y_j, \alpha(X_i, Y_j)), X_i \rangle = \langle \alpha([X_i, Y_j]_{\mathfrak{m}}, Y_j), X_i \rangle = \sum_{\substack{k=1\\k\neq j}}^{r} \frac{\lambda_k - \lambda_i}{\lambda_j - \lambda_k} \langle [Y_j, [X_i, Y_j]_k]_i, X_i \rangle. \tag{3.3}$$

Choosing $Z = [X_i, Y_j]_k$ and switching the roles of X and Y in (2.3) leads to

$$-(\lambda_j - \lambda_k)^2 \|[X_i, Y_j]_k\|^2 + (\lambda_j - \lambda_i)^2 \langle [Y_j, [X_i, Y_j]_k]_i, X_i \rangle = 0$$

which, when combined with (3.3), implies that

$$\langle R^d(X_i, Y_j)Y_j, X_i \rangle = \frac{2}{(\lambda_i - \lambda_j)^2} \sum_{\substack{k=1\\k\neq j}}^{r} (\lambda_i - \lambda_k)(\lambda_j - \lambda_k) \|[X_i, Y_j]_k\|^2. \tag{3.4}$$

We are ready to generalize [**6**, Proposition 3]:

PROPOSITION 3.1. *If G/H has a G-invariant Codazzi tensor field A with $\nabla A \neq 0$, the difference sectional curvature K^d assumes both positive and negative values.*

PROOF. We claim that

(3.5) there is a smallest integer $2 \leq \rho \leq r-1$, as well as integers $1 \leq \mu < \nu \leq r$, such that (a) $\mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\rho$ and (b) $\mathfrak{m}_\mu \oplus \mathfrak{m}_\nu$ are *not* subalgebras of $(\mathfrak{m}, [\cdot,\cdot]_{\mathfrak{m}})$.

If either (3.5-a) or (3.5-b) fails to hold, then $\langle [X_i, Y_j]_{\mathfrak{m}}, Z_k \rangle = 0$ whenever i, j, k are mutually distinct, so that $\nabla A = 0$ by Proposition 2.1. Indeed, if (a) fails then $\langle [X_i, Y_j]_{\mathfrak{m}}, Z_k \rangle = 0$ whenever $k > \max\{i, j\}$ as $[\mathfrak{m}_i, \mathfrak{m}_j]_{\mathfrak{m}} \subseteq \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_{\max\{i,j\}}$ is orthogonal to $\mathfrak{m}_k$, and we may apply (2.3). If (b) fails instead, then again $[\mathfrak{m}_i, \mathfrak{m}_j]_{\mathfrak{m}} \subseteq \mathfrak{m}_i \oplus \mathfrak{m}_j$ is orthogonal to $\mathfrak{m}_k$ whenever i, j, k are mutually distinct. This proves (3.5).

For ρ as in (3.5-a), minimality of ρ implies that $[\mathfrak{m}_i, \mathfrak{m}_j]_\rho = 0$ whenever $i, j < \rho$, and so $[\mathfrak{m}_i, \mathfrak{m}_\rho]_j = 0$ for distinct $i, j < \rho$ by (2.3) with $k = \rho$. Hence, (2.2) and (3.4) yield

$$K^d(\Pi) = \frac{2}{(\lambda_i - \lambda_\rho)^2} \sum_{k=\rho+1}^{r} (\lambda_i - \lambda_k)(\lambda_\rho - \lambda_k) \|[X_i, Y_\rho]_k\|^2 > 0$$

for $\Pi = \mathbb{R}X_i \oplus \mathbb{R}Y_\rho$ with $i < \rho$, $\|X_i\| = \|Y_\rho\| = 1$, and $[X_i, Y_\rho]_{\mathfrak{m}} \neq 0$.

Lastly, for μ, ν as in (3.5-b) chosen so that the difference $\nu - \mu$ is maximal, we have that $\mathfrak{m}_i \oplus \mathfrak{m}_j$ is a subalgebra of $(\mathfrak{m}, [\cdot,\cdot]_{\mathfrak{m}})$ for $1 \leq i \leq \mu < \nu \leq j \leq r$, provided that $i \neq \mu$ or $j \neq \nu$. This implies that $[\mathfrak{m}_k, \mathfrak{m}_\mu]_\nu = [\mathfrak{m}_\nu, \mathfrak{m}_k]_\mu = 0$ whenever $k < \mu$

or $k > \nu$, and thus $[\mathfrak{m}_\mu, \mathfrak{m}_\nu]_k = 0$ by (2.3) with $(\mu, \nu) = (i, j)$. Choosing unit vectors X_μ and Y_ν with $[X_\mu, Y_\nu]_\ell \neq 0$, for some $\ell \neq \mu, \nu$, it follows from (2.2) and (3.4) that

$$K^d(\Pi) = \frac{2}{(\lambda_\mu - \lambda_\nu)^2} \sum_{k=\mu}^{\nu} (\lambda_\mu - \lambda_k)(\lambda_\nu - \lambda_k) \|[X_\mu, Y_\nu]_k\|^2 < 0$$

for $\Pi = \mathbb{R}X_\mu \oplus \mathbb{R}Y_\nu$, as required. □

EXAMPLE 3.2. Recall that a homogeneous space G/H with a G-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ is called *naturally reductive* if it admits a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with the additional property that $\langle [X, Y]_\mathfrak{m}, Z\rangle + \langle Y, [X, Z]_\mathfrak{m}\rangle = 0$, for all $X, Y, Z \in \mathfrak{m}$. Rearranging the formula in [**2**, Proposition 5.7] we see that, in this case, $K^d(\Pi) = \|[X, Y]_\mathfrak{m}\|^2/4 \geq 0$, where $\{X, Y\}$ is any orthonormal basis for Π. By Proposition 3.1, every G-invariant Codazzi tensor field on such a naturally reductive homogeneous space is necessarily parallel.

For the next result, which generalizes [**6**, Proposition 4], we let M_i be the leaf passing through eH of the eigendistribution of A associated with λ_i, so that $T_{eH}M_i = \mathfrak{m}_i$. Each M_i is a totally geodesic submanifold of G/H equipped either with the Levi-Civita connection of $\langle \cdot, \cdot \rangle$ (by Proposition 2.1), or with the canonical connection ∇^0. This allows us to consider the difference Ricci and scalar curvatures Ric_i^d and s_i^d in (3.1) for each M_i. More precisely, given $Y_i, Z_i \in \mathfrak{m}_i$, the endomorphism $X \mapsto R^d(X, Y_i)Z_i$ of $\mathfrak{m}$ restricts to an endomorphism of $\mathfrak{m}_i$, whose trace is $\mathrm{Ric}_i^d(Y_i, Z_i)$. Then, the trace of Ric_i^d computed with $\langle \cdot, \cdot \rangle|_{\mathfrak{m}_i \times \mathfrak{m}_i}$ is s_i^d.

PROPOSITION 3.3. *If G/H has a G-invariant Codazzi tensor field, then:*

i) $\mathrm{Ric}^d(Y_j, Y_j) \leq \mathrm{Ric}_j^d(Y_j, Y_j)$ *for $j \in \{1, r\}$ and all $Y \in \mathfrak{m}$.*

ii) $\mathrm{s}_1^d + \cdots + \mathrm{s}_r^d = \mathrm{s}^d$.

PROOF. First, observe that the cyclic identity

$$\begin{aligned} &\frac{(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}{(\lambda_i - \lambda_j)^2} \langle [X_i, Y_j]_\mathfrak{m}, Z_k\rangle^2 + \frac{(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)}{(\lambda_j - \lambda_k)^2} \langle [Y_j, Z_k]_\mathfrak{m}, X_i\rangle^2 + \\ &\qquad + \frac{(\lambda_k - \lambda_j)(\lambda_i - \lambda_j)}{(\lambda_k - \lambda_i)^2} \langle [Z_k, X_i]_\mathfrak{m}, Y_j\rangle^2 = 0 \end{aligned} \tag{3.6}$$

holds for all $X, Y, Z \in \mathfrak{m}$ whenever i, j and k are mutually distinct, as a direct consequence of (2.8). Now, writing $d_i = \dim \mathfrak{m}_i$ and letting $\{E_{i,a}\}_{a=1}^{d_i}$ be an orthonormal basis for $\mathfrak{m}_i$, for each $i = 1, \ldots, r$, it follows from the definition of Ric_j^d and (3.4) that

$$\mathrm{Ric}^d(Y_j) = \mathrm{Ric}_j^d(Y_j) + 2 \sum_{\substack{i=1 \\ i \neq j}}^{r} \sum_{a=1}^{d_i} \sum_{\substack{k=1 \\ k \neq j}}^{r} \sum_{b=1}^{d_k} \frac{(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}{(\lambda_i - \lambda_j)^2} \langle [E_{i,a}, Y_j]_\mathfrak{m}, E_{k,b}\rangle^2 \tag{3.7}$$

for every $Y_j \in \mathfrak{m}_j$. Here, we write $\mathrm{Ric}^d(Y_j)$ as a shorthand for $\mathrm{Ric}^d(Y_j, Y_j)$, and similarly for Ric^d_j. The summand in the right side of (3.7) vanishes when $k = i$ and, relabeling dummy indices $(i, a) \leftrightharpoons (k, b)$ in one of the two copies of such summation, we see that (3.6) leads to

$$\mathrm{Ric}^d(Y_j) = \mathrm{Ric}^d_j(Y_j) - \sum_{\substack{i=1\\ i\neq j}}^{r} \sum_{a=1}^{d_i} \sum_{\substack{k=1\\ k\neq j}}^{r} \sum_{b=1}^{d_k} \frac{(\lambda_k - \lambda_j)(\lambda_i - \lambda_j)}{(\lambda_k - \lambda_i)^2} \langle [E_{k,b}, E_{i,a}]_{\mathfrak{m}}, Y_j \rangle^2. \tag{3.8}$$

Using (2.2) and the fact that $(\lambda_k - \lambda_j)(\lambda_i - \lambda_j)$ is a product of positive (or, negative) factors when $j = 1$ (or, $j = r$) for all i and k, (i) follows. Finally, setting $Y_j = E_{j,c}$ in (3.8) and summing over $1 \le c \le d_j$ and $1 \le j \le r$, we conclude that (ii) holds: the difference $\mathrm{s}^d_1 + \cdots + \mathrm{s}^d_r - \mathrm{s}^d$ equals the sum over mutually distinct indices i, j, k of terms appearing in (3.6), and therefore it must vanish.

□

A last consequence of Proposition 3.3 is the counterpart to [**6**, Proposition 5]:

COROLLARY 3.4. *Suppose that* Ric^d *itself is a Codazzi tensor field on* G/H*, with* $\nabla\mathrm{Ric}^d \neq 0$*. If* $\mathrm{s}^d_i \ge 0$ *for* $1 \le i \le r-1$*, then* $\mathrm{s}^d_r \neq 0$*. In particular, not all eigenspaces of* Ric^d *can be Abelian subalgebras of* $(\mathfrak{m}, [\cdot,\cdot]_{\mathfrak{m}})$*.*

PROOF. Item (i) of Proposition 3.3 for $A = \mathrm{Ric}^d$ reads $\mathrm{Ric}^d(Y_r, Y_r) \ge \lambda_r$ for all unit vectors $Y_r \in \mathfrak{m}_r$, so averaging over an orthonormal basis yields $\mathrm{s}^d_r/d_r \ge \lambda_r$. If it were to be $\mathrm{s}^d_r = 0$, (2.2) would imply that $\lambda_1 < \cdots < \lambda_r \le 0$, and hence $\mathrm{s}^d = d_1\lambda_1 + \cdots + d_r\lambda_r < 0$. However, it is clear from $\mathrm{s}^d_i \ge 0$, for $1 \le i \le r-1$, and item (ii) of Proposition 3.3, that $\mathrm{s}^d \ge 0$. The last claim now follows as $R^d_i = 0$ (and thus $\mathrm{s}^d_i = 0$) whenever $\mathfrak{m}_i$ is Abelian, as $\alpha|_{\mathfrak{m}_i \times \mathfrak{m}_i} = 0$ in view of (1.9) and Proposition 2.1. □

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