MATH2177 – AU22 – RECITATION DIARY

This is a diary for the **MATH2177 - Mathematical Topics for Engineers** recitation classes offered at OSU, on the Autumn 2022 term. No additional material or exercises will be added here, as it is meant to be a reasonably faithful reflection of what happens in class (although I cannot promise I won't add a remark or footnote here or there, elaborating further on things I particularly find interesting). Since it is unlikely that all three recitations will be 100% equal all the time, you may occasionally see an exercise that was not discussed in class (but this only means that it may have been discussed in one of the other sections). Most of the exercises and problems discussed will be taken from reference [1], as expected.

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1 August 23rd

This class is divided into four major parts:

(1) Multivariable Calculus;

(2) Linear Algebra;

(3) Ordinary Differential Equations;

(4) Fourier Series.

There's no need to justify Multivariable Calculus here (after all, this is a natural continuation of MATH1172). Linear Algebra is too useful of a subject for us to spend the entire semester without using it (although most of you may have to take an actual Linear Algebra class later). Ordinary Differential Equations: briefly speaking, those are equations where one solves for a function instead of a number, and there are derivatives involved in the given equation (the word "ordinary" refers to the fact that only functions of a single variable are involved; "partial differential equations" are those with partial derivatives involved). Fourier Series: you can vaguely think of those as a trigonometric analogue of the Taylor series you have seen before:

$$f(x) = \sum_{n\geq 0} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{versus} \quad f(x) \sim \frac{a_0}{2} + \sum_{n\geq 1}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

Just like the coefficients of a Taylor series are given in terms of derivatives of f, there are specific formulas for the coefficients a_n and b_n of a Fourier series, but we don't need to worry about that now. The pressing question, however, is how these four topics make sense together.

To answer it, consider the **Heat Flow Problem**: imagine you have a metal wire of length L, positioned in the x-axis of a cartesian plane, with endpoints at $x = 0$ and $x = L$. Assume that:

(i) we are given the initial temperature distribution $f(x)$ of the wire (i.e., a point of coordinate x in the wire has temperature equal to $f(x)$; in particular, non-constant f means that the temperature distribution along the wire is not uniform);

(ii) we are given the diffusivity coefficient β of the wire;

(iii) the temperature at the endpoints of the wire will always be the same throughout time (say, 0 in some appropriate scale).

Question: Can we predict future temperature distributions along the wire? (i.e., can we find a function $u(x,t)$ giving us the temperature at the point of coordinate x at the wire t minutes after the initial measurement?)

This sort of question is the basis of the mathematical modeling for problems of physical nature: find a function controlling the time evolution of the physical system you're interested in studying. Such a function is usually a solution of some partial differential equation.

Answer: Yes! It boils down to solving the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \beta \dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L. \end{cases} \tag{H}$$

It is, *prima facie*, a very complicated problem. Think back of the four major parts of this course (1)-(4) mentioned above. The time-dependent temperature distribution $u(x,t)$ is a function of more then one variable (1), dealing with the boundary conditions $u(0,t) = u(L,t) = 0$ becomes easier with the use of Linear Algebra (2), it allows us to reduce the partial differential equation to an ordinary differential equation (3), and the final answer can be conveniently expressed in terms of the Fourier Series (4) of the initial temperature distribution $f(x)$.

The ultimate goal of this class is to get you to combine all the tools listed above to solve problems such as the Heat Flow Problem (H).

With this in place, we'll start with a review of multivariable calculus. It essentially deals with functions of *several variables* (where "several" will usually be two or three[1]), such as

$$f(x,y) = x^2 + y, \quad f(x,y,z) = x\mathrm{e}^y + \sin z, \quad f(x,y,z,w) = \sin(xy) + \cos(zw),$$

as opposed to the functions $f(x)$ of a single variable studied in previous classes. The main program then was to find the critical points of a given function and study their *local nature* (that is, whether they're local maxima, local minima, or saddle/inflection points). The main tools to do so, in turn, were derivatives. For functions of several variables, we have *partial derivatives*

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}, \quad \frac{\partial f}{\partial w}, \cdots$$

These are computed from the usual rules from single-variable calculus (product rule, quotient rule, chain rule), by freezing (hence treating as constants) all the variables except for the one on which the differentiation process is acting.

The mantra repeated throughout the first semester of calculus was "the derivative is the slope of the tangent line". If we're dealing with a function of, say, two variables instead, the picture goes like this: the graph of f is a surface on three-dimensional space (henceforth denoted by $\mathbb{R}^3$), and the numerical value $f(x,y)$ is understood as the height of a point whose first two coordinates are x and y. Fix a vector v on the plane (to be called $\mathbb{R}^2$ from now on), starting at the point (x,y), and draw a vertical plane Π passing through the point (x,y) and containing the direction determined by

[1]For pedagogical reasons.

v. The plane Π intersects the surface $z = f(x,y)$ along a curve, completely contained in Π, which may be thought of the graph of a function (more precisely, given by the relation $g(t) = f((x,y) + tv)$). The derivative $g'(0)$ is the slope of the tangent line to the intersection curve, in the plane Π, and it is called the *directional derivative of f in the direction v, at the point (x,y)*, and it is denoted by $(\partial f/\partial v)(x,y)$.

Instead of thinking about directional derivatives along infinitely many directions, we recall the relation

$$\frac{\partial f}{\partial v}(x,y) = \nabla f(x,y) \cdot v,$$

where $\cdot$ stands for the *dot product* of vectors, and $\nabla f(x,y)$ is the gradient of f at the point (x,y). More generally:

Definition 1

Let f be a differentiable function of the variables $x_1, \ldots, x_n$. The **gradient** of f is the vector (field) ∇f defined by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

The gradient is a mathematical device to gather all information about the first-order derivatives of a function, into a single object: namely, a vector. Knowledge of the gradient is indeed knowledge about all directional derivatives of the function, due to the dot product relation (which remains true in all dimensions).

Recall that, in single-variable calculus, a number a is a critical point of a function f if $f'(a) = 0$. Picturing the graph of f, this condition is to be expected, as the tangent line to the graph of f at the candidates to local max/min is horizontal, the derivative is the slope of the tangent line, and the slope of a horizontal line is zero.

With this in place:

Definition 2

Let f be a differentiable function of the variables $x_1, \ldots, x_n$. Then $(a_1, \ldots, a_n)$ is a **critical point** of f if $\nabla f(a_1, \ldots, a_n) = (0, \ldots, 0)$ or if $\nabla f(a_1, \ldots, a_n)$ does not exist.

In other words, a critical point is one for which the gradient vector is the zero vector. In two variables, a point (a,b) is a critical point of f if $\nabla f(a,b) = (0,0)$. Meaning that we replace the single equation $f'(a) = 0$ with the more elaborate system $\nabla f(a_1, \ldots, a_n) = (0, \ldots, 0)$ (one equation for each variable). Or yet, such system is obtained by setting all the partial derivatives of f equal to zero (which by the dot product relation is the same as requiring all directional derivatives to be zero).

Example 1

Find the critical points of the function $f(x,y) = 1 + x^2 + y^2$.

The graph of this function is called a **paraboloid**, and it is obtained by rotating the parabola $y = 1 + x^2$ around the z-axis. Geometrically, it should be clear that there is only one critical point, $(0,0)$, and that is is a global minimum for the function. Sometimes it is not so simple to get the answer so quickly by using geometric intuition. Doing it directly, we compute that $\nabla f(x,y) = (2x, 2y)$, and this equals $(0,0)$ if and only if $(x,y) = (0,0)$. Observe that the $(0,0)$'s indicated in different colors play different roles!

Example 2

Find the critical points of the function $f(x,y) = (3x-2)^2 + (y-4)^2$.

The graph of this function is very similar to the one in the previous example. This can be made clearer by letting $u = 3x-2$ and $v = y-4$, so that (abusing notation) $f(u,v) = u^2 + v^2$. This means that the graph of the function considered here is a paraboloid, up to this change of variables (which amounts to an offset and a stretching of the x-axis). Geometrically, we see that the critical point is described by $u = v = 0$, which means that $x = 2/3$ and $y = 4$. Thus there is only one critical point $(2/3, 4)$, which is a global minimum for the function. Changing variables like this will be useful not only to try and gain geometric intuition for situations like this (where what you have in front of you is similar to something you have already seen before, but not quite equal to it), but it will also be a very important tool when dealing with change of variables for computing double integrals.

Doing it directly, however, we have that $\nabla f(x,y) = (6(3x-2), 2(y-4))$, and this equals $(0,0)$ if and only if $(x,y) = (2/3,4)$. Compare the choice of colors here with the one made in the previous example.

Example 3

Functions may not have critical points at all! Consider $f(x,y,z) = g(x,y) + z$, where $g(x,y)$ is the most horrible expression you can come up with. Then we have that $\nabla f = (*,*,1)$. As the last component of ∇f is never zero, then ∇f can never be the zero vector, and so f has no critical points.

Once critical points have been found, the next task is to decide their nature, i.e., whether they are local max/min or saddle/inflection points. In the situation where we had a critical point a of a function f of a single-variable, we knew that:

- If $f''(a) > 0$, then a is a local minimum (◡̈)
- If $f''(a) < 0$, then a is a local maximum (⌢̈)
- If $f''(a) = 0$, the test is inconclusive.

Remark. A very common mistake is to think that $f''(a) = 0$ means that a is a saddle/inflection point. This is not true. Case in point: for the function $f(x) = x^4$ and $a = 0$ ($f(0) = f'(0) = f''(0) = f'''(0) = 0$ but $f^{(4)}(0) > 0$ and a is a global minimum). If $f''(a) = 0$ but $f'''(a) \neq 0$, then a is a saddle point. If $f'''(a) = 0$, then we must look at the fourth derivative $f^{(4)}(a)$: if positive, then a is a local minimum, and if negative, a is a local maximum. If $f^{(4)}(a) = 0$, we must look at the fifth derivative. If $f^{(5)}(a) \neq 0$, then a is a saddle point. If $f^{(5)}(a) = 0$, we must look at the sixth derivative $f^{(6)}(a)$: if positive, then a is a local minimum, and if negative, a is a local maximum. This procedure repeats, alternating between even-order derivatives and odd-order derivatives, until a conclusion is obtained. The "proof" of this fact relies on the Taylor polynomial of f. More about this on office hours if anyone is interested.

Let's focus on the case where f is a function of two variables instead. The first derivative became the gradient vector. The second derivative must correspond to what comes after a vector: a matrix.

Definition 3

Let f be a twice-differentiable function of the variables x and y. The **Hessian matrix** of f is defined by

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix},$$

where we use the shorthands

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}.$$

We also let $D = f_{xx}f_{yy} - f_{xy}^2$ be the **determinant** of H.

Theorem 1

Let f be a twice-differentiable function of the variables x and y, and let (a, b) be a critical point of f. Then:

- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a local minimum.
- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a local minimum.
- If $D(a, b) < 0$, then (a, b) is a saddle point.
- If $D(a, b) = 0$, the test is inconclusive.

Remark. There is a version of the second derivative test for functions with more than two variables. The number of conditions to be considered increases with the number of variables. Justifying it requires more Linear Algebra than what we have available now. The keyword is "Sylvester's Criterion" (for positivity of matrices).

We'll continue to explore this on next class.

2 August 30th

We'll start picking up from last time, with a full example of how the Hessian second derivative test works.

Example 4

Find and classify the critical points of the function $f(x,y) = x^2 + xy^2 - 2x + 1$.

To find the critical points of f, we must compute the gradient of f and set it equal to the zero vector $(0,0)$. In this case, we have that the gradient equals $\nabla f(x,y) = (2x + y^2 - 2, 2xy)$, and we must consider the system

$$\begin{cases} 2x + y^2 - 2 = 0, \\ 2xy = 0. \end{cases}$$

Recall here the **key idea: whenever the product of two factors equals zero, one of them must necessarily be zero**. So, the second equation says that either $x = 0$ or $y = 0$. But we have no way to tell which one, and so we must consider both cases.

- Case 1: If $x = 0$, then $y^2 - 2 = 0$ and so $y = \pm\sqrt{2}$. This gives us the pair of critical points $(0, \sqrt{2})$ and $(0, -\sqrt{2})$.

- Case 2: If $y = 0$, then $2x - 2 = 0$ and so $x = 1$. This gives us the single critical point $(1, 0)$.

It remains to classify these critical points. To do so, we compute the Hessian of f as

$$H(x,y) = \begin{pmatrix} 2 & 2y \\ 2y & 2x \end{pmatrix},$$

as well as its determinant $D(x,y) = 4x - 4y^2$. Then:

- $(1,0)$: here, we have that $D(1,0) = 4 > 0$, so we can think of the single-variable second derivative test with f_{xx} playing the role of f''. Since we have that $f_{xx}(1,0) = 2 > 0$, we conclude that $(1,0)$ is a local minimum of f.

- $(0, \sqrt{2})$: this time, we have $D(0, \sqrt{2}) = -8 < 0$, so $(0, \sqrt{2})$ is a saddle point.

- $(0, -\sqrt{2})$: this time, we have $D(0, -\sqrt{2}) = -8 < 0$, so $(0, -\sqrt{2})$ is a saddle point.

The fact that the critical points $(0, \pm\sqrt{2})$ had the same local nature should not be a surprise: the relation $f(x,y) = f(x,-y)$ says that f is "even in the variable y" (which geometrically says that the graph of f is symmetric about the xz-plane in space).

Remark. Note that $H(x,y)$ is always a *symmetric matrix*, due to the equality $f_{xy} = f_{yx}$, which in words says that taking the derivative with respect to x first, and then y, produces the same result as doing things in the reverse order, namely, differentiating with respect to y first, and then x. This is sometimes called the Clairaut-Schwarz theorem[2]. Another observation which may be very useful for computing Hessians quickly is that, writing $\nabla f = (f_x, f_y)$, the first row of H is the gradient of the first component f_x, while the second row of H is the gradient of the second component f_y.

We continue with the discussion on maxima and minima of functions of two variables. Most of the time, the region R to be considered will be closed and bounded (those are called "compact", in short), and so may be thought of as having two regions: an interior $\operatorname{int}(R)$, and a boundary ∂R. **Candidates** to global max/min in $\operatorname{int}(R)$ are nothing more than the critical points of f which turn out to land there. Observe that the second derivative test is only good for determining the local behavior of critical points, not the global one. So, in the end of the day, one must still list all candidates and compute the value of the function on each of them.

The question still remains of how to study the boundary ∂R. Very frequently, it may be described as an equation of the form $g(x, y) = 0$, where g is nice enough. In this case, we have a specific technique to use:

Theorem 2 (Lagrange multipliers)

Let f and g be differentiable functions of the variables $x_1, \ldots, x_n$, with continuous partial derivatives. Assume that $\nabla g(x_1, \ldots, x_n)$ is not the zero vector whenever $g(x_1, \ldots, x_n) = 0$. If $(a_1, \ldots, a_n)$ is the global maximum or minimum of f restricted to the set $\{(x_1, \ldots, x_n) \mid g(x_1, \ldots, x_n) = 0\}$, there is a real number λ such that

$$\nabla f(a_1, \ldots, a_n) = \lambda \nabla g(a_1, \ldots, a_n),$$

i.e., the gradients of f and g are proportional in such point.

Back to the problem at hand, the strategy will be to solve the system

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y), \\ g(x, y) = 0. \end{cases}$$

The condition $g(x, y) = 0$ there is a crucial part of the problem, and it is always used in practice. Without it, one could find points *not on the given curve* for which the gradients of f and g are proportional, but we do not care about those. On the other hand, knowing the value of the Lagrange multiplier λ is not crucial. Sometimes it is convenient to solve for it as an intermediate step in finding x and y. If one manages to find all the candidates to max/min without solving for λ, there is nothing wrong.

For boundary regions ∂R which are not of the form $g(x, y) = 0$, the usual strategy is to parametrize each differentiable component of ∂R and use single-variable calculus strategies on each piece. Let's illustrate on the next example how this can be done.

[2]Whose formal assumption is that f is twice-differentiable and all the second order derivatives are continuous (or, more generally, defined on all points of an open set).

Example 5

Find the global maximum and global minimum values of the function

$$f(x,y) = x^2 + y^2 - 2y + 1$$

on the region $R = \{(x,y) \mid x^2 + y^2 \leq 4\}$.

We'll always organize our work in two parts. Observe that R describes a closed disk with center in $(0,0)$ and radius 2.

(a) Candidates on $\mathrm{int}(R)$: the interior consists of the points (x,y) satisfying the relation $x^2 + y^2 < 4$. By replacing the inequality $\leq$ with the strict inequality $<$, we are forgetting about the boundary circle and considering only the open disk bounded by it. We compute the gradient of f as $\nabla f(x,y) = (2x, 2y-2)$. Setting this equal to $(0,0)$, we obtain $(x,y) = (0,1)$. **Is the point $(0,1)$ in the interior of R?** Yes. The reason why we ask ourselves this is because f is defined everywhere (in particular, its domain is larger than R), so a priori it could be that critical points found here lie outside R. If this were to happen, such critical points would have to be ignored.

(b) Candidates on ∂R: we use Lagrange multipliers, letting $g(x,y) = x^2 + y^2 - 4$. In this case, $\nabla g(x,y) = (2x, 2y)$ cannot be the zero vector whenever we have $x^2 + y^2 = 4$, which means that the assumptions for using Lagrange multipliers are satisfied. Thus, we have that

$$\begin{cases} \nabla f(x,y) = \lambda g(x,y) \\ g(x,y) = 0 \end{cases} \implies \begin{cases} 2x = 2\lambda x \\ 2y - 2 = 2\lambda y \\ x^2 + y^2 = 4 \end{cases}$$

Consider the first equation $2x = 2\lambda x$. We would like to cancel x on both sides, but this step cannot be made if $x = 0$. Thus, we have cases to analyze.

- Case 1: if $x \neq 0$. Then $\lambda = 1$, and substituting this onto the second equation gives $2y - 2 = 2y$, and thus $-2 = 0$. This is clearly nonsense, which says that Case 1 does not happen, and so we get no candidates here.
- Case 2: if $x = 0$. In this case, we only have to solve for y, and then the third equation reads $0^2 + y^2 = 4$, so that $y = 2$ or $y = -2$. This case gave us two candidates, $(0,2)$ and $(0,-2)$.

(c) Candidates on ∂R without using Lagrange multipliers: consider the parametrization $\mathbf{r}(t) = (2\cos t, 2\sin t)$, defined for all t (in fact, $0 \leq t \leq 2\pi$ covers the circle). Then $f(\mathbf{r}(t)) = 5 - 4\sin t$. The maximum value of this expression is 9, for $t = 3\pi/2$, and we have $\mathbf{r}(3\pi/2) = (0,-2)$. The minimum along the curve is 1, for $t = \pi/2$, with $\mathbf{r}(\pi/2) = (0,2)$ (as we'll see, is not the global minimum of f because the critical point in $\mathrm{int}(R)$ gives a lower value). We have reobtained the candidates $(0,2)$ and $(0,-2)$.

Conclusion:

Candidates	Values of f
$(0,1)$	0
$(0,2)$	1
$(0,-2)$	9

Hence the global maximum is 9, realized at $(0,-2)$, while the global minimum is 0, realized at $(0,1)$.

Example 6

Find the global maximum and global minimum values of the function

$$f(x,y) = 2x^2 - 4x + 3y^2 + 2$$

restricted to the circle of equation $(x-1)^2 + y^2 = 1$.

Note that the circle has center at $(1,0)$ and radius 1. We can parametrize the circle with $\mathbf{r}(t) = (1+\cos t, \sin t)$, for t ranging over all real numbers. Write

$$\begin{aligned} h(t) &= f(\mathbf{r}(t)) \\ &= f(1+2\cos t, 2\sin t) \\ &= 2(1+\cos t)^2 - 4(1+\cos t) + 3(\sin t)^2 + 2 \\ &\overset{\text{(do it!)}}{=} 2 + \sin^2 t. \end{aligned}$$

Critical points of h will correspond under $\mathbf{r}$ to candidates to extremum points of f along the circle. The minimum value of h is 2 whenever $\sin^2 t = 0$, and its maximum value is 3 whenever $\sin^2 t = 1$. Since restricting $\mathbf{r}$ to the domain $[0,2\pi]$ already covers the circle and the circle has no corners, we directly obtain that the global maximum is 3, realized at $(1,1)$ and $(1,-1)$ (coming from $t = \pi/2$ and $t = 3\pi/2$), while the global minimum is 2, realized at $(2,0)$ and $(0,0)$ (coming from $t = 0$ and $t = \pi$).

We'll concluce pointing some directions for a couple of the homework problems:

Example 7

Find the global maximum and global minimum values of the function

$$f(x,y) = x^2 + y^2 - 2x - 2y$$

on the region R bounded by the triangle of vertices at $(0,0)$, $(2,0)$ and $(0,2)$.

The boundary ∂R has three sides:

$$\begin{aligned} A &= \{(0,y) \mid 0 \leq y \leq 2\} \\ B &= \{(x,0) \mid 0 \leq x \leq 2\} \\ C &= \{(x,y) \mid y = -x+2 \text{ and } 0 \leq x \leq 2\} \end{aligned}$$

We proceed with our analysis as before:

(a) Candidates on $\mathrm{int}(R)$: this time we have $\nabla f(x,y) = (2x-2, 2y-2)$, so setting this equal to $(0,0)$ gives $(x,y) = (1,1)$. **Is the point $(1,1)$ in the interior of R?** No. In fact, $(1,1)$ lies on side C, and so it is not considered a candidate **as far as** $\mathrm{int}(R)$ **is concerned**.

(b) Candidates on ∂R: we'll study each of the sides A, B and C separately.

- Side A: evaluating f alongside Side A, we're led to consider the composition $g(y) = f(0,y) = y^2 - 2y$, defined on the interval $[0,2]$. The candidates here will be the points corresponding to the endpoints of the interval $[0,2]$, and critical points of g inside the open interval $]0,2[$. The graph of g is a parabola which is concave up, and so $g'(y) = 2y - 2$ leads to $y = 1$. Side A thus gives us the candidates $(0,0)$, $(0,2)$ and $(0,1)$.
- Side B: can you draw conclusions from the work done for Side A given that the function f satisfies the symmetry $f(x,y) = f(y,x)$?
- Side C: what is the line equation for Side C? Consider the composition

  $$h(x) = f(x, \text{"line equation"}(x)),$$

  defined on $[0,2]$. As before the endpoints of $[0,2]$ gives us the candidates $(2,0)$ and $(0,2)$ (both repeated). Now find which candidates are coming from $h'(x) = 0$ repeating what was done for side (a).

Summarizing it:

Candidates	Values of f
$(0,0)$	?
$(2,0)$	?
$(0,2)$	?
$(0,1)$	?
from Side B	?
from Side C	?

Fill the table and read off it the desired conclusions.

Example 8

Find the global maximum and global minimum values of the function

$$f(x,y) = \sqrt{x^2+y^2-2x+2}$$

on the region $R = \{(x,y) \mid x^2+y^2 \leq 4 \text{ and } y \geq 0\}$.

Observe that $x^2+y^2-2x+2 = (x-1)^2+y^2+1 > 0$, so f is differentiable at all points. This means that when looking for critical points in the interior of R, you don't have to worry about points where $\nabla f(x,y)$ does not exist: this won't happen. Start finding all points for which $\nabla f(x,y) = (0,0)$. Are they in the interior of R? If yes, list them as candidates. If not, do not list them as candidates.

The next step is to study the boundary ∂R. Note that Lagrange multipliers may not be applied here as we cannot describe the region as the level set of a function g with nowhere-vanishing gradient. This means we must look at the two "components" of the boundary separately.

- The upper half-arc. It may be parametrized by $\mathbf{r}(t) = (\cos t, \sin t)$, with t ranging over a certain interval. What is such interval? What are the candidates for max/min of the function $h_1(t) = f(\cos t, \sin t)$ on this interval? To which points in ∂R do they correspond under $\mathbf{r}$?

- The line segment on the x-axis. It may be parametrized by $\mathbf{r}(t) = (t,0)$, with $-2 \leq t \leq 2$. What are candidates for max/min of the function $h_2(t) = f(t,0)$ on the interval $[-2,2]$. To which points in ∂R do they correspond under $\mathbf{r}$?

List all candidates in a table as in the previous example, fill in the values of f at the candidates, and find where the global max/min values are realized.

3 September 6th

Today we start with double integrals. Recall the geometric intuition: if $f > 0$, then $\iint_R f(x,y)\,\mathrm{d}A$ computes the volume bounded between the xy-plane and the graph of f, in the same fashion that $\int_a^b f(x)\,\mathrm{d}x$ computed the area under the graph of f. The main difficulty here, however, is to deal with bounds of integration when setting up iterated integrals.

Example 9

Compute $\displaystyle\iint_R y\cos(xy)\,\mathrm{d}A$, *where* $R = \{(x,y) \mid 0 \leq x \leq 1$ *and* $0 \leq y \leq \pi/3\}$.

Here, $\mathrm{d}A$ stands for the **infinitesimal area element**, which in rectangular coordinates is just given by $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$. The region R is simply a rectangle, as the bounds for x and y are all constants. Do not be mislead to think that R is a sector of a circle just because there's π there: R is not being described in polar coordinates, there's no θ anywhere. One could set up the iterated integrals as

$$\int_0^{\pi/3}\int_0^1 y\cos(xy)\,\mathrm{d}x\,\mathrm{d}y \quad\text{or}\quad \int_0^1\int_0^{\pi/3} y\cos(xy)\,\mathrm{d}y\,\mathrm{d}x.$$

Fubini's Theorem says that the order you choose does not matter, you will obtain the same result regardless of the choice made. Now, it could very well happen that one choice of order leads to a much easier computation than the other. In this case, the second option would require an unpleasant integration by parts, while the first one requires a simple u-substitution. Making $u = xy$, so $\mathrm{d}u = y\,\mathrm{d}x$ (as y is a constant from the perspective of x), we have that

$$\int y\cos(xy)\,\mathrm{d}y = \int \cos u\,\mathrm{d}u = \sin u = \sin(xy).$$

We don't bother with the constant of integration here because we're dealing with definite integrals, so it would dissapear anyway. Thus

$$\begin{aligned}\int_0^{\pi/3}\int_0^1 y\cos(xy)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{\pi/3} \sin(xy)\Big|_{x=0}^{x=1}\,\mathrm{d}x \\ &= \int_0^{\pi/3}\sin y\,\mathrm{d}x \\ &= -\cos y\Big|_{y=0}^{y=\pi/3} \\ &= -\frac{1}{2}+1 \\ &= \frac{1}{2}.\end{aligned}$$

Next, let's look at more general regions which are not rectangles.

Example 10

Set up iterated integrals for a generic continuous function $f(x,y)$ over the region R given in the picture.

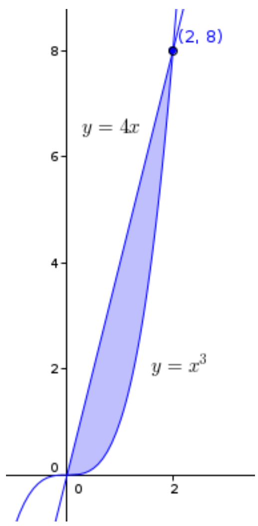
As usual, there are two orders we can set up.

- $\mathrm{d}x\,\mathrm{d}y$. Fix one value of the outermost variable, y. What is the range for the other variable x, as the horizontal line passing through y cuts the given region, from left to right? Here, we must express x as a function of y, so there's some small work to be done. The lower bound for x is $x = y/4$, coming from the line equation, and the upper bound is $x = \sqrt[3]{y}$, from the cubic equation. The variable y, in turn, goes from 0 to 8. Thus

$$\iint_R f(x,y)\,\mathrm{d}A = \int_0^8 \int_{y/4}^{\sqrt[3]{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

- $\mathrm{d}y\,\mathrm{d}x$. Fix one value of the outermost variable, x. What is the range for the other variable y, as the vertical line passing through x cuts the given region, upwards? Here, we must express y as a function of x, so there no work to be done. The lower bound for y is x^3, coming from the cubic equation, and the upper bound is $4x$, from the line equation. The variable x, in turn, goes from 0 to 2. Thus

$$\iint_R f(x,y)\,\mathrm{d}A = \int_0^2 \int_{x^3}^{4x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$



Here's another one:

Example 11

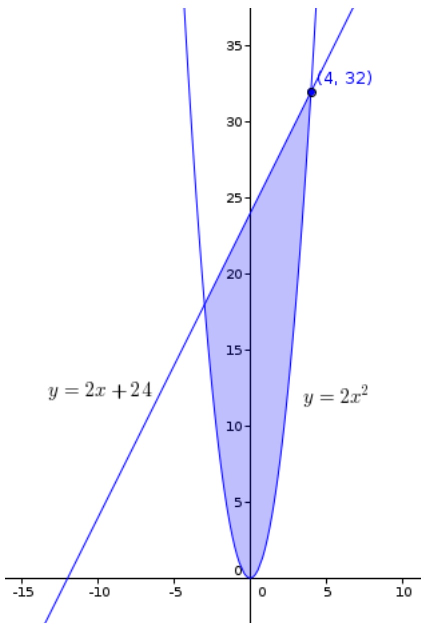
Set up iterated integrals for a generic continuous function $f(x,y)$ over the region R given *in the picture.*

One more time, there are two orders we can set up. The exercise did not give us the coordinates for the left intersection point between the graphs, but we need it to know the full bounds for x. To find it, we consider $2x^2 = 2x + 24$, which is readily simplified to $x^2 - x - 12 = 0$. We already know that one of the solutions is $x = 4$. Due to the coefficient 12, the other one is -3 or 3, but it clearly cannot be the latter. Hence, the coordinates of the remaining intersection point are $(-3, 18)$ (where 18 is obtained by plugging $x = -3$ into either $y = 2x^2$ or $y = 2x + 24$). Now, let's study what happens with both orders of integration:

- $\mathrm{d}y\,\mathrm{d}x$: Fix one value of the outermost variable, x. What is the range for the other variable y, as the vertical line passing through x cuts the given region, upwards? Here, we must express y as a function of x, so there no work to be done. The lower bound for y is $2x^2$, coming from the quadratic equation, and the upper bound is $2x + 24$, from the line equation. The variable x, in turn, goes from -3 to 4. Thus

$$\iint_R f(x,y)\,\mathrm{d}A = \int_{-3}^{4}\int_{2x^2}^{2x+24} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

- $\mathrm{d}x\,\mathrm{d}y$: This time, we're forced to break the region into two pieces, as once a value for the outermost variable y is fixed, the lower bound for x cannot be written as a single formula as a function of y, due to the "break" at the point $(-3, 18)$. We know that if $R = R_1 \cup R_2$ with $R_1 \cap R_2 = \varnothing$, then the double integral of f over R equals the sum[a] of the double integrals over R_1 and R_2. Let's say that R_1 is the part of R which lies inside the strip $0 \le y \le 18$, and R_2 is the one inside the strip $18 < y \le 32$.



Then we have that

$$\begin{aligned}\iint_R f(x,y)\,\mathrm{d}A &= \iint_{R_1} f(x,y)\,\mathrm{d}A + \iint_{R_2} f(x,y)\,\mathrm{d}A \\ &= \int_0^{18}\int_{-\sqrt{y/2}}^{\sqrt{y/2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{18}^{32}\int_{(y-24)/2}^{\sqrt{y/2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Namely, the "upper bound" for x, once y is fixed, is always $\sqrt{y/2}$, but the lower bound depends on whether $0 \leq y \leq 18$ or $18 < y \leq 32$: in the former case, it is $-\sqrt{y/2}$, and in the latter case it is $(y-24)/2$ (obtained from solving for x in terms of y in $y = 2x+24$).

[a]This is a two-variable version of the general rule $\int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t$ for single-variable integrals.

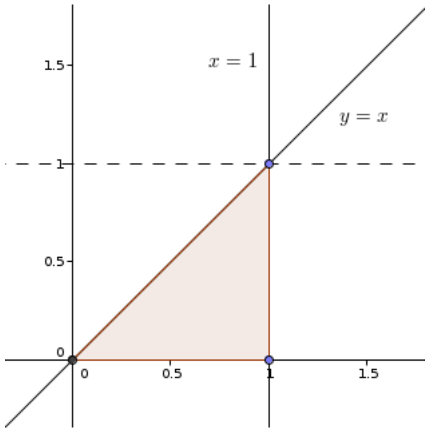
By now, you should be convinced that a convenient choice of order of integration is crucial to making things simpler (getting the feeling for which choice is best takes some practice and experience). There are situations, however, where one choice simply makes the problem impossible, and we're forced to switch the order.

Example 12

Compute $\displaystyle\int_0^1\int_y^1 \mathrm{e}^{x^2}\,\mathrm{d}x\,\mathrm{d}y$.

The function $f(x) = \mathrm{e}^{x^2}$ has no elementary anti-derivative, in the sense that its indefinite integral cannot be expressed in terms of well-known functions (such as polynomials, rational functions, exponentials, logarithms, and trigonometric functions). Knowing whether a given function of a single-variable has an elementary anti-derivative or not is not a simple task (keywords: Risch's Algorithm, and Differential Galois Theory). We will not concern ourselves with this. The extra tool we have in this case, is precisely to change the order of integration.

To draw the region of integration, one general strategy is: first recognize that the outermost bounds for y are 0 and 1, so whatever we draw will be inside the region where $0 \leq y \leq 1$. As for the innermost bounds, draw the curves described by the bounds, $x = y$ and $x = 1$. Namely, they're the usual diagonal, and a vertical line.



If the innermost original bounds were from 0 to y, the region of integration would be the upper triangle, as opposed to the lower one (as the picture indicates). Now:

$$\begin{aligned}\int_0^1\int_y^1 e^{x^2}\,dx\,dy &= \int_0^1\int_0^x e^{x^2}\,dy\,dx\\ &= \int_0^1 xe^{x^2}\,dx\\ &= \left.\frac{e^{x^2}}{2}\right|_0^1 = \frac{e-1}{2}.\end{aligned}$$

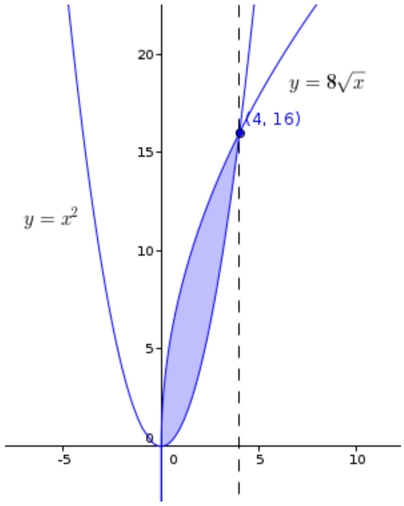
The x factor produced by realizing the integral with respect to y first saves the day.

Being able to sketch regions given algebraically is an important skill. Here's more practice:

Example 13

Sketch $R = \{(x,y) \mid 0 \le x \le 4$ and $x^2 \le y \le 8\sqrt{x}\}$ and set the iterated integral of a generic continuous function $f(x,y)$ over R in the order $dy\,dx$.

We immediately know that whatever we draw will remain inside the vertical strip $0 \le x \le 4$. As for $x^2 \le y \le 8\sqrt{x}$, forget for one moment that we're dealing with inequalities, and draw the bounds $y = x^2$ and $y = 8\sqrt{x}$ instead. Recall that the graph of $\sqrt{x}$ is obtained by reflecting the graph of x^2 about the diagonal line $y = x$, and that 8 is just a vertical stretching factor (made by design to make $(4,16)$ the rightmost intersection of the two curves).



Fixed x, the lower bound for y is x^2 and the upper bound is $8\sqrt{x}$. As x itself ranges from 0 to 4, we simply have that

$$\iint_R f(x,y)\,\mathrm{d}A = \int_0^4 \int_{x^2}^{8\sqrt{x}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

While sketching the regions of integration is always helpful, it is also possible to switch the order of integration without making any pictures (although I do not recommend doing so).

Example 14

Reverse the order of integration in $\displaystyle\int_0^1 \int_1^{e^y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$.

Let's do it algebraically, starting from $0 \le y \le 1$ and $1 \le x \le e^y$, and rewriting them in an equivalent form which makes the bounds for x constant (as the region of integration is not a rectangle, we will pay the price: the bounds for y will be functions of x instead of constants as well).

Since $y \le 1$, we have $e^y \le e^1 = e$. So, we already have $1 \le x \le e$. As for the bounds for y, applying ln (which is an increasing function and thus preserves inequalities) to $x \le e^y$, we obtain $\ln x \le y \le 1$ (the latter inequality given initially). Conclusion:

$$\int_0^1 \int_1^{e^y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_1^e \int_{\ln x}^1 f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

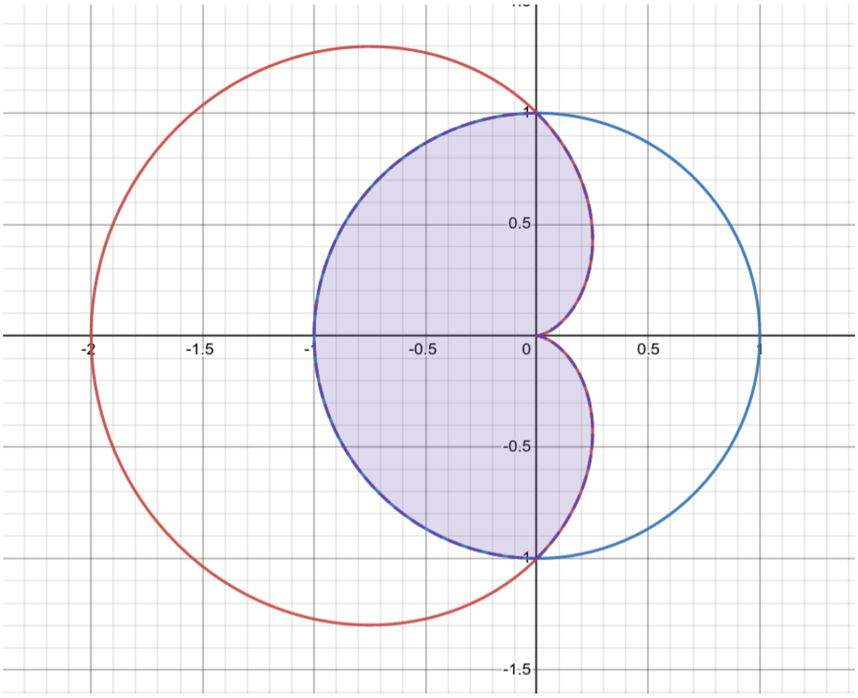
4 September 13th

Let's start with the setup for two of the most challenging problems in HW3.

Exercise 1

Find the area bounded between the unit circle $r = 1$ and the cardioid $r = 1 - \cos\theta$.

The "cardio" in "cardioid" hints at the origin of the word and that the curve described by $r = 1 - \cos\theta$ should resemble a heart. See the following picture:



For $0 \leq \theta < \pi/2$, we have that $0 \leq r < 1$, and $r = 1$ for $\theta = \pi/2$ (this indicates the intersection point $(0,1)$). For $\pi/2 < \theta < 3\pi/2$, we have that $r > 1$, so the cardioid doesn't intersect the circle (e.g., for $\theta = \pi$ we see that the point $(-2,0)$ is in the cardioid). For $\theta = 3\pi/2$, we again have $r = 1$ (this indicates the second intersection point $(0,-1)$). The "polar graph" changes at $\theta = \pi/2$ and $\theta = 3\pi/2$, so we should expect that more than one integral is needed here. The region considered is symmetric about the x-axis, so we may write its area as

$$A = 2\left(\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta + \int_{\pi/2}^{\pi}\int_0^1 r\,\mathrm{d}r\,\mathrm{d}\theta\right)$$

Don't forget the correction factor of r in $r\,\mathrm{d}r\,\mathrm{d}\theta$. Now think of "slices" as explained in class:

- For a generic θ between 0 and $\pi/2$, send forward a "radial slice". The lowest value of r in this slice is 0 and the highest is $1 - \cos\theta$ (it hits the cardioid before the circle).

- For a generic θ between $\pi/2$ and π, send forward a "radial slice". The lowest value of r in this slice is 0 and the highest is 1 (it hits the circle before the cardioid).

Exercise 2

Use a triple integral to compute the volume bounded between the plane $z = 0$ and the graph of $z = \sin y$, over the region $D = \{(x, y) : x \leq y \leq \pi$ and $0 \leq x \leq \pi\}$.

Let's set up the iterated integral in the order $\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$ (which is one of the most useful when we have a solid bounded by graphs above and below):

$$\iiint_D 1\,\mathrm{d}V = \int_?^? \int_?^? \int_?^? 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Recall that the when computing a volume with a triple integral, the function to be integrated is 1 *by default* (note the general mechanism: to find a volume one integrates $1\,\mathrm{d}V$, for an area one integrates $1\,\mathrm{d}A$, and for arclengths one integrates $1\,\mathrm{d}s$ – more on this last one in a week or two). Sketching the region yourself (or looking at what the book gives you), you can see that the lowest value of x that occurs in D is 0 and the highest is π. So we have

$$\int_0^\pi \int_?^? \int_?^? 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Fixed a generic value of x, what are the bounds for y in terms of x? This leads us to

$$\int_0^\pi \int_x^\pi \int_?^? 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Fixed generic values of x and y, what are the bounds for z in terms of x and y? We obtain that

$$\int_0^\pi \int_x^\pi \int_0^{\sin y} 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

is the integral to be computed. Time for a sanity-check: if you were to compute the volume under the graph of $\sin y$ with a double integral, you would have

$$\int_0^\pi \int_x^\pi \sin y\,\mathrm{d}y\,\mathrm{d}x.$$

Now observe that $\displaystyle\int_0^{\sin y} \mathrm{d}z = \sin y$. This general mechanism explains why we integrate "top minus bottom" when computing the volume of a solid bounded between two graphs, it's like

$$\iint_R \int_{\text{bottom}}^{\text{top}} 1\mathrm{d}z\,\mathrm{d}A = \iint_R (\text{top} - \text{bottom})\mathrm{d}A.$$

Before we go back to the regular schedule, let's register a useful shortcut (for specific situations):

Lemma

Let f be a continuous function on the rectangle $[a,b] \times [c,d]$, and assume that variables may be separated, i.e., that $f(x,y) = g(x)h(y)$, where g is a continuous function on $[a,b]$ and h is a continuous function on $[c,d]$. Then

$$\iint_{[a,b]\times[c,d]} f(x,y)\,\mathrm{d}A = \left(\int_a^b g(x)\,\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right).$$

In other words, **when the region of integration is a rectangle and variables can be separated on the integrand**, the integral of a product equals the product of the integrals.

Proof: Just compute

$$\begin{aligned}\iint_{[a,b]\times[c,d]} f(x,y)\,\mathrm{d}A &= \int_a^b \int_c^d g(x)h(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_a^b g(x)\left(\int_c^d h(y)\,\mathrm{d}y\right)\,\mathrm{d}x \\ &= \left(\int_a^b g(x)\,\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right),\end{aligned}$$

where in the first equalilty we used Fubini's Theorem, on the second one we pulled $g(x)$ out of the inner integral relative to y, and on the third equal sign we pulled out the **number** (and not function!) $\int_c^d h(y)\,\mathrm{d}y$ out of the outer integral relative to x. $\square$

Finally, let's talk about polar coordinates. What you need to know is:

- $x = r\cos\theta$;
- $y = r\sin\theta$;
- $x^2 + y^2 = r^2$;
- $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$.

Note:

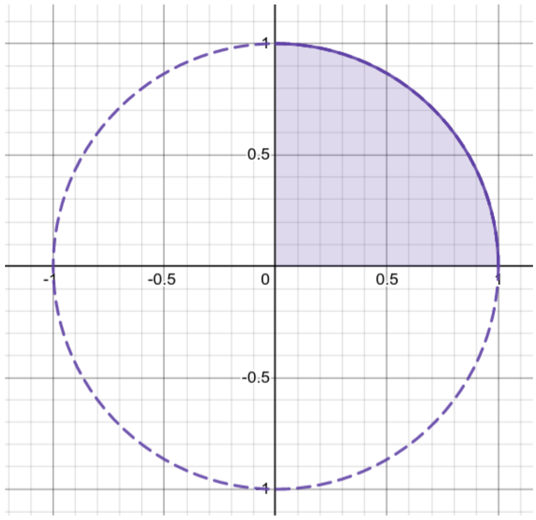
It is absolutely crucial to not forget r in $r\,\mathrm{d}r\,\mathrm{d}\theta$! This is probably the most common mistake students make in multivariable calculus!

Remembering this, you should be able to solve essentially every problem involving polar coordinates. We start with the following problem:

Example 15

Compute $\displaystyle\iint_R \frac{\mathrm{d}A}{\sqrt{16-x^2-y^2}}$, *where* $R = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4, x \geq 0, y \geq 0\}$.

First of all, we sketch the region of integration:



With this, we move on to the change to polar coordinates:

$$\begin{aligned}
\iint_R \frac{\mathrm{d}A}{\sqrt{16-x^2-y^2}} &= \int_0^{\pi/2}\int_0^2 \frac{{\color{red}r}\,\mathrm{d}r\,\mathrm{d}\theta}{\sqrt{16-r^2}} \\
&= \left(\int_0^{\pi/2}\mathrm{d}\theta\right)\left(\int_0^2 \frac{r}{\sqrt{16-r^2}}\,\mathrm{d}r\right) \\
&\overset{(*)}{=} \frac{\pi}{2}\left(-\sqrt{16-r^2}\right)\Big|_0^2 \\
&= \frac{\pi}{4}(4-2\sqrt{3}) \\
&= 2\pi - \pi\sqrt{3}.
\end{aligned}$$

On $(*)$, we could have done an u-substitution to flesh out more details, as in $u = 16-r^2$, so $\mathrm{d}u = -2r\,\mathrm{d}r$, and thus

$$\int \frac{r}{\sqrt{16-r^2}}\,\mathrm{d}r = \int \frac{1}{\sqrt{u}}\left(-\frac{\mathrm{d}u}{2}\right) = -\frac{1}{2}\int u^{-1/2}\,\mathrm{d}u = -u^{1/2} = -\sqrt{16-r^2}.$$

Note that by doing this u-substitution on the side, **we don't risk writing the integrand as a function of u but keeping the bounds for the old variable r; this is NOT ok even if you know you'll come back to the initial variable later.**

Next, a fun example (which you should see at least once in your life):

Example 16

We know from single-variable calculus that the integral

$$\int e^{-x^2}\,dx$$

cannot be solved, in the sense that there is no elementary anti-derivative for e^{-x^2}. If you don't remember this or don't believe me, set up a timer on your phone for, say, 15 minutes (but no longer!) and try to solve it yourself. Failure builds up the character. So, let's take the impossible and make it worse. Consider

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx.$$

Indulging the lack of self-love of yours truly, let's not stop here and square it:

$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2.$$

Now, the fun begins.

$$\begin{aligned}\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 &= \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\\ &= \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right)\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\,dx\,dy\end{aligned}$$

Performing a change of variables to polar coordinates, we continue:

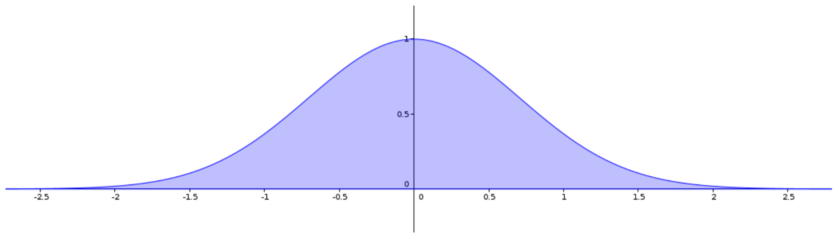
$$\begin{aligned}\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 &= \int_0^{2\pi}\int_0^{+\infty} e^{-r^2}\,{\color{red}r}\,dr\,d\theta\\ &\overset{(*)}{=} \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{+\infty} re^{-r^2}\,dr\right)\\ &= 2\pi\left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_0^{+\infty}\\ &= \pi(0-(-1))\\ &= \pi,\end{aligned}$$

where on $(*)$ we used an obvious variant of the lemma regarding the integral of a product over a rectangle. Note how the correction factor r coming from the

Jacobian saved the day. We conclude (as the original definite integral is positive to begin with) that

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

This integral appears in statistics, when studying random variables $X \sim \mathcal{N}(\mu, \sigma^2)$ with normal probability distribution (here, μ is the mean and σ^2 is the variance); the graph of $f(x) = \mathrm{e}^{-x^2}$ is the "bell curve" you might be already familiar with:



We have effectively shown that the area under the bell curve equals $\sqrt{\pi}$.

Let's also practice setting up bounds of integration for triple integrals, like done last week in class for double integrals.

Example 17

Write an iterated integral in the order $\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$ *for* $\iiint_D f(x,y,z)\,\mathrm{d}V$*, where* f *is a generic continuous function and* D *is the sphere centered at the origin* $(0,0,0)$ *with radius* 9*.*

Observe that we are **not** being asked to compute any integrals here. The continuous function f is arbitrary! If $f = 1$ and we computed the integral, the result would be the volume. If f were positive everywhere and interpreted as a mass distribution, the integral would be the total mass. But in general, f doesn't necessarily have any relation to the sphere whatsoever. What matters here are the bounds we need to find:

$$\iiint_D f(x,y,z)\,\mathrm{d}V = \int_?^? \int_?^? \int_?^? f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

We always start with the outermost variables, whose bounds must always be both constants. What are the lowest and highest values of x occuring in D? This leads us to

$$\iiint_D f(x,y,z)\,\mathrm{d}V = \int_{-9}^{9} \int_?^? \int_?^? f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Now, fixed a value of x, we slice D with a vertical plane parallel to the yz-plane, but passing through the point $(x,0,0)$. The resulting cross-section is a circle on the yz-plane, described by the equation $y^2 + z^2 = 81 - x^2$ (since the original equation of the sphere was $x^2 + y^2 + z^2 = 81$ and x is fixed, we just subtract it on both sides). On this cross-section, what are the lowest and highest values of y (the next variable, in order)? Setting $z = 0$, we obtain

$$\iiint_D f(x,y,z)\,\mathrm{d}V = \int_{-9}^{9}\int_{-\sqrt{81-x^2}}^{\sqrt{81-x^2}}\int_{?}^{?} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

With x still fixed and y now fixed as well, what are the lowest and highest values of z that appear in the intersection of the circle $y^2+z^2 = 81-x^2$ and the "vertical" (in the yz-plane) line passing through $(y,0)$? We finally obtain

$$\iiint_D f(x,y,z)\,\mathrm{d}V = \int_{-9}^{9}\int_{-\sqrt{81-x^2}}^{\sqrt{81-x^2}}\int_{-\sqrt{81-x^2-y^2}}^{\sqrt{81-x^2-y^2}} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

5 September 20th

Let's start with the setup for one of the homework problems:

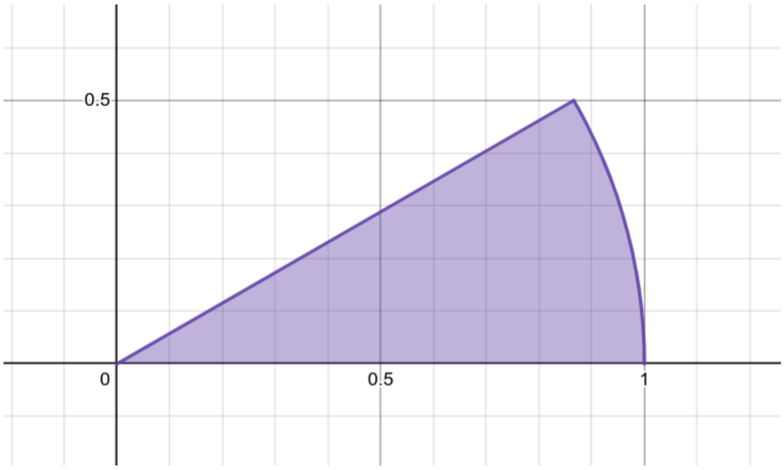
Exercise 3

Describe the region of integration for $\int_{-1}^{1}\int_{0}^{1/2}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *in cylindrical coordinates.*

For each value of z between -1 and 1, the bounds for the double integral

$$\int_{0}^{1/2}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y$$

over the "z-slice" don't depend on z, which means that all slices look the same. The angle between the line of equation $x = \sqrt{3}y$ and the x-axis is $\pi/6$ (since $\tan(\pi/6) = \sqrt{3}/3 = 1/\sqrt{3}$ and $y = (1/\sqrt{3})y$), and we have that:



To wit, for each fixed value of y between 0 and $1/2$, x *starts* at $\sqrt{3}y$ and *stops* at $\sqrt{1-y^2}$. We have that

$$\int_{-1}^{1}\int_{0}^{1/2}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{0}^{2\pi}\int_{0}^{\pi/6}\int_{0}^{1} f(r,\theta,z)r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$

Exercise 4

Change the order of integration on $\int_{1}^{4}\int_{z}^{4z}\int_{0}^{\pi^2} \frac{\sin\sqrt{yz}}{x^{3/2}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$ *so it works.*

Let's do an elimination process. The variables y and z are in equal standing in the expression $\sin\sqrt{yz}/x^{3/2}$, and there's no way to integrate something like $\int \sin\sqrt{u}\,\mathrm{d}u$, so we must have $\mathrm{d}x$ first to have any hope of succeeding. Between the two choices $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and $\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y$, the latter seems better, as the outermost bounds $\int_1^4$ for z will remain unchanged. Now, we must look at

$$\int_1^4 \left(\int_?^? \int_?^? \frac{\sin\sqrt{yz}}{x^{3/2}}\,\mathrm{d}x\,\mathrm{d}y \right) \mathrm{d}z.$$

However, for any fixed value of z, the region of integration for the inner double integral is just a rectangle, and so we may just switch the order of the differentials without worrying about the bounds:

$$\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin\sqrt{yz}}{x^{3/2}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z = \int_1^4 \int_0^{\pi^2} \int_z^{4z} \frac{\sin\sqrt{yz}}{x^{3/2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Note that if, for each value of z, the corresponding cross-sections were not simple rectangles, this would not have been so simple!

Next, let's review how spherical coordinates work. Suppose that (x, y, z) is a point in space, and consider the following setup:

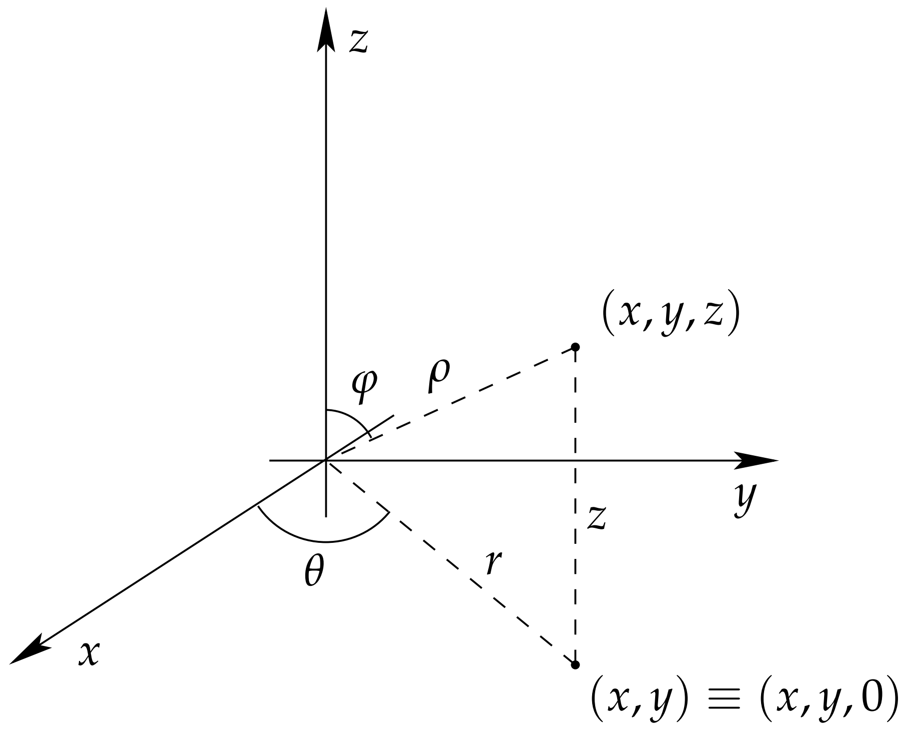


Figure 1: Spherical coordinates in $\mathbb{R}^3$.

Thinking about polar coordinates in the xy-plane, we may write $x = r\cos\theta$ and $y = r\sin\theta$. To relate r with ρ and φ, see the following right triangle:

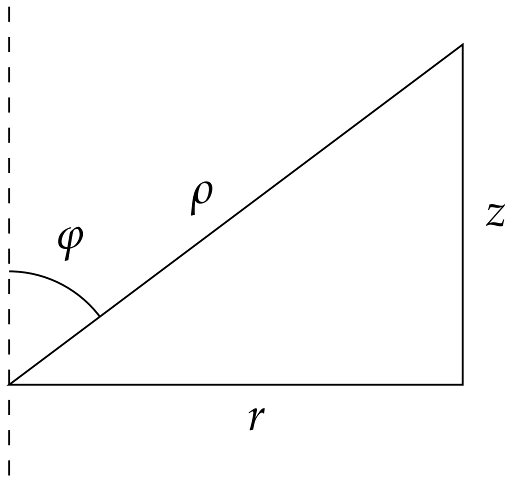


Figure 2: Eliminating the variable r.

Since we have $r = \rho \sin\varphi$ and $z = \rho\cos\varphi$, we may substitute it to obtain

$$\begin{cases} x = \rho \sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi. \end{cases}$$

We also note that $\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$, by the Pythagorean Theorem applied twice. To cover all of space, we must have

$$\rho \geq 0, \quad 0 \leq \theta < 2\pi \quad \text{and} \quad 0 \leq \varphi \leq \pi.$$

Observe that if φ took values bigger than π, we would be counting some points in space twice. The correction between differentials is $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2 \sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta$. Let's put all of it together in an example:

Example 18

Compute $\displaystyle\iiint_D \frac{\mathrm{d}V}{(x^2+y^2+z^2)^{3/2}}$, *where D is the solid bounded by the spheres of radius* 1 *and* 2 *centered at the origin.*

We directly have that

$$1 \leq \rho \leq 2, \quad 0 \leq \theta \leq 2\pi, \quad \text{and} \quad 0 \leq \varphi \leq \pi.$$

This is because the smallest distance from a point in D to the origin is 1, while the largest is 2. We are not omissing any directions around the z-axis, so the interval for θ is full and, lastly, φ goes from 0 to π (because stopping at $\pi/2$ would cover only the upper half of D). With this in place, we proceed:

$$\iiint_D \frac{\mathrm{d}V}{(x^2+y^2+z^2)^{3/2}} = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \frac{\rho^2\sin\varphi}{(\rho^2)^{3/2}}\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta$$

$$
\begin{aligned}
&= \int_0^{2\pi}\int_0^{\pi}\int_1^2 \frac{\sin\varphi}{\rho}\,d\rho\,d\varphi\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{\pi}\sin\varphi\,d\varphi \int_1^2 \frac{d\rho}{\rho} \\
&= 2\pi\cdot 2\cdot \ln 2 \\
&= 4\pi\ln 2.
\end{aligned}
$$

Let's now review how change of variables work. It is essentially a multivariable version of an u-substitution. Then, we had that if $u = g(x)$, then $du = g'(x)\,dx$, and then

$$\int_a^b f(u)\,du = \int_c^d f(g(x))g'(x)\,dx,$$

where $g(c) = a$ and $g(d) = b$. Two things happened here: we needed to update the interval of integration, and pay attention to the correction in the differential. The same thing will happen with more than one variable, with the only difference being that computing this correction is more complicated. Roughly, what happens is that

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \quad\text{and}\quad du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy.$$

We will occasionally need to pay attention to both. Here, we have that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix},$$

and same for $\partial(u,v)/\partial(x,y)$. The bars here stand for "taking the absolute value of the determinant". How can you be sure you're not mixing things up? The following illegal cancellations should happen (in your head, never in the paper):

$$dx\,dy = \left|\frac{\partial(x,y)}{\cancel{\partial(u,v)}}\right| \cancel{du\,dv} \quad\text{and}\quad du\,dv = \left|\frac{\partial(u,v)}{\cancel{\partial(x,y)}}\right| \cancel{dx\,dy}.$$

An expression like

$$dx\,dy = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| du\,dv,$$

where things "don't want to cancel each other", is **wrong**. In any case, it is often easier to compute one of $|\partial(u,v)/\partial(x,y)|$ or $|\partial(x,y)/\partial(u,v)|$ over the other (depending on how the change of variables was set up or given to you). The strategy here is to always go for the easier one and use the inverse relation

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$$

if needed. Let's put all of this together in one last example, emphasizing all the steps to be carried out.

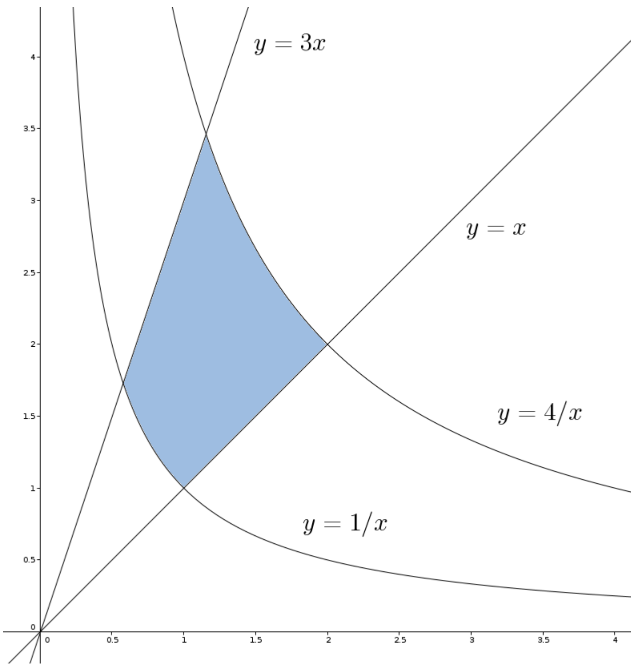
Example 19

Compute the integral

$$\iint_R e^{xy}\, dA,$$

where R is the region in the first quadrant bounded by the hyperbolas $y = 1/x$ and $y = 4/x$, and the lines $y = x$ and $y = 3x$.

- Step 1: sketch the region R. You just have to remember that the graph of $y = 1/x$ is a branch of a hyperbola and that the 4 in $y = 4/x$ is just a rescaling factor.



- Step 2: rewrite the bounds of R in the form "something" = "constant" and hopefully read the new variables and their bounds from there. Namely, we have that

$$\begin{array}{ccc} y = 1/x & \to & xy = 1 \\ y = 4/x & \to & xy = 4 \\ y = x & \to & y/x = 1 \\ y = 3x & \to & y/x = 3 \end{array}$$

  This suggests letting $u = xy$ and $v = y/x$. Immediately, the new bounds are $1 \leq u \leq 4$ and $1 \leq v \leq 3$. Who is u and who is v is not relevant here: the rectangle in the uv-plane corresponding to R will just come out rotated, and the negative sign you get in the Jacobian determinant disappears because of the absolute value present when we write $du\, dv$ in terms of $dx\, dy$ or vice-versa.

- Step 3: compute the Jacobian determinant. As we have u and v in terms of x and y, it's easier to begin with

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} = \frac{y}{x} - x\left(-\frac{y}{x^2}\right) = 2\frac{y}{x} = 2v.$$

As $v \geq 0$, we have $|2v| = 2v$, so

$$\mathrm{d}u\,\mathrm{d}v = 2v\,\mathrm{d}x\,\mathrm{d}y \implies \mathrm{d}x\,\mathrm{d}y = \frac{1}{2v}\,\mathrm{d}u\,\mathrm{d}v.$$

- Step 4: plug everything into the original integral and solve it.

$$\iint_R \mathrm{e}^{xy}\,\mathrm{d}A = \int_1^3\int_1^4 \frac{\mathrm{e}^u}{2v}\,\mathrm{d}u\,\mathrm{d}v = \left(\int_1^3 \frac{1}{2v}\,\mathrm{d}v\right)\left(\int_1^4 \mathrm{e}^u\,\mathrm{d}u\right) = \frac{\ln 3}{2}(\mathrm{e}^4 - \mathrm{e}).$$

6 September 27th

We'll start going over the setup for a couple of homework problems.

Exercise 5

Compute the integral $\iint_R xy\,\mathrm{d}A$, *where* R *is the region bounded by the ellipse whose equation is* $9x^2+4y^2=36$. *Use* $x=2u$ *and* $y=3v$.

Here, the idea is to note that the ellipse equation becomes $u^2+v^2=1$, so the region R in the xy-plane corresponds to the unit circle S in the uv-plane. The next step would be to compute the Jacobian determinant of the change-of-variables. To do so, note that the correct relation is

$$\mathrm{d}x\,\mathrm{d}y = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\,\mathrm{d}v.$$

In this case, we have that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \implies \mathrm{d}x\,\mathrm{d}y = 6\,\mathrm{d}u\,\mathrm{d}v.$$

With this in place, we have that

$$\iint_R xy\,\mathrm{d}x\,\mathrm{d}y = \iint_S (2u)(3v)(6\,\mathrm{d}u\,\mathrm{d}v) = \iint_S 36uv\,\mathrm{d}u\,\mathrm{d}v.$$

For some strange reason people seem to be under the impression that the bounds for this last integral are simply

$$\int_{-1}^{1}\int_{-1}^{1} 36uv\,\mathrm{d}u\,\mathrm{d}v,$$

but this is **wrong**: all-constant bounds for u and v would mean that the ellipse got sent via the change-of-variables $(x,y)\mapsto(u,v)$ to a rectangle, but we know that S is a circle instead. The correct expression would be

$$\int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} 36uv\,\mathrm{d}u\,\mathrm{d}v,$$

but the point of using polar coordinates is precisely to avoid annoying bounds such as those. A further change of coordinates $(u,v)\mapsto(r,\theta)$ with $\mathrm{d}u\,\mathrm{d}v = r\,\mathrm{d}r\,\mathrm{d}\theta$ does the job now.

It is possible, however, to cut the middle man and use "fake polar coordinates", by noting that

$$9x^2+4y^2=36 \implies \frac{x^2}{4}+\frac{y^2}{9}=1 \implies \left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$$

suggests setting $x = 2r\cos\theta$ and $y = 3r\sin\theta$. This time, the equation $r = 1$ corresponds not to the unit circle, but to the given ellipse. And we have that $\mathrm{d}x\,\mathrm{d}y = 6r\,\mathrm{d}r\,\mathrm{d}\theta$, since 6 is the product of the stretching factors 2 and 3. More precisely, one could compute

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{bmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{bmatrix} = \begin{bmatrix} 2\cos\theta & -2r\sin\theta \\ 3\sin\theta & 3r\cos\theta \end{bmatrix},$$

so that

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = (2\cos\theta)(3r\cos\theta) - (-2r\sin\theta)(3\sin\theta) = 6r,$$

by using that $\cos^2\theta + \sin^2\theta = 1$. Of course, you're supposed to solve this problem in the subpar way the book suggests with the intermediate variables (u,v), since it was explicitly said so in the problem statement.

Now, we move on to understand what line integrals are really about. Formally, they can be defined as certain limits of Riemann-like sums. What you need to keep in mind, however, is that there are two types of line integrals: scalar line integrals of functions, and line integrals of vector fields.

- Scalar case: $\int_C f\,\mathrm{d}s$. For curves in the plane, $\int_C f\,\mathrm{d}s$ equals the area of the "curtain" determined by the curve C inside the domain of f, and the graph surface $z = f(x,y)$ in space. "Stretching" the curve C like a string, the curve C can be thought of an "axis", and the integral computes the area of a graph over this axis. So, scalar line integrals are very similar to single-variable integrals, in this sense. When $f = 1$, the integral $\int_C \mathrm{d}s$ gives simply the length of the curve C. If $f > 0$ is regarded as a mass density, then the integral $\int_C f\,\mathrm{d}s$ is the total mass of the string C. In particular, with line integrals we can compute the center of mass/gravity of a string with non-uniform mass distribution, as the point

$$\left(\frac{\int_C xf\,\mathrm{d}s}{\int_C f\,\mathrm{d}s}, \frac{\int_C yf\,\mathrm{d}s}{\int_C f\,\mathrm{d}s}, \frac{\int_C zf\,\mathrm{d}s}{\int_C f\,\mathrm{d}s}\right).$$

- Vector case: $\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$. Let's think of basic mechanics: if a *constant* force F moves a block (with a certain mass) along a linear path, with displacement d, the work realized by the force to perform this action is $W = Fd$. The assumptions that the force F is constant and that the displacement happens along a linear path are very restrictive. Say that the initial position of the block is at $x = a$ and that the final one is at $x = b$, with $a < b$. Then $W = \int_a^b F\,\mathrm{d}x$ (the constant force may be pulled out of the integral, while $\int_a^b \mathrm{d}x = b - a = d$). This suggests that the work done by a (generally non-constant) force field $\boldsymbol{F}$ on space to move a block from an initial position to a final position, along some now curvilinear path C, should simply be $W = \int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$. The generalization was

$$\int_a^b \to \int_C, \quad F \to \boldsymbol{F}, \quad \mathrm{d}x \to \mathrm{d}\boldsymbol{r}.$$

We still need to understand what the notations for line integrals mean. First, we observe that the dot $\cdot$ in $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ is actually a dot product, between the vector field $\boldsymbol{F}$ and $\mathrm{d}\boldsymbol{r}$, the "infinitesimal tangent vector to C". We write $\mathrm{d}\boldsymbol{r} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$. Why does it make sense to form a vector with those differentials? If there's any justice in the world, we should be able to write $\mathrm{d}\boldsymbol{r} = \boldsymbol{r}'(t)\,\mathrm{d}t$. But then

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{r}'(t)\,\mathrm{d}t = (x'(t), y'(t), z'(t))\,\mathrm{d}t = (x'(t)\,\mathrm{d}t, y'(t)\,\mathrm{d}t, z'(t)\,\mathrm{d}t) = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z).$$

As for the remaining differential $\mathrm{d}s$, a similar reasoning goes. Writing $\mathrm{d}s = s'(t)\,\mathrm{d}t$, it remains to understand what $s'(t)$, or more generally $s(t)$, means. This turns out to be a standard notation for the arclength function of a curve. Namely, given a parametrization $\boldsymbol{r}(t)$ of C on some interval $[a,b]$, the integral

$$s(t) = \int_a^t \|\boldsymbol{r}'(\tau)\|\,\mathrm{d}\tau$$

computes the arclength of C from the initial point $\boldsymbol{r}(a)$ to the chosen instant $\boldsymbol{r}(t)$ (in particular, $s(b)$ is the full length of C). The Fundamental Theorem of Calculus says that $s'(t) = \|\boldsymbol{r}'(t)\|\,\mathrm{d}t$. We now have our **dictionary of differentials** completed:

$$\begin{aligned}
\mathrm{d}\boldsymbol{r} &= (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) \\
\mathrm{d}s &= \|\boldsymbol{r}'(t)\|\,\mathrm{d}t \\
\mathrm{d}x &= x'(t)\,\mathrm{d}t \\
\mathrm{d}y &= y'(t)\,\mathrm{d}t \\
\mathrm{d}z &= z'(t)\,\mathrm{d}t
\end{aligned}$$

As for how to ahead and compute line integrals, generally, we can organize ourselves with steps, as in Example 19 (p. 31).

Step 0: If dealing with a vector line integral $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$, writing the components of $\boldsymbol{F}$ as (f, g, h), evaluate the dot product

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_C (f, g, h) \cdot (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) = \int_C f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z.$$

Step 1: Find a parametrization $\mathrm{r}(t)$, $a \leq t \leq b$, for C. Sometimes, the problem gives $\boldsymbol{r}(t)$ to you. Other times, you have to figure it out yourself (but it's generally not too difficult).

Step 2: Set up all the differentials appearing in the problem as something times $\mathrm{d}t$, by using the little dictionary above.

Step 3: Plug $x = x(t)$, $y = y(t)$, $z = z(t)$ into the given integral, as well as all the differentials you have set up in Step 2. You have now obtained a single-variable integral $\int_a^b \cdots\,\mathrm{d}t$. Solve it.

Of course, everything here was described for functions and fields in space. When working in two dimensions only, just ignore the variable z and the third components of everything.

Example 20

Compute the integral

$$\oint_C xy\,\mathrm{d}s,$$

where C is the unit circle in the plane with parametrization $\boldsymbol{r}(t) = (\cos t, \sin t)$, defined on $0 \leq t \leq 2\pi$.

First, we note that the circle on the integral, as in $\oint_C$ as opposed to $\int_C$, is **just a reminder** that the curve C on which we integrate over is closed. It makes absolutely no difference on how we'll solve it. Step 0 is unneeded, it's a scalar line integral. Step 1 is also unneeded, as the problem gave us $\boldsymbol{r}(t)$. For Step 2, we have that $\boldsymbol{r}'(t) = (-\sin t, \cos t)$, so $\|\boldsymbol{r}'(t)\| = 1$ for all t. This means that $\mathrm{d}s = \mathrm{d}t$. Finally, Step 3:

$$\oint_C xy\,\mathrm{d}s = \int_0^{2\pi} \cos t \sin t\,\mathrm{d}t = \frac{1}{2}\int_0^{2\pi} \sin(2t)\,\mathrm{d}t = 0.$$

Example 21

Compute the integral

$$\int_C (x+y)\,\mathrm{d}x,$$

where C is the upper half-circle (centered at the origin) with radius 2, oriented counterclockwise.

This is not really a scalar line integral, but a vector line integral in disguise, with Step 0 already performed. One may see it as $\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$ with the field $\boldsymbol{F}$ given by $\boldsymbol{F}(x,y) = (x+y, 0)$. In any case, for Step 1, we may take $\boldsymbol{r}(t) = (2\cos t, 2\sin t)$, with $0 \leq t \leq \pi$. For Step 2, there's only one differential to be considered, $\mathrm{d}x$: since $x = 2\cos t$, we have $\mathrm{d}x = -2\sin t\,\mathrm{d}t$. Now, Step 3 becomes just

$$\begin{aligned}\int_C (x+y)\,\mathrm{d}x &= \int_0^{\pi} (2\cos t + 2\sin t)(-2\sin t\,\mathrm{d}t) \\ &= -4\int_0^{\pi} (\cos t \sin t + \sin^2 t)\,\mathrm{d}t \\ &= -2\pi.\end{aligned}$$

Example 22

Compute $\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$, where $\boldsymbol{F}(x,y) = (x,y)$ and C is parametrized by $\boldsymbol{r}(t) = (4t, t^2)$, with $0 \leq t \leq 1$.

We're dealing with a vector line integral, so let's do Step 0:

$$\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_C (x,y)\cdot(\mathrm{d}x, \mathrm{d}y) = \int_C x\,\mathrm{d}x + y\,\mathrm{d}y.$$

There's no need to do Step 1, as the problem gave us $\boldsymbol{r}(t)$. As for Step 2, the only differentials appearing here[a] are $\mathrm{d}x$ and $\mathrm{d}y$. So we just note that $x = 4t$ and $y = t^2$ immediately give us that $\mathrm{d}x = 4\,\mathrm{d}t$ and $\mathrm{d}y = 2t\,\mathrm{d}t$. With this in place, Step 3 is straightforward:

$$\begin{aligned}\int_C x\,\mathrm{d}x + y\,\mathrm{d}y &= \int_0^1 4t(4\,\mathrm{d}t + t^2(2t\,\mathrm{d}t)) \\ &= \int_0^1 (16t + 2t^2)\,\mathrm{d}t \\ &= 8 + \frac{2}{3} \\ &= \frac{26}{3}.\end{aligned}$$

[a] If a vector field in three dimensions has zero as one of its components, the corresponding differential will simply not appear: the zero component kills it with the dot product.

Lastly, we'll explain the true reason for the name "conservative" when discussing conservative fields, and review the consequences of this definition. We go back for a moment to classical mechanics. Imagine that a particle in three-dimensional Euclidean space $\mathbb{R}^3$, of mass $m > 0$, moves in space subject to the action of a force field $\boldsymbol{F}$. The trajectory $\boldsymbol{r} = \boldsymbol{r}(t)$ satisfies Newton's force equation $\boldsymbol{F}(\boldsymbol{r}(t)) = m\boldsymbol{r}''(t)$ (the old formula $F = ma$ from your childhood). If $\boldsymbol{F}$ is a conservative field and we write $\boldsymbol{F} = -\nabla\varphi$ for some potential function φ (the negative sign will be addressed in an instant and is not relevant in this class), the total energy function

$$E(t) = \frac{m\|\boldsymbol{r}'(t)\|^2}{2} + \varphi(\boldsymbol{r}(t))$$

is constant along $\boldsymbol{r}$, since

$$\begin{aligned}E'(t) &= m\boldsymbol{r}''(t)\cdot\boldsymbol{r}'(t) + \nabla\varphi(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t) \\ &= \big(m\boldsymbol{r}''(t) + \nabla\varphi(\boldsymbol{r}(t))\big)\cdot\boldsymbol{r}'(t) \\ &= \big(m\boldsymbol{r}''(t) - \boldsymbol{F}(\boldsymbol{r}(t))\big)\cdot\boldsymbol{r}'(t) \\ &= \boldsymbol{0}\cdot\boldsymbol{r}'(t) \\ &= 0.\end{aligned}$$

In words: if $\boldsymbol{F}$ is conservative, there is a potential function φ giving rise to a **conserved quantity**, the total energy. If the field $\boldsymbol{F}$ were not conservative, we would not have a potential function, and so we would not be able to set the total energy function. If one wrote $\boldsymbol{F} = \nabla\varphi$ instead of $-\nabla\varphi$, the conserved quantity would be the kinetic energy **minus** the potential energy (as opposed to plus), which is psychologically uncomfortable.

In any case, this condition has nice mathematical consequences, and the Fundamental Theorem of Calculus comes back with a vengeance:

Theorem 3

Let φ be a differentiable function of n variables and C be a curve, from a point A to a point B. Then

$$\int_C \nabla\varphi \cdot \mathrm{d}\boldsymbol{r} = \varphi(B) - \varphi(A).$$

In particular, the result does not depend on the actual curve C, but only on the endpoints A and B. And when C is a closed curve (i.e., $A = B$), then

$$\oint_C \nabla\varphi \cdot \mathrm{d}\boldsymbol{r} = 0.$$

Remark. Compare this with $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$, where F is an anti-derivative of f. One should think of φ as an anti-derivative of $\boldsymbol{F}$ (in the gradient sense). Of course, the point is that it is not always true that a potential φ exists. When this does happen, the path-independence is used in practice as follows: imagine you're given a line integral of a conservative field $\boldsymbol{F}$ over a curve C which is terribly unpleasant to deal with. Since the field $\boldsymbol{F}$ is conservative, you are free to completely ignore the curve C, and replace it with another curve C', as simple as possible (such as the line segment joining the endpoints of the original C).

If $\boldsymbol{F} = (f, g)$ equals a gradient $\nabla\varphi = (\varphi_x, \varphi_y)$, then we necessarily have $f_y = g_x$, since $f_y = \varphi_{xy} = \varphi_{yx} = g_x$. This means that if $\boldsymbol{F} = (f, g)$ does **not** satisfy that $f_y = g_x$ then $\boldsymbol{F}$ is **not** equal to any gradient field, and therefore one has to use the techniques discussed in the beginning of class (with parametrizing, setting up differentials, etc). Recall that to find φ (when possible), the brief algorithm is: integrate $f = \varphi_x$ with respect to x, and the constant of integration is a function of the remaining variable, $c(y)$. Now differentiate both sides of what you have obtained with respect to y, and use $g = \varphi_y$. Then find $c(y)$.

7 October 4th

We move on to the second part of this course: linear systems and matrices. The goal here is to understand the general mechanism to solve linear systems with several equations and variables (as opposed to small systems with only two or three equations and variables). The main tool we'll use to do so will be matrices. The program we'll try to carry out here is to:

$$\begin{aligned} \text{start with linear system} &\Longrightarrow \text{convert it to a matrix} \\ &\Longrightarrow \text{get a simpler matrix} \Longrightarrow \text{get a simpler system,} \end{aligned}$$

and then draw conclusions about the original system from the original one. To understand how to convert between systems and matrices (that is, the first and last arrows above), let's consider the next two exercises. We'll address the middle arrow and explain what we mean by a "simpler matrix" soon.

Exercise 6

What are the coefficient matrix and the augmented matrix for each of the following linear systems?

$$\text{(a)} \begin{cases} x_1 + 3x_2 - x_3 = 1 \\ 2x_1 + 5x_2 + x_3 = 5 \\ x_1 + x_2 + x_3 = 3 \end{cases} \qquad \text{(b)} \begin{cases} x_1 + x_2 + x_3 - x_4 = 1 \\ -x_1 + x_2 - x_3 + x_4 = 3 \\ -2x_1 + x_2 + x_3 - x_4 = 2 \end{cases}$$

For item (a) we have that the coefficient and augmented matrices are, respectively,

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 2 & 5 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{array}\right].$$

Note that we have one row for each equation, and one column per variable in the coefficient matrix. This is a general phenomenon. To obtain the augmented matrix, we *augment* the coefficient matrix with an extra column, containing the right-hand-sides of the equations in the system. When there is no coefficient in front of a variable, as in x_2, it means $1x_2$. When dealing with augmented matrix, the last column is usually separated from the rest with a line or dotted line, just to remind us that it does not correspond to any variable. For item (b), we have that the the coefficient and augmented matrices are, respectively,

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -2 & 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 3 \\ -2 & 1 & 1 & -1 & 2 \end{array}\right].$$

How does the reverse process work?

Exercise 7

If the following two matrices are the augmented matrices for two systems, write the systems explicitly.

$$\text{(a)}\ \left[\begin{array}{cccc|c} 2 & 3 & -1 & 1 & 1 \\ -2 & -3 & 5 & \pi & \sqrt{2} \\ 1/2 & 5 & 0 & 0 & 2022 \end{array}\right] \qquad \text{(b)}\ \left[\begin{array}{ccc|c} 0 & 1 & \mathrm{e} & -2 \\ 4 & \sqrt[3]{2} & -1 & 0 \\ 2 & 0 & \ln(5) & 3 \end{array}\right]$$

This time, we set up one equation per row, reading the coefficients in order, and recalling that the last column after the divide correspond to the right-hand-sides of the equations in the systems. For item (a) we obtain the system

$$\begin{cases} 2x_1 + 3x_2 - x_3 + x_4 & = 1 \\ -2x_1 - 3x_2 + 5x_3 + \pi x_4 & = \sqrt{2} \\ \frac{1}{2}x_1 + 5x_2 & = 2022, \end{cases}$$

while for item (b) we have that

$$\begin{cases} \phantom{4x_1 +{}} x_2 + \mathrm{e}x_3 & = -2 \\ 4x_1 + \sqrt[3]{2}x_2 - x_3 & = 0 \\ 2x_1 \phantom{{}+ \sqrt[3]{2}x_2} + \ln(5)x_3 & = 3. \end{cases}$$

There are two things to observe here.

- The first one is that while most examples you'll see on the book only have nice, integer coefficients, there is nothing saying that this must be the case. The word "linear" in linear system refers to the fact that each equation will be a "linear combination" of variables, as in the first equation $2x_1 + 3x_2 - x_3 + x_4 = 1$ of the first system, but the numeric coefficients can be anything (such as π, e, $\ln(5)$, $\cos(2)$, etc., all legit real numbers).

- The second one is that zeros in the coefficient matrix mean that the corresponding variables are simply absent from the corresponding equation. This is a *good thing*. Having fewer variables to worry about makes dealing with the system easier.

Remark. This is a convenient moment to introduce one piece of terminology: a linear system is called **homogeneous** if **all** the right-hand-sides of the equations in the system are equal to zero. For example, none of the systems presented so far in today's entry are homogeneous. It turns out that understanding homogeneous systems is slightly easier than understanding non-homogeneous one, and solutions to non-homogeneous systems can be found by first finding solutions to the "associated homogeneous system" (obtained by replacing all the right-hand-sides with zeros). More on this later. A similar idea will appear again when we move on to the third part of this class, discussing differential equations (they can be divided into homogeneous and non-homogeneous, and homogeneous ones are nicer).

Following up the idea of the second bulleted point above, we can return to the idea of going from the augmented matrix representing a system to a simpler matrix. The idea will be to create as many zeros as possible. More precisely, we will consider matrices in **reduced row echelon form (RREF)**, which look like this:

$$\begin{bmatrix} 1 & 0 & * & 0 & 0 & * & * \\ 0 & 1 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The **pivots**, which are the first non-zero entries in each row, must all be equal to 1. On each column where a pivot 1 appears, all entries other than the pivot itself must be zero. The pivots must form a "staircase" shape (hence the name "echelon"). The word "reduced", in turn, refers to the fact that all entries in a given column with a pivot, above the pivot, are zero. If a row consists only of zeros, it must be on the bottom of the matrix.

But this cannot be done in a haphazard way. How can one ensure that the RREF of a matrix will correspond to a system with the same solutions as the original one? To ensure that the solutions of the original system will not be affected, one casts a given matrix into RREF by using the following **elementary operations** are allowed:

(1) switch two rows;

(2) multiply any row by a non-zero number;

(3) add to any row a multiple of another row.

When solving a system, we will always follow this procedure of putting the **augmented** matrix into RREF (as opposed to the coefficient matrix), as doing operations between rows is morally the same as doing operations between equations in the system (with the obvious advantage that we won't keep writing variable names x_1, x_2, etc., all the time), and when doing operations between equations, those operations happen on the right-hand side of the equations involved too.

Remark. There are several ways of solving linear systems (one of them being "Cramer's method", for example), but dealing with matrices in RREF is preferred from a computational viewpoint for being more efficient, in the sense that solving a linear system using this algorithm is what takes a computer the fewer number of operations to do. Moreover, RREF is the answer to the natural question "could I do something else to make the matrix simpler?" or "am I missing something?". If the matrix is in RREF, the answer is "no": you did everything possible to simplify the matrix. And, again, *the reason why we'll only stick with the elementary operations above is because they make sure that the matrix in RREF obtained in the end does represent a system equivalent to the original one.*

Exercise 8

Decide whether the following matrix is in RREF and, if not, put it in RREF.

$$\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & -2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The pivots of this matrix are, respectively, 1, 2 and 1. Since we have a pivot which is not equal to 1, the matrix is not in RREF. Each step we carry out has a specific objective, and whatever happens with the rest of the matrix, happens. Our first goal is to turn that pivot 2 into a 1, and for that we may divide the whole second row by 2 (this is an allowed elementary operation):

$$R_2 := \frac{1}{2}R_2 \qquad \sim \qquad \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note how we're borrowing notation from computer science to keep track of the operations performed. This is very important to do as it improves organization and readability. You should take this seriously on HW assignments and exams.

By the way, it may be worth mentioning that there is also something called **echelon form**: it should have the same "staircase" shape as in RREF with all pivots being 1 and rows of zeros being in the bottom, but elements above a pivot don't need to be zero. So, the matrix is now in echelon form, but not RREF. Recognizing when a matrix is in echelon form but not RREF tells you that you're halfway through the process; it's a checkpoint: all that's left to do now is to take out the trash above the pivots. This is done systematically, again from left to right, but now from bottom to top. The next goal should be to eliminate the 2 in the first row of the matrix.

$$R_1 := R_1 - 2R_2 \qquad \sim \qquad \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since on the third column there is no pivot, nothing can be done about the non-zero entries there. We move on to clean up what's above the pivot on the fourth column. As both R_1 and R_2 will interact with R_3, but not with each other, we can do two steps at once:

$$\begin{matrix} R_1 := R_1 - R_3 \\ R_2 := R_2 + \frac{3}{2}R_3 \end{matrix} \qquad \sim \qquad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have obtained the RREF.

Let's see how to put all of this together to solve a more complicated system.

Exercise 9

Solve, if possible, the following system:

$$\begin{cases} x_1 + x_2 - x_5 = 1 \\ x_2 + 2x_3 + x_4 + 3x_5 = 1 \\ x_1 - x_3 + x_4 + x_5 = 0 \end{cases}$$

Like in Exercise 6, let's set up the augmented matrix for this system:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 1 & 0 & -1 & 1 & 1 & 0 \end{array}\right]$$

This matrix is obviously not in RREF; we don't even have the "staircase" shape. The first 1 in the first row, however, is the pivot, and so we need to eliminate all entries below it. This can be achieved as follows:

$$R_3 := R_3 - R_1 \qquad \sim \qquad \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & -1 & -1 & 1 & 2 & -1 \end{array}\right].$$

We have cleaned up everything below the first pivot, so we move on to the next. The pivot in the second row is the 1 appearing in the second column as well, so we need to eliminate the -1 below it. This is done by:

$$R_3 := R_3 + R_2 \qquad \sim \qquad \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

Observe that this -1 could also have been eliminated via $R_3 := R_3 + R_1$, but this operation would produce a 1 in the bottom left corner of the matrix! The reason why this happened is because we're trying to make an operation involving a row whose pivot has already had everything cleaned up below it. The moral of the story here is that each step to be carried has a laser-like focus, a single goal, and if you do an operation which undoes something that should have been already ok by that point, you have done something wrong. As a matter of fact, the matrix is now in echelon form, but not RREF. We must proceed to clean up the spaces above the pivots, going from left to right, as usual. Above the pivot in the second row, we have a 1, which is the next target:

$$R_1 := R_1 - R_2 \qquad \sim \qquad \left[\begin{array}{ccccc|c} 1 & 0 & -2 & -1 & -4 & 0 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

Observe that this operation did not destroy the "staircase" shape because we have only started to clean up spaces above the pivots once the matrix was already in echelon form. We move on to the next pivot, on the third row: we must clean up the -2 and 2 above it. As in the previous example (and this turns out to be a general phenomenon), since both R_1 and R_2 will interact with R_3, but not with each other, we can do the two steps at once:

$$\begin{array}{c} R_1 := R_1 + 2R_3 \\ R_2 := R_2 - 2R_3 \end{array} \quad \sim \quad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 6 & 0 \\ 0 & 1 & 0 & -3 & -7 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

The matrix is now in RREF, and it corresponds to the system

$$\begin{cases} x_1 \qquad\qquad + 3x_4 + 6x_5 = 0 \\ \qquad x_2 \qquad - 3x_4 - 7x_5 = 1 \\ \qquad\qquad x_3 + 2x_4 + 5x_5 = 0, \end{cases}$$

which is equivalent to the original one (that is, this system and the first one have the same solution set). The variables x_4 and x_5 (corresponding to the columns in RREF which could not be cleaned up – as we didn't have pivots to use) are **free**, in the sense that the remaining variables x_1, x_2 and x_3 may be written in terms of x_4 and x_5. We may change notation, say, to $t_1 = x_4$ and $t_2 = x_5$, effectively **parametrizing the solution set of the system**, and writing

$$S = \{(-3t_1 - 6t_2, 3t_1 + 7t_2 + 1, -2t_1 - 5t_2, t_1, t_2) \in \mathbb{R}^5 \mid t_1, t_2 \in \mathbb{R}\}.$$

The solution set S is a 2-dimensional plane in $\mathbb{R}^5$, not passing through the origin $(0,0,0,0,0)$ of $\mathbb{R}^5$. Every point of such plane corresponds to a solution of the system. For example, choosing $t_1 = 1$ and $t_2 = 2$, we obtain the solution $(-15, 18, -12, 1, 2)$ of the original system. This is the same as saying that plugging

$$x_1 = -15, \quad x_2 = 18, \quad x_3 = -12, \quad x_4 = 1 \quad \text{and} \quad x_5 = 2$$

on the equations of the original system, the right-hand sides come out to be 1, 1, and 0. Every time you have even a single free variable, the system has infinitely many solutions. These variables are called "free" because you're free to choose values to substitute into it, thus generating different solutions to the system. If there are infinitely many solutions (in fact, **uncountably many**, in a very precise sense), you are not supposed to try and list them one by one. This is why it is important to understand how free variables work and how to describe your solution set in a "parametric form": it will carry all the information you need in a succint way.

8 October 11th

We continue to explore linear systems and the RREF of matrices.

Exercise 10

Consider the following linear system:

$$\begin{cases} x_1 + x_2 + x_3 - x_4 = 1 \\ -x_1 + x_2 - x_3 + x_4 = 3 \\ -2x_1 + x_2 + x_3 - x_4 = 2 \end{cases}$$

(a) Determine if $(2, 1, 3, 3)$ is a solution of the system. Explain.

(b) Determine if $\left(-\frac{1}{3}, 2, 0, \frac{2}{3}\right)$ is a solution of the system. Explain.

(c) Solve the system and describe its solution set.

What does it mean for something to be a solution of a system? It means that if we substitute it on the left-hand side of all equations in the system, we must obtain the results on the right-hand sides. If even one equation does not work out, the proposed point is not a solution. For item (a), $(2, 1, 3, 3)$ is not a solution since (for example) the first equation does not work: $2 + 1 + 3 - 3 = 3 \neq 1$. For item (b), a straightforward calculation shows that we do have a solution. Finally, for item (c), we proceed with the usual program: write the augmented matrix for the system, put it into RREF, read of the reduced system, and recognize its solutions and free variables.

The augmented matrix is:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 3 \\ -2 & 1 & 1 & -1 & 2 \end{array}\right].$$

The first 1 in the first row is the pivot, so we must eliminate the -1 and -2 below it. This is done as follows:

$$\begin{array}{c} R_2 := R_2 + R_1 \\ R_3 := R_3 + 2R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 0 & 4 \\ 0 & 3 & 3 & -3 & 4 \end{array}\right].$$

Moving on to the second row, we have that 2 is the pivot. We normalize this row (by dividing it by 2), and the eliminate the 3 below the pivot:

$$\begin{array}{c} R_2 := \frac{1}{2}R_2 \\ R_3 := R_3 - 3R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 3 & -3 & -2 \end{array}\right].$$

As for the final row, we have that 3 is the pivot. Normalize it:

$$R_3 := \frac{1}{3}R_3 \qquad \sim \qquad \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 & -2/3 \end{array}\right].$$

Now, our matrix is in echelon form, but not RREF yet. It remains to clean up what's above the pivots. This is done from left to right.

$$R_1 := R_1 - R_2 \qquad \sim \qquad \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 & -2/3 \end{array}\right].$$

Next, we clean up what's above the third pivot, to finally obtain the RREF of the original augmented matrix:

$$R_1 := R_1 - R_3 \qquad \sim \qquad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 & -2/3 \end{array}\right].$$

Observe that the fourth column does not contain a pivot, and therefore corresponds to a **free variable**. This automatically means that the system has infinitely many solutions. You are not expected to write all of them one by one, so we need an intelligent way of conveying this information at once. This is done by simply writing out the **solution set** S. And saying that x_4 is a free variable means we should write all the other remaining variables (called **dependent variables**) in terms of x_4. The reduced system is:

$$\begin{cases} x_1 & & & = -1/3 \\ & x_2 & & = 2 \\ & & x_3 - x_4 & = -2/3 \end{cases}$$

Letting $t = x_4$ be the free variable (changing the kernel letter from x to t is just a device to remind you that free variables play a different role than dependent variables, revisit the solution for Exercise 9, p. 43), we have that the solution set here is

$$S = \left\{ \left(-\frac{1}{3}, 2, -\frac{2}{3} + t, t \right) \in \mathbb{R}^4 \mid t \in \mathbb{R} \right\}.$$

Since we have only one free variable, the "dimension" of S (whatever that means) ought to be 1. Indeed, writing things in vector form as

$$\begin{bmatrix} -1/3 \\ 2 \\ -2/3 + t \\ t \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2 \\ -2/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

we see that S is a line in $\mathbb{R}^4$, passing through the point $(-1/3, 2, -2/3, 0)$, with direction given by the vector $(0,0,1,1)$. This is just a parametric equation for a line inside a four-dimensional space.

For the next problem, recall that a linear system is called **compatible** if it has solutions, and **incompatible** if it has no solutions. A compatible linear system always has either a unique solution, of infinitely many solutions (there is no other possibility).

Exercise 11

For each system, find the value(s) of c for which they have no solution:

$$\text{(a)} \begin{cases} x_1 + 3x_2 = 4 \\ 2x_1 + 6x_2 = c \end{cases} \qquad \text{(b)} \begin{cases} 3x_1 + cx_2 = 3 \\ cx_1 + 3x_2 = 5 \end{cases}$$

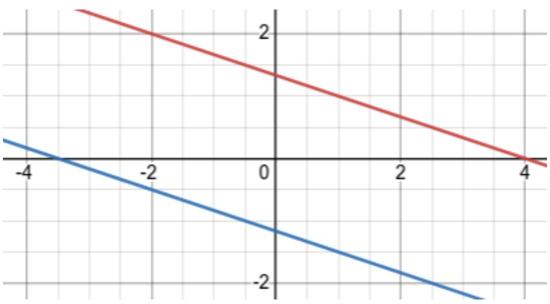
Hint: Try to solve them and see which values of c threaten a division by zero.

Let's treat both systems separately.

(a) Set up the augmented matrix and put it into RREF, by doing $R_2 := R_2 - 2R_1$:

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 6 & c \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 0 & c-8 \end{array}\right].$$

If $c - 8 = 0$ (i.e., if $c = 8$), the last matrix is already in RREF, and we have that the second column does not contain a pivot, meaning that x_2 is a free variable, and the system is compatible (having infinitely many solutions). If $c - 8 \neq 0$ (i.e., if $c \neq 8$), we'll have a row of the form $\left[\begin{array}{cc|c} 0 & 0 & \text{nonzero} \end{array}\right]$, which says that the system is incompatible. Geometrically, what happens is that in the x_1x_2-plane, the lines described by the equations $x_1 + 3x_2 = 4$ and $2x_1 + 6x_2 = c$ are always parallel, and hence equal or disjoint. They are equal **precisely** when $c = 8$, and disjoint otherwise. The solutions of the system are the intersections between the lines. See the following figure.



Play around with the values of c yourself at

https://www.desmos.com/calculator/vxqc7v8lf2.

(b) Again, start setting up the augmented matrix as

$$\left[\begin{array}{cc|c} 3 & c & 3 \\ c & 3 & 5 \end{array}\right].$$

Normalize the first row to obtain a pivot of 1, and then eliminate the c below it by doing $R_2 := R_2 - cR_1$:

$$\left[\begin{array}{cc|c} 1 & c/3 & 1 \\ c & 3 & 5 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & c/3 & 1 \\ 0 & 3 - \frac{c^2}{3} & 5 - c \end{array}\right].$$

Now, there are cases to look at. Observe that $3 - c^3/3 = 0$ is equivalent to having $c = \pm 3$. But for those values of c, we have that $5 - c \neq 0$, and so the system is incompatible due to the presence of a row of the form $\left[\begin{array}{cc|c} 0 & 0 & \text{nonzero} \end{array}\right]$ (as in the previous item). It is worth pointing out that if, instead of $5 - c$, we had some function of c whose value were equal to zero for $c = 3$ (resp. $c = -3$), then $c = 3$ (resp. $c = -3$) would **not** make the system incompatible – to wit, the last matrix above would be in RREF, with the second column corresponding to a free variable. And if $c \neq \pm 3$, one may proceed to normalize the second row as to obtain something of the form

$$\left[\begin{array}{cc|c} 1 & * & * \\ 0 & 1 & * \end{array}\right],$$

which always corresponds to a compatible system (why?).

Geometrically, we observe that the lines in the x_1x_2-plane whose equations are $3x_1 + cx_2 = 3$ and $cx_1 + 3x_2 = 5$ are parallel precisely when $c = \pm 3$ and, in this case, they are disjoint. It's Desmos time again:

https://www.desmos.com/calculator/4ls7umhvtj.

Exercise 12

Consider the three lines whose equations are

$$x_1 + 2x_2 = 1, \qquad 2x_1 + 4x_2 = 2, \qquad \text{and} \qquad -x_1 - 2x_2 = -1.$$

Determine algebraically the number of points of intersection, without actually plotting the lines.

Recall yet again that a point lies on a line if its coordinates satisfy the line equation. Asking for a point lying in the intersection of the three lines is asking for a point whose coordinates satisfy all three line equations. So this is just a geometric way of asking how many solutions does the system

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 2 \\ -x_1 - 2x_2 = -1 \end{cases}$$

have. Set up the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 2 \\ -1 & -2 & -1 \end{array}\right]$$

and, eliminating what's below the pivot 1 in the first row, we may see the miracle happen:

$$\begin{matrix} R_2 := R_2 - 2R_1 \\ R_3 := R_3 + R_1 \end{matrix} \quad \sim \quad \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The above matrix is immediately in RREF and has not only one, but two free variables! Therefore we have infinitely many solutions. This means that we have infinitely many intersection points. It turns out that the three given equations describe the same line (or, alternatively, they describe three lines which happen to all overlap one on top of the other).

Before moving on, here's a quick review on matrix algebra. Namely, the question is "what can we do with matrices?". We can:

- add two matrices **of the same size**, as in

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix},$$

  entrywise.

- multiply a matrix (of any size) by a real number, as in

$$3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix},$$

  again entrywise.

- take **transposes** of matrices, which means that rows become columns and vice-versa, as in

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \implies A^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

- multiply a $n \times m$ matrix A by a $m \times k$ matrix B, to obtain a $n \times k$ matrix AB, whose (i, j)-entry equals the **dot product** between the i-th row of A and the j-th column of B. Idea: $(n \times m)(m \times k) = n \times k$.

Here's a reason about why matrix multiplication is like that: it all comes back to the big idea of representing a given linear system with a matrix, studying the matrix instead, and drawing conclusions about the original system from, for example, the reduced row echelon form of the obtained matrix. More precisely, think of the silliest case possible, where we have only one equation and one variable: $ax = b$. If we want to study linear systems (say, with n variables and m equations) with sort of the same notation, $A\boldsymbol{x} = \boldsymbol{b}$, we need to make sense of what does it mean to multiply the $m \times n$ matrix A with the $n \times 1$ column vector $\boldsymbol{x}$, to obtain the $m \times 1$ column vector $\boldsymbol{b}$. Write the system explicitly:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

The entries of the product $A\boldsymbol{x}$ should be the entries of $\boldsymbol{b}$, but the system itself gives the expression fot the entries of $\boldsymbol{b}$ in terms of A and $\boldsymbol{x}$. The definition

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$$

is thus **forced** upon us, in the sense that if we want to make sense of the system in the form $A\boldsymbol{x} = \boldsymbol{b}$, there is only one possible choice for what the vector $A\boldsymbol{x}$ must be. To multiply A by a second matrix B which is $n \times k$, one applies the above for each column of B — and this is the matrix multiplication you have first seen in class. In particular, this should justify why the number of columns of A must equal the number of rows of B for this product to make sense.

Here are some pitfalls to avoid, regarding matrix multiplication:

- matrix multiplication is **non-commutative**, i.e., AB is not equal to BA, in general; one way to convince yourself quickly of this is that unless both A and B are square matrices of the same size, in general only one of the products AB or BA is well-defined, while the other is not, so it doesn't even make sense to compare them (and to make it worse, there are examples of square matrices A and B for which $AB \neq BA$). This means that while one does $(a+b)^2 = a^2 + 2ab + b^2$ for real numbers a and b, one must write $(A+B)^2 = A^2 + AB + BA + B^2$ in full, with no further simplifications available.

- one cannot "cancel" non-zero matrices like real numbers: if $AB = AC$ and $A \neq 0$, we **cannot conclude** that $B = C$. In fact, this only works provided that A is non-singular!

- the identity matrix (which plays the role of 1 for real numbers) is not the matrix full of 1's, but instead the matrix with 1's in the diagonal and zeros everywhere else.

Exercise 13

If A is a 2×2 matrix, we consider

$$B = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix},$$

and $A^\top + B = C$, what is A?

If we were dealing with real numbers, the first natural step to solve for A would be to subtract B on both sides, leading to $A^\top = C - B$. Now, observe that doing a transposition twice in a row returns a matrix to its original state. This means that we can apply $^\top$ on both sides of $A^\top = C - B$ to obtain $A = (C - B)^\top$. **Now, once we have used as much matrix algebra as possible, we substitute the actual matrices**. We obtain

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

9 October 18th

We'll start with a follow-up from Exercise 9 (p. 43) to emphasize one last time how to express solutions of (often homogeneous) systems in vector form. The advantage of doing this is recognizing a "basis" for the solution set and writing arbitrary solutions are linear combinations of such "basis vectors". The free variables appear as the coefficients in such linear combination, and the number of free variables is the "dimension" of the solution set.

Exercise 14

The following matrix is the augmented matrix for a system of linear equations:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Give the vector form for the general solution.

The usual strategy is to start with a system, set up its augmented matrix, put it into RREF, and read off the solutions directly from the simpler system. In this case, note that the given matrix is already in RREF (there is a staircase pattern, all the pivots are equal to 1, and all entries above the pivots are equal to 0). Since the second and fourth columns contain no pivots, we know that x_2 and x_4 are free variables (and therefore the system has infinitely many solutions). Observe that the last column of the augmented matrix consisting only of zeros says that the linear system in question is homogeneous (and hence automatically consistent, even if we knew nothing about free variables). We simply write

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 + 2x_4 \\ x_2 \\ -2x_4 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_4 \\ 0 \\ -2x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

With this in place, we may (renaming $t_1 = x_2$ and $t_2 = x_4$) express the solution set in a more concise way:

$$S = \left\{ t_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \mid t_1, t_2 \in \mathbb{R} \right\}.$$

The solution set S is a 2-dimensional plane, spanned by the vectors $(1,1,0,0,0)$ and $(2,0,-2,1,0)$, inside 5-dimensional Euclidean space $\mathbb{R}^5$.

We move on to discuss some matrix algebra in more detail.

Exercise 15

(a) Find, if possible, the values of x, y, and z for which the matrix

$$A = \begin{bmatrix} 7 & x+y & x+z \\ 1 & 8 & 2 \\ 3 & y+z & 9 \end{bmatrix}$$

is symmetric (i.e., satisfies $A = A^\top$).

(b) Do the same for when A is skew-symmetric (i.e., satisfies $A = -A^\top$) instead.

(a) "Symmetry" of a (square!) matrix really means "symmetry about its diagonal". So, we must set $x+y=1$, $y+z=2$, and $x+z=3$. It is worth noting that all three ways of expressing this system convey exactly the same information:

$$\text{(i)} \begin{cases} x+y \phantom{{}+z} &= 1 \\ \phantom{x+{}} y+z &= 2 \\ x \phantom{{}+y} +z &= 3 \end{cases}, \qquad \text{(ii)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \end{array}\right], \qquad \text{(iii)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

The third way (i.e., the vector form $A\boldsymbol{x} = \boldsymbol{b}$ of a system) makes it apparent that, once it is known that the coefficient matrix has an inverse (or, in other words, is nonsingular), the solution to the system exists and is unique. Namely, equal to

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Finding inverses of square matrices, whenever they exist, can also be done with the use of Gaussian Elimination (i.e., the algorithm we run to put a matrix into RREF) – it is called *Gauss-Jordan inversion*. See the worksheet from the previous week and practice sheet for the second midterm for more details on that. It doesn't really matter which way you choose to solve this system, one eventually finds that $x = 1$, $y = 0$, and $z = 2$.

(b) There is something rather subtle going on here. If the matrix A were to be skew-symmetric, we would necessarily have that $x+y = -1$, $y+z = -2$, and $x+z = -3$. The solutions, based on item (a), are immediately equal to $x = -1$, $y = 0$, and $z = -2$. But for such values, we have that

$$A = \begin{bmatrix} 7 & -1 & -3 \\ 1 & 8 & 2 \\ 3 & -2 & 9 \end{bmatrix} \quad \text{and} \quad -A^\top = \begin{bmatrix} -7 & -1 & -3 \\ 1 & -8 & 2 \\ 3 & -2 & -9 \end{bmatrix}.$$

They are not equal! So, what went wrong here? All entries match, except for the diagonal ones. And that's where the problem lies: **all the diagonal entries**

of a skew-symmetric matrix must necessarily be equal to zero. Making $j = i$ in the relation $a_{ji} = -a_{ij}$ (which means "look at the diagonal") leads to the equality $a_{ii} = -a_{ii}$, and so $a_{ii} = 0$. To summarize the conclusion: since the diagonal entries of A are not all equal to zero, there are no values of x, y and z for which A is skew-symmetric. If the diagonal entries of A were all equal to zero, however, the suggested strategy would work and $x = -1$, $y = 0$, and $z = -2$ would be the only answer.

Now, recall that it is not always true that any two matrices may be multiplied. Namely, the (i, j)-th entry of a product AB, whenever it makes sense, must be equal to the dot product between the i-th entry of A and the j-th column of B. For this to make sense, such rows and columns **must have the same size**. Therefore, only products like $(n \times m)(m \times k)$ make sense, with output having size $n \times k$. Let's drill all of this with the next problem.

Exercise 16 (Matrix multiplications may not "compile")

Consider the three matrices

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \\ 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 7 & 4 & 3 \\ 6 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 1 & 4 & 0 \\ 6 & 1 & 3 & 5 \\ 2 & 4 & 2 & 0 \end{bmatrix}.$$

(a) Among all $3^2 = 9$ products A^2, AB, BA, AC, CA, B^2, BC, CB, and C^2, decide which ones are well-defined and which ones are not. Evaluate the ones which are well-defined.

(b) If the transposes $A^\top$, $B^\top$, and $C^\top$, enter the fray, there are a priori $6^2 = 36$ products to form. Are the products $A^\top B$, $BC^\top$, $C^\top A$, and $CA^\top$ well-defined? Evaluate those which are.

(a) Let's look at it systematically, thinking that A has size 3×2, B has size 3×3, and C has size 3×4.

- A^2: we have $(3 \times 2)(3 \times 2)$, so it's undefined.
- AB: we have $(3 \times 2)(3 \times 3)$, so it's undefined.
- AC: we have $(3 \times 2)(3 \times 4)$, so it's undefined.
- BA: we have $(3 \times 3)(3 \times 2)$, so BA is defined and has size 3×2. We have that
$$BA = \begin{bmatrix} 1 & 2 & 1 \\ 7 & 4 & 3 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 7 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 13 & 21 \\ 43 & 53 \\ 20 & 12 \end{bmatrix}.$$
- CA: we have $(3 \times 4)(3 \times 2)$, so it's undefined.

- B^2: we have $(3 \times 3)(3 \times 3)$, so B^2 is defined and has size 3×3. We have that
$$B^2 = \begin{bmatrix} 1 & 2 & 1 \\ 7 & 4 & 3 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 7 & 4 & 3 \\ 6 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 21 & 10 & 8 \\ 53 & 30 & 22 \\ 12 & 12 & 7 \end{bmatrix}.$$
- BC: we have that $(3 \times 3)(3 \times 4)$, so BC is defined and has size 3×4. We have that
$$BC = \begin{bmatrix} 1 & 2 & 1 \\ 7 & 4 & 3 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 & 0 \\ 6 & 1 & 3 & 5 \\ 2 & 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 7 & 12 & 10 \\ 44 & 23 & 46 & 20 \\ 14 & 10 & 26 & 0 \end{bmatrix}.$$
- CB: we have $(3 \times 4)(3 \times 3)$, so it's undefined.
- C^2: we have $(3 \times 4)(3 \times 4)$, so it's undefined.

(b) $\ddot\smile$

We conclude today's discussion with the next problem, trying to lose the fear of doing analytic geometry with vectors in $\mathbb{R}^n$ for $n > 3$.

Exercise 17

Consider in $\mathbb{R}^4$ the vectors
$$\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{y} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 2 \end{bmatrix}.$$

(a) Compute the norms (i.e., lengths) $\|\boldsymbol{x}\|$, $\|\boldsymbol{y}\|$, and $\|\boldsymbol{x}+\boldsymbol{y}\|$.

(b) Is it true that $\|\boldsymbol{x}+\boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2$? Try to give a geometric explanation.

(c) Compute the "inner product" $\boldsymbol{x}^\top \boldsymbol{y}$ and the "outer product" $\boldsymbol{x}\boldsymbol{y}^\top$. What is the *sum of the diagonal terms*[a] of the matrix $\boldsymbol{x}\boldsymbol{y}^\top$?

(a) Computing "magnitudes" of vectors in $\mathbb{R}^n$ works just like what we had in $\mathbb{R}^2$ and $\mathbb{R}^3$. We have that
$$\begin{aligned} \|\boldsymbol{x}\| &= \sqrt{1^2 + (-1)^2 + 2^2 + 3^2} = \sqrt{15}, \\ \|\boldsymbol{y}\| &= \sqrt{2^2 + 0^2 + (-3)^2 + 2^2} = \sqrt{17}, \\ \|\boldsymbol{x}+\boldsymbol{y}\| &= \sqrt{3^2 + (-1)^2 + (-1)^2 + 5^2} = 6. \end{aligned}$$

(b) It is not true, since $17 + 15 = 32 \neq 36$. You should think of the proposed equality $\|\boldsymbol{x} + \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2$ as a **Pythagorean identity**, which is not satisfied by the given vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ because they are not orthogonal. To wit, we have that their dot product is nonzero:

$$\boldsymbol{x} \cdot \boldsymbol{y} = 1 \cdot 2 + (-1) \cdot 0 + 2 \cdot (-3) + 3 \cdot 2 = 2.$$

It is worthwhile to note that the correct equality would be given by the **Law of Cosines** (as a consequence of $\boldsymbol{x} \cdot \boldsymbol{y} = \|\boldsymbol{x}\|\|\boldsymbol{y}\| \cos\theta$, where θ is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$), since

$$\begin{aligned} \|\boldsymbol{x} + \boldsymbol{y}\|^2 &= (\boldsymbol{x} + \boldsymbol{y}) \cdot (\boldsymbol{x} + \boldsymbol{y}) \\ &= \boldsymbol{x} \cdot \boldsymbol{x} + \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{y} \cdot \boldsymbol{x} + \boldsymbol{y} \cdot \boldsymbol{y} \\ &= \|\boldsymbol{x}\|^2 + 2\boldsymbol{x} \cdot \boldsymbol{y} + \|\boldsymbol{y}\|^2 \\ &= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\|\boldsymbol{x}\|\|\boldsymbol{y}\| \cos\theta. \end{aligned}$$

The above reduces to the proposed Pythagorean identity when $\cos\theta = 0$ (which is equivalent to $\boldsymbol{x}$ and $\boldsymbol{y}$ being orthogonal.

(c) In general, $\boldsymbol{x}^\top \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y}$ (since a 1×1 matrix is nothing more than just a number), so that $\boldsymbol{x}^\top \boldsymbol{y} = 2$ was already shown when explaining the previous item. As for the outer product, just note that its (i, j)-entry is $x_i y_j$, so that

$$\boldsymbol{x}\boldsymbol{y}^\top = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 & 2 \\ -2 & 0 & 3 & -2 \\ 4 & 0 & -6 & 4 \\ 6 & 0 & -9 & 6 \end{bmatrix}.$$

Then we have that $\operatorname{tr}(\boldsymbol{x}\boldsymbol{y}^\top) = 2$ as well. That $\operatorname{tr}(\boldsymbol{x}\boldsymbol{y}^\top) = \boldsymbol{x}^\top \boldsymbol{y}$ is true for any vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in any dimension.

[a] This is called the **trace** of a (square) matrix.

10 October 25th

The last concept we need to discuss in this part of the course is the one of **linear independence**. Geometrically in low dimensions, we have that:

- Two vectors in the plane $\mathbb{R}^2$ are linearly independent if they are not collinear.
- Two vectors in space $\mathbb{R}^3$ are linearly independent if they are not collinear.
- Three vectors in space $\mathbb{R}^3$ are linearly independent if they are not coplanar, or not all collinear.

How to make sense of this in higher dimensions, where pictures are no longer available?

Definition 4

Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \in \mathbb{R}^n$ be vectors. We say that they are **linearly independent** if whenever $a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k = \boldsymbol{0}$ (here, $a_1, \ldots, a_k$ are real numbers), we must necessarily have $a_1 = \cdots = a_k = 0$. If they are not linearly independent, we call them **linearly dependent**.

Briefly, saying that a collection of vectors is linearly independent is saying that none of the vectors considered is a linear combination of the others. Similarly, saying that a collection of vectors is linearly dependent is saying that there is at least one of them which may be expressed as a linear combination of the others.

How to make sense of the condition $a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k = \boldsymbol{0} \implies a_1 = \ldots = a_k = 0$ in terms of what we already know? Note that $a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k = \boldsymbol{0}$ is, in disguise, a homogeneous linear system for the unknowns $a_1, \ldots, a_k$! Linear independence means that such system has only the trivial solution. More precisely, this system can be written in vector form as

$$[\boldsymbol{u}_1 | \cdots | \boldsymbol{u}_k] \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $[\boldsymbol{u}_1 | \cdots | \boldsymbol{u}_k]$ is the matrix whose columns are $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$. This suggests the following **algorithm** for deciding when a given collection of vectors is linearly independent:

1. Set the given vectors as columns of a matrix.
2. Put it into RREF, or at least reduce it to "triangular" form, to identify the **rank** (i.e., the number of pivots).
3. If the number of pivots is less than the number of vectors considered (which happens whenever you get a row of zeros), they are linearly dependent[3]. If the number of pivots equals the number of vectors considered, they are linearly independent. You don't have to worry about the number of pivots being greater than the number of vectors considered, since this is impossible to happen (why?).

[3] In more detail: since there is a free variable, you get nontrivial solutions for $a_1, \ldots, a_k$, which correspond to a nontrivial linear combination of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ resulting in $\boldsymbol{0}$.

Before moving on, we might as well register the following definition:

Definition 5

A square matrix A is called **nonsingular** if the only solution to the system $A\boldsymbol{x} = \boldsymbol{0}$ is the trivial solution $\boldsymbol{x} = \boldsymbol{0}$. We call A **singular** if it is not nonsingular.

There are several equivalences regarding the notion of "nonsingular matrix":

Theorem 4

For a given square matrix A, the following conditions are equivalent:

(i) A is nonsingular.

(ii) The columns of A are linearly independent.

(iii) $\det A \neq 0$.

(iv) The inverse matrix A^{-1} exists.

One usually uses the word "singular" to refer to something bad[4]. Not having unique solutions to systems is something "bad", as well as having vectors being linearly dependent. This should help you not confuse the meanings of "singular" versus "nonsingular" while keeping item (ii) – the most useful one for us here – of the above Theorem in mind.

With this in place, let's put the algorithm in practice with the next:

Exercise 18

Consider the (column) vectors

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \quad \boldsymbol{u}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Are $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}_3$ linearly independent?

(b) Are $\boldsymbol{u}_2$, $\boldsymbol{u}_3$, and $\boldsymbol{u}_4$ linearly independent?

We start with (a). Applying the algorithm described above. Namely, we consider the matrix

$$[\boldsymbol{u}_1|\boldsymbol{u}_2|\boldsymbol{u}_3] = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ -1 & -3 & 3 \end{bmatrix}.$$

[4]For example, a singularity in a spacetime is a region on which the laws of Physics break down.

Then

$$\begin{matrix} R_2 := R_2 - 2R_1 \\ R_3 := R_3 + R_1 \end{matrix} \quad \sim \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 6 \\ 0 & -1 & 2 \end{bmatrix}.$$

Now

$$R_2 := R_2 - 3R_1 \quad \sim \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix}.$$

having a row full of zeros says that $\boldsymbol{u}_1, \boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are linearly **dependent**.

For item (b), the same idea works. To make the calculations a bit easier, we note that the order of the vectors is completely irrelevant in the definition of linear independence (which makes perfect sense, as the rank of a matrix is not affected by permuting columns), so we may just as well place $\boldsymbol{u}_4$ as the first vector and set up

$$[\boldsymbol{u}_4|\boldsymbol{u}_2|\boldsymbol{u}_3] = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 4 \\ 0 & -3 & 3 \end{bmatrix}.$$

Then

$$R_2 = R_2 - R_1 \quad \sim \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & -3 \\ 0 & -3 & 3 \end{bmatrix},$$

and next

$$R_3 := R_3 - 3R_2 \quad \sim \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & 12 \end{bmatrix}.$$

This last matrix is in triangular form, and has three pivots (not all equal to 1, though). This means that the matrix has full rank, and so $\boldsymbol{u}_2, \boldsymbol{u}_3$ and $\boldsymbol{u}_4$ are linearly **independent**.

Remark. In the above problem, it follows from item (ii) of Theorem (p. 59), together with the fact that elementary row operations do not change whether a matrix is singular or nonsingular, that the matrices

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ -1 & -3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 6 \\ 0 & -1 & 2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

are all singular, while

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 4 \\ 0 & -3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & -3 \\ 0 & -3 & 3 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & 12 \end{bmatrix}$$

are all nonsingular. It is also worth pointing out that while here all matrices were square, this is not important for applying the algorithm. For instance, if one has 7 vectors in $\mathbb{R}^{10}$, one would set up a 10×7 matrix and reduce it to the point where the rank could be identified – if the rank equals 7, the vectors would be linearly independent, while rank less than 6 would mean that the vectors are linearly dependent.

Exercise 19

Consider in $\mathbb{R}^4$ the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \quad \text{and} \quad w = \begin{bmatrix} 1 \\ -1 \\ c \\ 1 \end{bmatrix}$$

Determine the value of c for which w may be written as a linear combination of v_1 and v_2. What is the combination?

Note that w being a linear combination of v_1 and v_2 is equivalent to saying that v_1. v_2, and w are linearly dependent[a]. Then, we would like to find the value of c for which the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & -1 \\ -1 & 3 & c \\ 1 & 0 & 1 \end{bmatrix}$$

has rank less than 3. In fact, since v_1 and v_2 are not multiples of each other, we can immediately say that this rank has to be equal to 2. In other words, no matter what we do to the above matrix, we will end up with at least two pivots. But what is the value of c for which the third "pivot" is not actually a pivot? Go on:

$$\begin{matrix} R_2 := R_2 - 2R_1 \\ R_3 := R_3 + R_2 \\ R_4 := R_4 - R_1 \end{matrix} \quad \sim \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 3 & c+1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The normalize the second row and eliminate the 3 below the pivot:

$$\begin{matrix} R_2 := -R_2 \\ R_3 := -3R_2 \end{matrix} \quad \sim \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & c-8 \\ 0 & 0 & 0 \end{bmatrix}.$$

The conclusions so far are:

- If $c \neq 8$, the matrix has rank 3, and v_1, v_2 and w are linearly **independent**. In this case, it is impossible to write w as a linear combination of v_1 and v_2.

- If $c = 8$, the matrix has rank 2, and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{w}$ are linearly **dependent**. Since $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are not multiples of each other, it is possible to write $\boldsymbol{w}$ as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

So, assume that $c = 8$, and write $\boldsymbol{w} = a\boldsymbol{v}_1 + b\boldsymbol{v}_2$ for real number a and b to be found. Explicitly, this means that

$$\begin{bmatrix} 1 \\ -1 \\ 8 \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 \\ -1 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ 2a-b \\ -a+3b \\ a \end{bmatrix},$$

but a system with 4 equations and 2 variables, generally speaking, does not want to have any solutions. By setting $c = 8$, however, we ensure that such a solution will exist. It should be clear that $a = 1$ and $b = 3$ does the job here, so $\boldsymbol{w} = 1\boldsymbol{v}_1 + 3\boldsymbol{v}_2$.

[a] In fact, this is related to a more general phenomenon. If you have a collection $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ which is linearly independent and there is a vector $\boldsymbol{w}$ which is **not** a linear combination of them, then the new collection $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{w}$ is again linearly independent.

Let's conclude today's discussion with a review:

Exercise 20

True or False?

(a) (T/F) If a linear system of equations has more than one solution, then it has infinitely many.

(b) (T/F) If A has size 3×5 matrix and B has size 3×4, then AB is defined and has size 5×4.

(c) (T/F) We have that $AB = AC \implies B = C$, whenever A, B and C are square matrices and $A \neq 0$.

(d) (T/F) If the augmented matrix of the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has a pivot in its last column, the given system has no solution.

(e) (T/F) If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is linearly independent, so is $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

(f) (T/F) For any square matrices A and B, we have that $(AB)^\top = A^\top B^\top$.

(g) (T/F) If a linear system has more variables than equations, then it must be consistent.

(h) (T/F) If A is a matrix with orthogonal columns, then $A^\top A$ is a diagonal matrix.

(i) (T/F) If A is a $n \times m$ matrix and B is the $m \times m$ square matrix consisting only of 1's, then $AB = A$.

(j) (T/F) A non-homogeneous linear system with 3 equations and 4 variables having $(2,0,2,2)$ as a solution has infinitely many solutions.

(a) True. Any linear system has either no solutions, one solution, or infinitely many. If you know the system has more than one solution, the only option left is that is has infinitely many solutions.

(b) False. The product is undefined, since $(3\times {\color{red}5})({\color{red}3}\times 4)$ doesn't "compile".

(c) False. It is not enough to have that $A\neq 0$, we need A to be nonsingular (and this is the main reason people care about nonsingular matrices). Namely, the correct statement is: if A, B and C are square matrices and A is nonsingular, then $AB = AC \implies B = C$.

(d) True. If the augmented matrix $[A|\boldsymbol{b}]$ has a pivot in its last column, it means that in the row where such pivot appears, all entries before it must be zeros. And we have seen that a row of the form $\begin{bmatrix} 0 & \cdots & 0 & | & * \end{bmatrix}$ with $*\neq 0$ indicates that the system is incompatible.

(e) True. Imagine that the RREF of the matrix $[v_1|v_2|v_3|v_4]$ has rank 4, and looks like

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the RREF of the submatrix $[v_1|v_2|v_3]$ is indicated in red:

$$\begin{bmatrix} {\color{red}1} & {\color{red}0} & {\color{red}0} & 0 \\ {\color{red}0} & {\color{red}1} & {\color{red}0} & 0 \\ {\color{red}0} & {\color{red}0} & {\color{red}1} & 0 \\ {\color{red}0} & {\color{red}0} & {\color{red}0} & 1 \\ {\color{red}0} & {\color{red}0} & {\color{red}0} & 0 \\ {\color{red}\vdots} & {\color{red}\vdots} & {\color{red}\vdots} & \vdots \\ {\color{red}0} & {\color{red}0} & {\color{red}0} & 0 \end{bmatrix}.$$

It has rank 3, so we are done.

(f) False. The correct formula is $(AB)^\top = B^\top A^\top$, in the reverse order. By the way, this is not something particular for transpositions, it works for inverses too. Namely, if A and B are both invertible (square) matrices, then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(g) False. It would be true, however, if the system considered were **homogeneous** (because then we would have a free variable). There are several nonhomogeneous counter-examples, such as

$$\begin{cases} x+y+z=0 \\ x+y+z=1, \end{cases} \qquad \text{or} \qquad \begin{cases} 2x+3y-4z-w=1 \\ x-4y+3x+2w=2, \\ 0x+0y+0z+0w=1 \end{cases}$$

etc.

(h) True. Recall that computing matrix products boils down to computing dot products between rows of the first matrix and columns of the second. But the rows of $A^\top$ are the columns of A, so that

$$A^\top A = \begin{bmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_1 \cdot \boldsymbol{a}_n \\ \vdots & \ddots & \vdots \\ \boldsymbol{a}_n \cdot \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \cdot \boldsymbol{a}_n \end{bmatrix}$$

If the columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ of A are orthogonal, all off-diagonal entries in the above matrix are zero, so $A^\top A$ is diagonal.

(i) False. The identity matrix has 1's in the diagonal and 0's everywhere else, **not** 1's everywhere. Almost any matrix A you choose will work as a counter-example, such as

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & * \\ * & * \end{bmatrix}.$$

(j) True. Provided a nonhomogeneous system has at least one solution, it will have the same number of solutions as the associated homogeneous system. Since we know that a **homogeneous** system with 3 equations and 4 variables has infinitely many solutions (since there is a free variable), so does the original given system.

11 November 1st

We now start with the third part of this course: **Ordinary Differential Equations** ("ODE"s, for short). The idea is simple: we have equations involving derivatives, but instead of solving for a number or vector, we solve for a function instead. Consider a simple example: $y'(t) = 2t$, and imagine you want to solve for y. Integrating, we obtain that $y(t) = t^2 + c$ for some constant of integration $c \in \mathbb{R}$. Keeping track of such constants is now crucial (and perhaps the reason why you kept losing points in Calculus 1 for forgetting the $+c$; it was all a prelude to this moment), for different choices of c lead to different solutions. As we have infinitely many choices of c, we see that even the simplest differential equation will have infinitely many solutions.

The adjective "ordinary" refers to the fact that all the functions involved are functions of a single variable, and that there are no partial derivatives of anything in play. If this were to be the case, we would be dealing with **Partial Differential Equations** ("PDE"s, for short) instead.

In any case, the point remains that solving ODEs can be very hard and one usually resorts to softwares or numerical methods to understand the behavior of solutions to an ODE. We will focus on very specific types of ODEs which we can indeed solve. To understand when this is the case, some vocabulary is useful. Let's always organize our equations by placing in the left side all the terms involving y, and on the right side all the terms not involving y.

- **order:** the order of a differential equation is the highest derivative that appears.
- **linearity:** a differential equation is linear if its left side is a linear combination (with function coefficients) of y, y', y'', etc.
- **homogeneity:** a differential equation is homogeneous if its right side equals zero.

We care about this because: the higher the order, the harder the equation should be to solve; we like linear things better than non-linear things, and homogeneous equations are generally easier to deal with (because we like zeros).

Let's practice the terminology:

Exercise 21

Classify the following differential equations (order/linearity/homogeneity):

(a) $y'' - 4y' + 2y = 10t^2$.

(b) $y' - 2y^3 = -4t$.

(c) $y''' - \cos(t)y'' + e^t y = 0$.

(a) It's a second order equation because the term with the highest derivative is y''. It is linear because the left side is a linear combination of y, y' and y'' (with coefficients 2, -4 and 1, respectively). It is non-homogeneous as the right side does not equal the zero function.

(b) It's a first order equation because the term with the highest derivative is y'. It is non-linear because of the y^3 term (the non-linear operation of taking a cube is being applied to the function variable y). It is non-homogeneous for hopefully obvious reasons.

(c) It's a third order equation because the term with the highest derivative is y'''. It is linear because the left side is a linear combination of y, y', y'' and y''' (with coefficients $\mathrm{e}^t, 0, -\cos(t)$ and 1, respectively). It is clearly homogeneous. Note here that the functions $\cos(t)$ and e^t are nonlinear functions, but of the variable t, not the variable y (which is what we care about when deciding whether a given differential equation is linear or not).

Remark. Warning: choosing particular values of t does not tell you anything about the nature of the differential equation. For example, saying that the equation in item (a) is homogeneous for $t = 0$ and nonhomogeneous for $t \neq 0$ is **nonsense**. What matters is if what's on the right side is the **zero function** or not.

Next, we will focus on **second order linear homogeneous ODEs, with constant coefficients**. This means, for instance, that the ideas we will present next are good for trying to solve an equation whose left side looks like the one in item (a) of the previous exercise, but not something like in item (c) (which, as a matter of fact, is very difficult to solve). The reason why we are going straight to second order equations, as opposed to first order ones, is that first order equations are "easy", in the sense that morally, all one has to do is one integration. And this is not enough when dealing with second order equations.

To further motivate what will come next, consider the simplest equation ever: $y'(t) = y(t)$. The solutions are clearly $y(t) = c\mathrm{e}^t$, for some constant $c \in \mathbb{R}$. If we considered $y'(t) = ry(t)$ instead, the solutions would be $y(t) = c\mathrm{e}^{rt}$. There is no reason for us to expect exponentials to come again to the rescue in the second order case, so we will think backwards. The idea is to **try** an exponential $y = \mathrm{e}^{rt}$, and find the values of r for which this is indeed a solution. Namely, $y' = r\mathrm{e}^{rt}$ and $y'' = r^2\mathrm{e}^{rt}$, so

$$0 = ay'' + by' + cy = ar^2\mathrm{e}^{rt} + br\mathrm{e}^{rt} + c\mathrm{e}^{rt} = (ar^2 + br + c)\mathrm{e}^{rt} \implies ar^2 + br + c = 0,$$

as $\mathrm{e}^{rt} \neq 0$ may be cancelled.

So, we can solve the differential equation provided we can solve the **characteristic equation** $ar^2 + br + c = 0$. Note that this idea works even if the order of the differential equation is bigger than two, but then solving the characteristic equation may be difficult. When solving a quadratic equation, only three possibilities may happen:

Theorem 5

Consider the differential equation $ay'' + by' + cy = 0$, with $a, b, c \in \mathbb{R}$, and $a \neq 0$. If r_1 and r_2 are the solutions of the **characteristic equation** $ar^2 + br + c = 0$, then:

(i) if r_1 and r_2 are both real and distinct, the general solution of the given ODE is $y = c_1\mathrm{e}^{r_1 t} + c_2\mathrm{e}^{r_2 t}$, with $c_1, c_2 \in \mathbb{R}$.

(ii) if $r := r_1 = r_2$ is a real double root, the general solution of the given ODE is

$y = c_1 e^{rt} + c_2 t e^{rt}$, with $c_1, c_2 \in \mathbb{R}$.

(iii) if r_1 and r_2 are complex (and hence conjugate to each other[a]), the general solution of the given ODE is $y = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$, with $c_1, c_2 \in \mathbb{R}$, where $r_1 = \alpha + i\beta$.

[a]Complex roots of a real polynomial always come in conjugate pairs.

Of course, memorizing the above is a waste of time. Working case by case from here on saves you brain power. On the following examples, we will simultaneously explore this idea, as well as the fact that once suitable initial conditions have been imposed, the solution to the so-called **Initial Value Problem** ("IVP", for short) becomes unique.

Exercise 22 (Two distinct real roots)

(a) *Find the general solution of* $y'' - 3y' - 18y = 0$.

(b) *Find the unique solution with initial conditions* $y(0) = 0$ *and* $y'(0) = 4$.

For item (a), we start setting up the characteristic equation $r^2 - 3r - 18 = 0$. It may be factored as $(r-6)(r+3) = 0$, which says that the characteristic roots are $r_1 = 6$ and $r_2 = -3$. Therefore, we know that $y_1 = e^{6t}$ and $y_2 = e^{-3t}$ are two solutions. They are **linearly independent** because one is not a (real) multiple of the other. However, **since the given ODE is linear and homogeneous, the dimension of the space of solutions equals the order of the equation**. This says that taking linear combinations of y_1 and y_2 does, in fact, produce all solutions of this ODE. In other words, the general solution is

$$y = c_1 e^{6t} + c_2 e^{-3t}, \text{ with } c_1, c_2 \in \mathbb{R}.$$

For item (b), imposing **two** initial conditions (at the same point) allows us to solve for the **two** coefficients c_1 and c_2. The relations $y(0) = 0$ and $y'(0) = 4$ becomes the linear system

$$\begin{cases} c_1 + c_2 = 0 \\ 6c_1 - 3c_2 = 4 \end{cases} \implies c_1 = \frac{4}{9} \quad \text{and} \quad c_2 = -\frac{4}{9}.$$

The solution of the given IVP is

$$y = \frac{4}{9}e^{6t} - \frac{4}{9}e^{-3t}.$$

Let's see next what happens in the case where two complex conjugate roots appear. Two additional facts are crucial to understand this case:

Theorem 6

Real and imaginary parts of a **complex solution** to a **real** linear homogeneous ODE are **real solutions**.

Theorem 7 (Euler's Formula)

For any real number θ, we have $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$.

We move on (phrasing things in a lazier way, but with the same content as in Example 22):

Exercise 23 (Complex conjugate roots)

Solve the IVP:

$$\begin{cases} y'' + 9y = 0 \\ y(0) = 8, \quad y'(0) = -8 \end{cases}$$

The strategy will always be the same: first find the general solution of the differential equation alone, and then use the given initial conditions to solve for c_1 and c_2. The characteristic equation is simply $r^2 + 9 = 0$, whose roots are $\pm 3\mathrm{i}$. We may just focus on one of them[a], say $3\mathrm{i}$. This says that $\mathrm{e}^{3\mathrm{i}t}$ is a complex solution. By Euler's Formula,

$$\mathrm{e}^{3\mathrm{i}t} = \cos(3t) + \mathrm{i}\sin(3t),$$

so Theorem 6 says that $y_1 = \cos(3t)$ and $y_2 = \sin(3t)$ are real solutions. They're clearly linearly independent, so the general solution of the differential equation is

$$y = c_1\cos(3t) + c_2\sin(3t), \quad c_1, c_2 \in \mathbb{R}.$$

With this in place, we move on to impose the initial conditions. Computing the derivative as $y' = -3c_1\sin(3t) + 3c_2\cos(3t)$, we see that $y(0) = 8$ and $y'(0) = -8$ together give us

$$c_1 = 8 \quad \text{and} \quad 3c_2 = -8,$$

so the unique solution to the IVP is

$$y = 8\cos(3t) - \frac{8}{3}\sin(3t).$$

[a] Because complex roots of a real polynomial always come in conjugate pairs. So $3\mathrm{i}$ and $-3\mathrm{i}$ carry the same amount of information about the original differential equation.

There is one last case to study.

Exercise 24 (Real double root)

Solve the IVP:

$$\begin{cases} y'' - 2y' + y = 0 \\ y(0) = 4, \quad y'(0) = 0 \end{cases}$$

The characteristic equation of the given ODE in this case is just $r^2 - 2r + 1 = 0$, so that $(r-1)^2 = 0$ says that $r = 1$ is a real double root. Hence $y_1 = \mathrm{e}^t$ is one solution, but since the equation has order 2, we need a second solution y_2, linearly independent from y_1, to span the whole solution space via linear combinations. Indeed, repeating y_1 and writing $c_1\mathrm{e}^t + c_2\mathrm{e}^t$ just leads to $(c_1+c_2)\mathrm{e}^t$, but c_1+c_2 is as arbitrary as c_1 and c_2, so it really counts as one single degree of freedom. One attempt to create linear independence is to replace c_1+c_2 with c_1+c_2t, which suggests that $y_2 = t\mathrm{e}^t$ works as a second solution independent from y_1 and that the general solution is

$$y = c_1\mathrm{e}^t + c_2t\mathrm{e}^t, \quad c_1, c_2 \in \mathbb{R}.$$

This is indeed the case by Theorem 5. Imposing initial conditions works the same as the other cases, so we may compute $y' = c_1\mathrm{e}^t + c_2\mathrm{e}^t + c_2t\mathrm{e}^t$ by the product rule. Thus, $y(0) = 4$ and $y'(0) = 0$ together read as the system

$$\begin{cases} c_1 = 4 \\ c_1 + c_2 = 0 \end{cases} \implies c_1 = 4 \quad \text{and} \quad c_2 = -4,$$

and the unique solution to the IVP is

$$y = 4\mathrm{e}^t - 4t\mathrm{e}^t.$$

As mentioned before, this mechanism can be used to solve linear homogeneous ODEs with constant coefficients of **any order**, say

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0, \quad \text{with } a_1, \ldots, a_n \in \mathbb{R},\ a_n \neq 0,$$

provided one can solve the characteristic equation

$$a_nr^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0$$

instead. Once this equation has been completely factored, one builds the general solution from the factors obtained, using what has been discussed so far. Let's illustrate this with one more complicated last example.

Exercise 25 (Dealing with higher order)

Determine the general solution of a 5th order linear homogeneous ordinary differential equation whose characteristic equation is factored as

$$(r-2)(r-3)^2(r-(4+5\mathrm{i}))(r-(4-5\mathrm{i}))=0.$$

Let's understand each factor separately:

- The term $(r-2)$ provides $y_1 = \mathrm{e}^{2t}$.
- The term $(r-3)^2$ provides $y_2 = \mathrm{e}^{3t}$ and $y_3 = t\mathrm{e}^{3t}$.
- The term $(r-(4+5\mathrm{i}))$ provides the complex solution $\mathrm{e}^{(4+5\mathrm{i})t}$, so using Euler's formula to write

$$\mathrm{e}^{(4+5\mathrm{i})t} = \mathrm{e}^{4t}\mathrm{e}^{5\mathrm{i}t} = \mathrm{e}^{4t}(\cos(5t)+\mathrm{i}\sin(5t)) = \mathrm{e}^{4t}\cos(5t)+\mathrm{i}\mathrm{e}^{4t}\sin(5t)$$

  gives us the real solutions $y_4 = \mathrm{e}^{4t}\cos(5t)$ and $y_5 = \mathrm{e}^{4t}\sin(5t)$.

We conclude that the general solution is

$$y = c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{3t} + c_3t\mathrm{e}^{3t} + c_4\mathrm{e}^{4t}\cos(5t) + c_5\mathrm{e}^{4t}\sin(5t),$$

with $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$.

If you want an extra reference for these things, I particularly like Chapters 3 and 4 of [2].

12 November 8th

As a follow up from last class, we now move on to discuss **non-homogeneous** second order linear ODE's with constant coefficients. Namely, we consider

$$ay'' + by' + cy = f,$$

where $a, b, c \in \mathbb{R}$ with $a \neq 0$ (or else the equation would be of first order instead), but with the right side being equal to some arbitrary function f. Here's what we need to know about it:

Theorem 8

The general solution of $ay'' + by' + cy = f$ is $y = y_p + y_h$, where y_h is the general solution of the associated homogeneous equation $ay'' + by' + cy = 0$, and y_p is any particular solution of the original non-homogeneous equation.

In other words, to solve such a non-homogeneous equation, we first consider its homogeneous version and find y_h, by methods already studied. As for finding y_p, however, the answer is dissapointing: we have to guess it. Of course, by "guess" we mean a "reasonable" guess, in the sense that if f is a polynomial, trigonometric function, exponential, etc., we'll try to find y_p of the same type. This method of guessing **almost** always works. We will explore this in the next examples, and also see how could guessing could go wrong.

Exercise 26

Determine the general solution of the non-homogeneous linear differential equation $y'' - 9y = 2t + 1$.

We will always proceed with two steps, first finding the general solution y_h of the associated homogeneous equation $y'' - 9y = 0$. The characteristic equation is simply $r^2 - 9 = 0$, which may be factored as $(r-3)(r+3) = 0$. As this characteristic equation appeared by looking for the values of r for which e^{rt} was actually a solution of the homogeneous equation, we obtain two linearly independent solutions e^{3t} and e^{-3t}, so that $y_h = c_1\mathrm{e}^{3t} + c_2\mathrm{e}^{-3t}$, with $c_1, c_2 \in \mathbb{R}$. It remains to find y_p. As $2t+1$ is a polynomial of degree 1, we try to make $y_p = At + B$ a polynomial of degree 1 as well. The goal is to find A and B for which y_p is actually a solution of the original non-homogeneous equation. Plugging it into the differential equation, we have that $y_p'' - 9y_p = -9y_p = -9At - 9B = 2t + 1$, so that $A = -2/9$ and $B = -1/9$. So, the particular solution is $y_p = -2t/9 - 1/9$, and the general solution to the original non-homogeneous is

$$y = -\frac{2}{9}t - \frac{1}{9} + c_1\mathrm{e}^{3t} + c_2\mathrm{e}^{-3t}, \qquad c_1, c_2 \in \mathbb{R}.$$

Exercise 27

Determine the general solution of the non-homogeneous linear differential equation $y'' - 4y' - 32y = 6e^{-3t}$.

Let's start by finding the general solution y_h of the associated homogeneous equation $y'' - 4y' - 32y = 0$ first. Its characteristic equation is $r^2 - 4r - 32 = 0$, which may be factored as $(r-8)(r+4) = 0$. This implies, as usual, that we have $y_h = c_1 e^{8t} + c_2 e^{-4t}$, with $c_1, c_2 \in \mathbb{R}$. As the non-homogeneous term in the original equation is an exponential, we try $y_p = Ae^{-3t}$, and substitute it together with its derivatives $y_p' = -3Ae^{-3t}$ and $y_p'' = 9Ae^{-3t}$ into the original equation to find the value of A which makes y_p into a solution. We have that

$$9Ae^{-3t} - 4(-3Ae^{-3t}) - 32Ae^{-3t} = 6e^{-3t} \implies 9A + 12A - 32A = 6,$$

so $A = -6/11$, since e^{-3t} may be cancelled everywhere. Therefore, the particular solution is $y_p = (-6/11)e^{-3t}$, and the general solution of the original non-homogeneous equation is

$$y = -\frac{6}{11}e^{-3t} + c_1 e^{8t} + c_2 e^{-4t},$$

with $c_1, c_2 \in \mathbb{R}$.

Exercise 28

Determine the general solution of the following non-homogeneous linear differential equations:

(a) $y'' - y = 3\sin(2t)$.

(b) $y'' + y = 3\cos t$.

(a) We start considering the associated homogeneous equation $y'' - y = 0$, whose characteristic equation $r^2 - 1 = 0$ has roots 1 and -1, to obtain that its general solution is $y_h = c_1 e^t + c_2 e^{-t}$, with $c_1, c_2 \in \mathbb{R}$. Now, as the non-homogeneous term is trigonometric, we may try $y_p = A\sin(2t) + B\cos(2t)$. Of course one might expect that $B = 0$ as the right side of the equation doesn't have any y' term, but for more complicated equations it is safer to keep both the sin and cos terms, as the derivative of each of them equals the other, and things balance themselves out (this is not a formal argument, but just a rough intuition). Anyway, we have that $y_p'' = -4A\sin(2t) - 4A\cos(2t)$, and thus

$$y_p'' - y_p = 3\sin(2t) \implies -5A\sin(2t) - 5B\cos(2t) = 3\sin(2t),$$

so that $A = -3/5$ and $B = 0$ (as expected). We conclude that the solution of the original non-homogeneous equation is

$$y = -\frac{3}{5}\sin(2t) + c_1 e^t + c_2 e^{-t},$$

with $c_1, c_2 \in \mathbb{R}$.

(b) To find y_h, we consider $y'' + y = 0$ instead, whose characteristic equation is $r^2 + 1 = 0$, with roots $r = \pm i$. This says that e^{it} is a complex solution, but we would like to have two linearly independent real solutions instead. They are obtained by using Euler's formula to write $e^{it} = \cos t + i \sin t$, and using that the real and imaginary parts will be real solutions (as the equation is linear; revisit Theorem 6, p. 67). So, $y_h = c_1 \cos t + c_2 \sin t$, with $c_1, c_2 \in \mathbb{R}$. As for y_p, the obvious guess is $y_p = A \sin t + B \cos t$. Plugging this into the original non-homogeneous equation leads to a disaster: $0 = 3\cos t$, which is complete nonsense.

This disaster, however, is known as **resonance**. Namely, the reason why the initial guess has failed is because the right hand side (which in this case is $3\cos t$) was already a solution of the associated homogeneous equation (choose $c_1 = 0$ and $c_2 = 3$ in the formula for y_h). In vague terms, this creates an "artificial double-root effect" on the homogeneous equation, so in the same way that when r was a double root of the characteristic equation $ar^2 + br + c = 0$ we had to, in addition to the solution e^{rt}, consider a second solution te^{rt}, we multiply our old guess for y_p by t and try again.

Take two: let $y_p = At \sin t + Bt \cos t$. Plugging this into the original non-homogeneous equation leads (after 30 seconds or so of calculations) to

$$2A\cos t - 2B \sin t = 3\cos t \implies A = \frac{3}{2} \quad \text{and} \quad B = 0.$$

It was to be expected that terms with $t\cos t$ and $t \sin t$ should completely dissapear: the introduction of the extra factors of t was just meant to kill the resonance, but the original right side did not have any such terms. In any case, we conclude that the general solution of the original non-homogeneous equation is

$$y = \frac{3t}{2}\sin t + c_1 \cos t + c_2 \sin t,$$

with $c_1, c_2 \in \mathbb{R}$.

Remark. At this point, it is worth pointing out that if you were asked to solve an Initial Value Problem (IVP) with non-homogeneous ODE, you would have to, on top of everything already done, use the given initial conditions to find c_1 and c_2 (by solving a linear system). It works exactly the same as in the examples done in last recitation.

Let's conclude with one final practice for guesses:

Exercise 29

What are the correct trial solutions for the following non-homogeneous linear differential equations? Is there resonance happening? Do **not** solve the equations.

(a) $y'' + 2y' + 2y = 5\mathrm{e}^{-2t}\cos t$.

(b) $y'' + y' - y = 3t^4 - 3t^3 + t$.

(c) $y'' - y = 25t\mathrm{e}^{-t}\sin(3t)$.

(d) $y^{(3)} - 3y'' + 3y' - y = 2022\mathrm{e}^t$.

(a) Here, we have that $5\mathrm{e}^{-2t}\cos t$ has the form "polynomial of degree 0" times "exponential" times "trigonometric function". So the first guess ought to be

$$y_p = A\mathrm{e}^{-2t}\cos t + B\mathrm{e}^{-2t}\sin t.$$

The paranoia created by the previous exercise now dictates that we should ask ourselves whether resonance happens. The trick here is to think of the term $\mathrm{e}^{-2t}\cos t$ as a single block, coming from the complex pair $r = -2 \pm \mathrm{i}$. Is $-2+\mathrm{i}$ a root of the associated characteristic equation $r^2 + 2r + 2 = 0$? If yes, there is resonance. If not, there is no resonance. In this case, there is no resonance.

(b) Since $3t^4 - 3t^3 + t$ is a polynomial of degree 4, the guess is that the trial solution should also be a polynomial of degree 4, namely,

$$y_p = At^4 + Bt^3 + Ct^2 + Dt + E.$$

If resonance were to happen, part of this trial solution should be a solution of the associated homogeneous equation. The only terms under this risk are Dt and E, in case 0 were a root or double root of the characteristic equation $r^2 + r - 1 = 0$. Since it is not, there is no resonance.

(c) This is similar to item (a), in that $25t\mathrm{e}^{-t}\sin(3t)$ has the form "polynomial of degree 0" times "exponential" times "trigonometric function". So the first guess ought to be

$$y_p = (At + B)\mathrm{e}^{-t}\cos(3t) + (Ct + D)\mathrm{e}^{-t}\sin(3t).$$

We can think of the terms $\mathrm{e}^{-t}\cos(3t)$ and $\mathrm{e}^{-t}\sin(3t)$ as coming from the complex pair $-1 \pm 3\mathrm{i}$. Since $-1 + 3\mathrm{i}$ is not a root of the associated characteristic equation $r^2 - 1 = 0$, there is no resonance.

(d) The first guess here is $y_p = A\mathrm{e}^t$. The exponential e^t comes from the characteristic root $r = 1$. Is $r = 1$ a root of the characteristic equation $r^3 - 3r^2 + 3r - 1 = 0$? Yes, there is resonance! In fact, it is a **triple** root (as the equation factors as $(r-1)^3 = 0$), so we multiply the initial guess by t^3 to obtain the correct guess $y_p = At^3\mathrm{e}^t$.

13 November 15th

We had a free review before the third midterm, answering questions at random.

14 November 22nd

Back in August 23^{rd}, we introduced the **Heat Flow Problem** as an attempt to motivate everything else that was going to happen in this class. It is a particular case of a **Partial Differential Equation** ("PDE", for short). They are considerably harder to deal with than everything we have seen so far. The time has come to solve it.

Recall the setup:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \beta \dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

- Given: a wire of length L with diffusivity constant β and initial temperature distribution $f(x)$;
- Assumption: the temperature of the wire at its endpoints – located at $x = 0$ and $x = L$ – will remain fixed (and equal to, say, 0°C);
- Goal: predict future temperature distributions.

The strategy for solving such a starts with a simple technique, called **separation of variables**. Namely, we write $u(x,t) = X(x)T(t)$ for some single-variable functions X and T, and see what the PDE says about X and T. More precisely, we want to replace the PDE for u with ODEs for X and T. We will illustrate the procedure with a concrete heat flow problem.

Example 23

Solve the following heat flow problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = 3\dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = \sin x - 6\sin(4x), & 0 < x < \pi \end{cases}$$

We start making a separation of variables, $u(x,t) = X(x)T(t)$. The PDE itself becomes $X(x)T'(t) = 3X''(x)T(t)$. We then have that

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{3T(t)} = -\lambda \in \mathbb{R},$$

as the left side doesn't depend on t and the right side doesn't depend on x. The reason for the negative sign on λ is a matter of convenience (as to make plus signs appear next and create "positive eigenvalues" for the boundary value problem involving X). In any case, we have the equations

$$X''(x) + \lambda X(x) = 0 \quad \text{and} \quad T'(t) + 3\lambda T(t) = 0,$$

and we'll use the first one to find out which values of λ may actually occur when looking for nontrivial solutions. To do so, we also need to see what the initial condition $u(0,t) = u(\pi,t) = 0$ means in terms of X and T. But this is easy: it becomes just the initial condition $X(0) = X(\pi) = 0$ (or else $T(t) = 0$ and thus $u(x,t) = 0$, which makes no physical sense). So:

- If $\lambda < 0$, then $X(x) = c_1 e^{\sqrt{-\lambda}t} + c_2 e^{-\sqrt{\lambda}t}$, so $X(0) = X(\pi) = 0$ gives us that $c_1 = c_2 = 0$, and so $X = 0$, which is no good.
- If $\lambda = 0$, then we have $X(x) = c_1 + c_2 x$, so $X(0) = X(\pi) = 0$ gives us that $c_1 = c_2 = 0$, and so $X = 0$, which is again no good.
- If $\lambda > 0$, then $X(x) = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t)$. Now $X(0) = 0$ means that $c_1 = 0$, while $X(\pi) = 0$ says that $\sin(\pi\sqrt{\lambda}) = 0$ (provided $c_2 \neq 0$). This can only occur if $\pi\sqrt{\lambda} = n\pi$ for some integer n, meaning that we must have that $\lambda = n^2$.

So, for each integer n, we have found a solution $X_n(x) = a_n \sin(nx)$, where a_n is a real number (we will keep relabeling constants as needed). With this in place, what is the corresponding $T_n(t)$? Solving the first order ODE $T_n'(t) = -3n^2 T_n(t)$ leads to $T_n(t) = b_n e^{-3n^2 t}$ for some second real constant b_n. With this in place, setting $c_n = a_n b_n$, we have that

$$u_n(x,t) = X_n(t)T_n(t) = c_n e^{-3n^2 t} \sin(nx).$$

For each n, this function u_n satisfies almost everything required to solve the heat flow problem, except for the initial distribution condition $u(x,0) = \sin x - 6\sin(4x)$. To achieve this, we consider the series $u = \sum_{n \geq 1} u_n$ and find the c_n's to make this work (as the PDE itself and the endpoint conditions are homogeneous, any linear combination – finite or infinite – of u_n's will also result in a solution). Namely, setting

$$\sum_{n \geq 1} c_n e^{-3n^2 t} \sin(nx)\Bigg|_{t=0} = \sum_{n \geq 1} c_n \sin(nx) = \sin x - 6\sin(4x)$$

tells us that $c_1 = 1$ and $c_4 = -6$ does the trick (with $c_n = 0$ for every n not equal to 1 or 4). The desired solution is

$$u(x,t) = e^{-3t} \sin x - 6e^{-48t} \sin(4x).$$

A few comments are in order.

First, when looking at possibilities for λ we will always have that $\lambda \leq 0$ leads to $X = 0$. When solving concrete problems, one could directly jump to the case where

$\lambda > 0$, but it is still instructive to eliminate $\lambda \leq 0$ to keep track of what's going on in each step of the solution.

Second, abstractly solving the Heat Flow Problem carrying the abstract parameters β and L, one arrives at what we'll call a **prototype solution to the Heat Flow Problem**:

$$u(x,t) = \sum_{n\geq 1} c_n \mathrm{e}^{-\beta(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right).$$

You have a choice to make now: remember all the several little steps described in the example above, or memorize this very unpleasant formula.

Third, if $f(x)$ is not a linear combination of sines, finding c_n's may be tricky. Here's where Fourier series come in: almost all functions $f(x)$ may be expressed as an infinite linear combination of sines and, once this is done, reading the coefficients c_n becomes much easier. More precisely, a function $f(x)$ on an interval $0 < x < L$ may be written as

$$f(x) = \sum_{n\geq 1} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficients b_n are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

Solving such integrals almost always requires **integration by parts** (and this is why I keep insisting for you to review it). In this case, it will turn out that

$$u(x,0) = f(x) \implies \sum_{n\geq 1} c_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n\geq 1} b_n \sin\left(\frac{n\pi x}{L}\right) \implies c_n = b_n.$$

The same strategy works to solve, for example, a **vibrating string problem**:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L, \\ \dfrac{\partial u}{\partial t}(x,0) = g(x), & 0 < x < L, \end{cases}$$

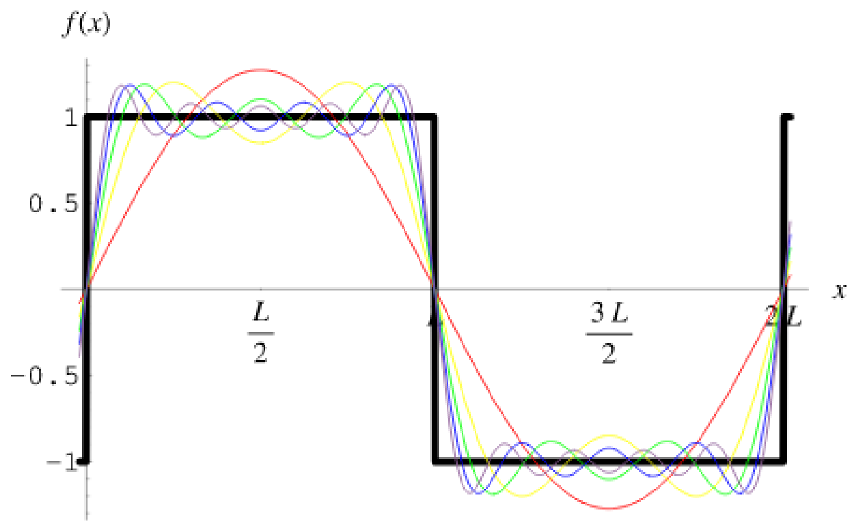
With the only difference being that now the ODE for $T_n(t)$ has order 2 instead of 1, and thus there are two sequences of coefficients to solve for (as opposed to just c_n's), and those are found by looking at the Fourier series of both $f(x)$ and $g(x)$.

15 November 29th

We finally explore a bit of Fourier series here. To get some first intuition, let's compare it with a type of series we have already studied before:

- **Taylor series:** its partial sums (Taylor polynomials) approximate a function, near a given point, with higher and higher degree polynomial expressions.
- **Fourier series:** its partial sums (trigonometric polynomials) approximate a function by superposing more and more "waves" of various frequencies.

Imagine something like the following picture[5]:



In the above picture, the indicated waves correspond to partial sums of a Fourier series. But how to actually compute them? The Fourier expansion[6] of a function $f(x)$ on a symmetric interval $[-L, L]$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n\geq 1}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right),$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

Here, the formula for a_n holds for $n = 0, 1, 2, \ldots$, while the formula for b_n holds for $n = 1, 2, \ldots$. There are many things to unpack here. First, there's no b_0 terms because trying to plug $n = 0$ into the formula for b_n gives just 0. Second, the coefficient a_0 plays a different role than the coefficients a_n for $n \geq 1$, and must always be addressed separately. One reason for this is that their "qualitative" behavior is very distinct.

[5]Taken from `https://mathworld.wolfram.com/FourierSeriesSquareWave.html`.

[6]"Fourier series" and "Fourier expansion" are used interchangeably. "Fourier transform" is something completely different, though.

For example, the constant function 1 regarded as a wave has no amplitude, and its usual antiderivative, x, is a polynomial, while $\cos(n\pi x/L)$ has varying amplitute and its usual antiderivative is trigonometric. The factor of 2 simply makes things work and is explained by general symmetry reasons we'll have opportunity to explore in detail soon. Lastly, when expressing the Fourier series of a function, one uses the approximate sign $\sim$ instead of an equality sign, as a reminder that convergence issues of Fourier series are more subtle than convergence issues for, say, Taylor series.

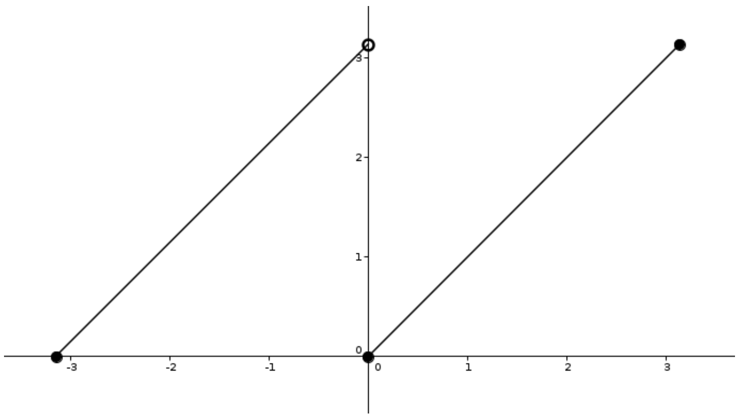
It's also convenient to note that while the argument $n\pi x/L$ doesn't look exactly friendly, in most problems we have to deal with, the number L will be an integer multiple of π, which makes things more tractable.

Example 24

Compute the Fourier expansion of the piecewise function f given by

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \pi, \\ x + \pi, & \text{if } -\pi \leq x < 0. \end{cases}$$

We may start trying to get some intution for what the graph of this function looks like. See the next figure.



We must simply compute a_0, a_n, and b_n, and insert the results in a series. From the figure, we immediately have that

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi} \cdot \frac{\pi^2}{2} \cdot 2 = \pi,$$

as we know that integrals of positive functions compute areas under graphs, and we have two triangles with both base and height equal to π. As for a_n and b_n with $n \geq 1$, we must break the integral from $-\pi$ to π into two integrals, so we can actually use the concrete expressions for $f(x)$ (on each subinterval) given to us.

For example:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\cos(nx)\,\mathrm{d}x + \int_{0}^{\pi} f(x)\cos(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} (x+\pi)\cos(nx)\,\mathrm{d}x + \int_{0}^{\pi} x\cos(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \pi\cos(nx)\,\mathrm{d}x + \cancelto{0}{\int_{-\pi}^{\pi} x\cos(nx)\,\mathrm{d}x}\right) \\
&= \int_{-\pi}^{0}\cos(nx)\,\mathrm{d}x \\
&= \frac{\sin(nx)}{n}\bigg|_{-\pi}^{0} \\
&= 0.
\end{aligned}$$

Again, breaking the original integral into two was needed so we could use the concrete expressions given for $f(x)$ on each interval. Being able to join things back together on the $\int_{-\pi}^{\pi} x\cos(nx)\,\mathrm{d}x$ was a convenient coincidence due to the fact that x appeared in both expressions defining $f(x)$. This integral is zero for symmetry reasons: **the integral of an odd function over a symmetric interval vanishes** (namely, x is odd and $\cos(nx)$ is even, so the product $x\cos(nx)$ is odd). Dealing with b_n's is similar, this time using that **the integral of an even function over a symmetric interval equals twice the integral over the right (or left) half of the interval**. We have that:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\sin(nx)\,\mathrm{d}x + \int_{0}^{\pi} f(x)\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} (x+\pi)\sin(nx)\,\mathrm{d}x + \int_{0}^{\pi} x\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \pi\sin(nx)\,\mathrm{d}x + \int_{-\pi}^{\pi} x\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \pi\sin(nx)\,\mathrm{d}x + 2\int_{0}^{\pi} x\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(-\frac{\pi}{n}\cos(nx)\bigg|_{-\pi}^{0} + 2\left(-\frac{x}{n}\cos(nx)\bigg|_{0}^{\pi} + \frac{1}{n}\int_{0}^{\pi}\cos(nx)\,\mathrm{d}x\right)\right) \\
&= \frac{1}{\pi}\left(-\frac{\pi}{n}(1-(-1)^n) + 2\left(-\frac{\pi}{n}(-1)^n + \cancelto{0}{\frac{\sin(nx)}{n^2}\bigg|_{0}^{\pi}}\right)\right)
\end{aligned}$$

$$= -\frac{1}{n}(1-(-1)^n + 2(-1)^n)$$
$$= -\frac{(1+(-1)^n)}{n}$$

by using that $\cos(n\pi) = (-1)^n$ for any integer n. In green, we have also used integration by parts with

$$u = x \qquad \mathrm{d}v = \sin(nx)\,\mathrm{d}x$$
$$v = -\frac{1}{n}\cos(nx) \qquad \mathrm{d}u = \mathrm{d}x$$

In any case, we have obtained the Fourier expansion

$$f(x) \sim \frac{\pi}{2} - \sum_{n\geq 1} \frac{1+(-1)^n}{n}\sin(nx) \quad \text{on } [-\pi,\pi],$$

as desired. Since $1+(-1)^n$ equals 0 when n is odd and 2 when n is even, we may set $n = 2k$ and rewrite our answer as

$$f(x) \sim \frac{\pi}{2} - \sum_{k\geq 1} \frac{1}{k}\sin(2kx) \quad \text{on } [-\pi,\pi],$$

after simplifying $2/(2k) = 1/k$.

In the above example, we were able to find the Fourier expansion of a function $f(x)$ which was not continuous (namely, the figure shows a jump discontinuity at $x = 0$). This is another important difference between Taylor series and Fourier series. For Taylor series, the function must have all derivatives existing at the chosen center point, while a Fourier series does not require the choice of a center point (although one could reasonably argue that the center in this case is 0) or derivatives to exist. The only thing we must be able to do is to compute the relevant integrals, but integrals are insentitive to a countable number of discontinuities.

16 December 6th

Let's start by recalling the different types of Fourier expansions.

- **"Full" Fourier series:** This is done for a function $f(x)$ defined on a **symmetric** interval $[-L, L]$, and we have

$$f(x) \sim \frac{a_0}{2} + \sum_{n\geq 1} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

  where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

  The formula for a_n is good for $n = 0, 1, 2...$ and the one for b_n is good for $n = 1, 2, ...$ Computing all these coefficients is generally time-consuming, so you should always look for symmetries. Namely, if $f(x)$ is even, then $b_n = 0$ for all n, since the full Fourier series of an even function should not have odd/sin terms. Similarly, if $f(x)$ is odd, then $a_n = 0$ for all n, since the full Fourier series of an odd function should not have even/cos terms. In other words, symmetries of a function reflect into symmetries of its Fourier series.

- **Sine series:** This is done for a function $f(x)$ defined on a **half-interval** $[0, L]$, and we have

$$f(x) \sim \sum_{n\geq 1} b_n \sin\left(\frac{n\pi x}{L}\right),$$

  where

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \qquad n = 1, 2, ...$$

  This is important when dealing with heat flow problems, as the initial temperature distribution is defined on an interval $[0, L]$, and not $[-L, L]$.

- **Cosine series:** This is done for a function $f(x)$ defined on a **half-interval** $[0, L]$, and we have

$$f(x) \sim \frac{a_0}{2} + \sim \sum_{n\geq 1} a_n \cos\left(\frac{n\pi x}{L}\right),$$

  where

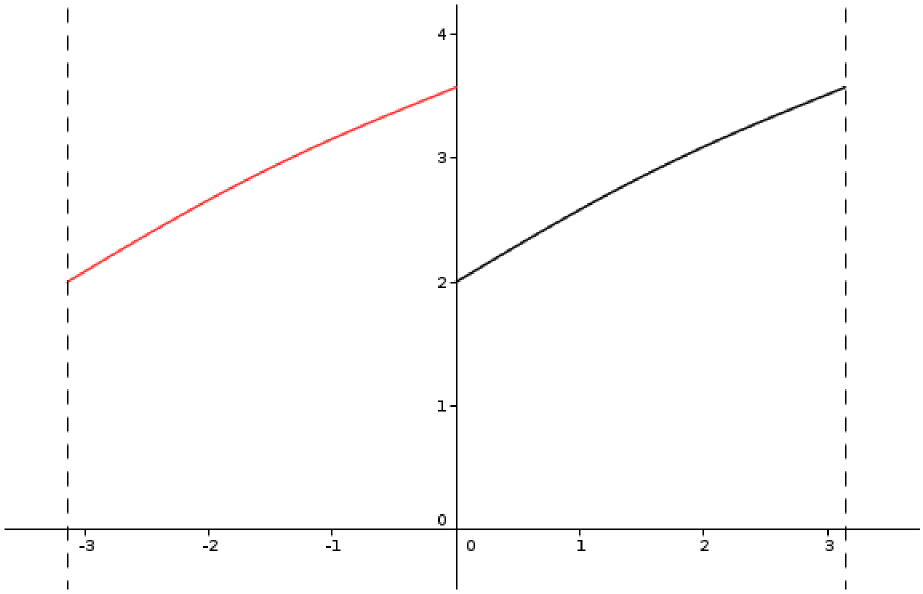
$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \qquad n = 0, 1, 2, ...$$

  This sort of expansion is useful for solving problems similar to heat flow problems, but on which the boundary condition $u(0, t) = u(L, t) = 0$ is replaced[7] with $(\partial u/\partial x)(0, t) = (\partial u/\partial x)(L, t) = 0$ – the resulting prototype solution has cosines instead of sines, and the rest is history.

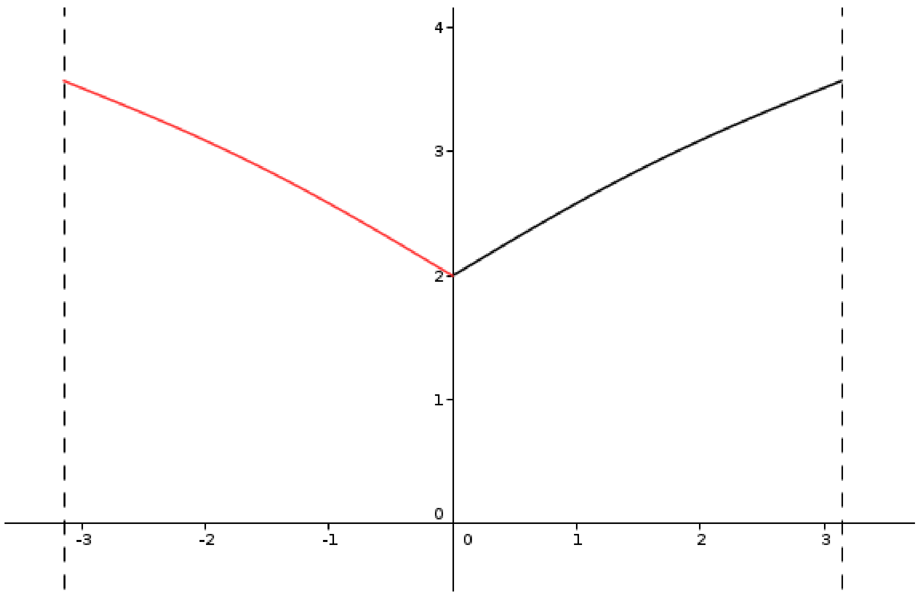
[7] **Exercise:** what are the eigenvalues λ and eigenfunctions X to the equation $X''(x) + \lambda X(x) = 0$ subject to $X'(0) = X'(L) = 0$ instead of $X(0) = X(\pi) = 0$?

Note that asking yourself whether a function defined on a half-interval $[0, L]$ is even or odd makes no sense: it would have to be defined on a symmetric interval $[-L, L]$ to begin with. There is no way to simplify the (already adjusted!) formulas for sine series and cosine series. The sine series and cosine series of a function $f(x)$ on $[0, L]$ are essentially restrictions of the full Fourier series of the odd or even extensions of $f(x)$ to $[-L, L]$. For example:

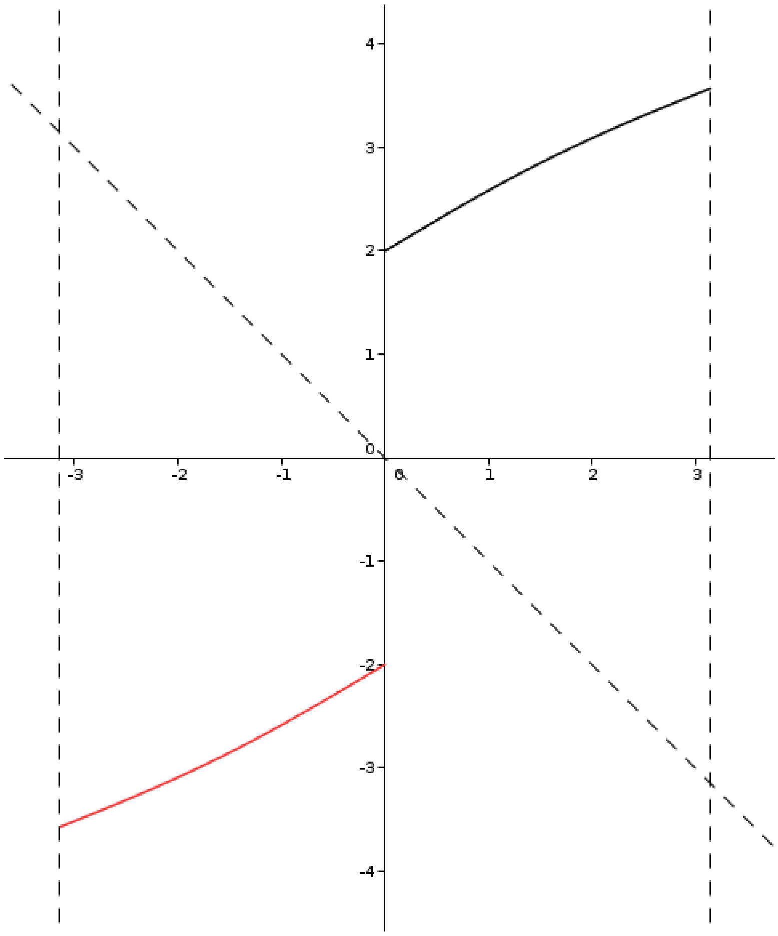
- **Periodic extension.** Copy and paste the function defined on $[0, L]$ to the interval $[-L, 0]$:



- **Even extension.** Flip it across the y-axis:



- **Odd extension.** Reflect the function across the origin:

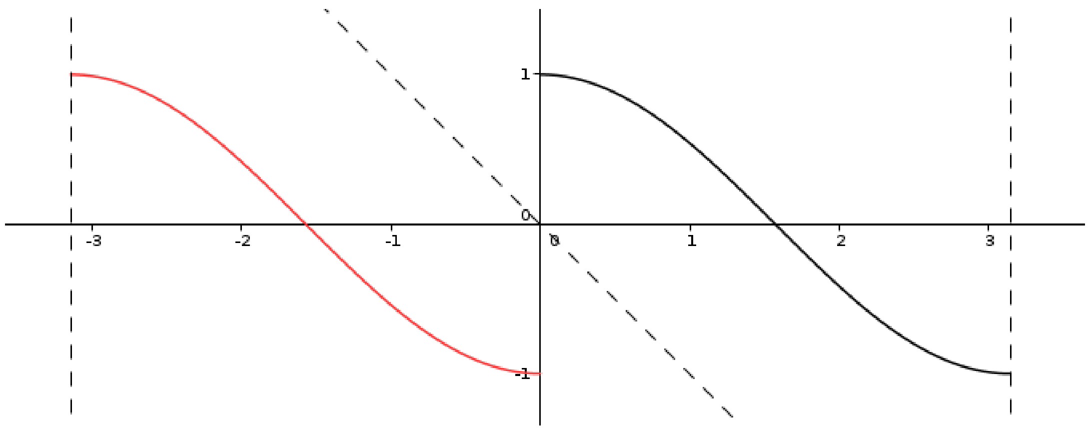


With this in place, let's see what happens with a concrete example:

Example 25

Determine the Fourier sine-expansion of $\cos x$ *on* $[0, \pi]$.

Let's start by understanding what is the odd extension $\widetilde{\cos x}$ of $\cos x$ looks like (and noting that the fact that $\cos x$, **when considered on the full interval** $[-\pi, \pi]$ **to begin with**, was already even, is irrelevant):



In this particular case, the periodic extension and the odd extension agree. We already know that $a_n = 0$ for all n by symmetry reasons. As for the coefficients b_n, we have that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \widetilde{\cos x}\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} \widetilde{\cos x}\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} \cos x\sin(nx)\,\mathrm{d}x,$$

since $\widetilde{\cos x}\sin(nx)$ is even (as the product of the two odd functions $\widetilde{\cos x}$ and $\sin(nx)$), and $\widetilde{\cos x} = \cos x$ for x in the right interval $[0,\pi]$ (this is what it means to say that $\widetilde{\cos x}$ is an **extension** of $\cos x$). Of course, when solving problems like this, you don't have to go and write the above step carrying the extension of the given function: go ahead and compute b_n (or a_n, for even extensions) with the extra coefficient of 2 and the integral being carried only over the "right" interval. The point of doing this step here is illustrating that while the textbook [1] presents different formulas for Fourier expansions, Fourier sine-expansions, and Fourier-cosine expansion, they're really the same thing, and no extra effort on memorizing things should be made.

In any case, it remains to compute this integral. For that, we must rely on product-to-sum trigonometric identities:

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \cos x\sin(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}(\sin((n+1)x)+\sin((n-1)x))\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} \sin((n+1)x)+\sin((n-1)x)\,\mathrm{d}x \\
&= -\frac{1}{\pi}\left(\frac{\cos((n+1)x)}{n+1}+\frac{\cos((n-1)x)}{n-1}\right)\Bigg|_0^{\pi} \\
&= -\frac{1}{\pi}\left(\frac{(-1)^{n+1}-1}{n+1}+\frac{(-1)^{n-1}-1}{n-1}\right) \\
&\overset{(*)}{=} \frac{1+(-1)^n}{\pi}\left(\frac{1}{n+1}+\frac{1}{n-1}\right) \\
&= \frac{2n(1+(-1)^n)}{\pi(n^2-1)},
\end{aligned}$$

where in $(*)$ we have used that $(-1)^{n+1} = (-1)^{n-1}$ (as the powers differ by an even number) and distributed the negative sign to carry the simplification. This means that we have

$$\cos x \sim \sum_{n\geq 1} \frac{2n(1+(-1)^n)}{\pi(n^2-1)}\sin(nx) \quad \text{on } [0,\pi].$$

We conclude this course by seeing how things come full circle:

Example 26

Solve the following heat flow problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = 10\dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0, & \\ u(x,0) = \cos x, & 0 < x < \pi \end{cases}$$

We start making a separation of variables, $u(x,t) = X(x)T(t)$. The PDE itself becomes $X(x)T'(t) = 10X''(x)T(t)$. We then have that

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{10T(t)} = -\lambda \in \mathbb{R},$$

as the left side doesn't depend on t and the right side doesn't depend on x. We obtain

$$X''(x) + \lambda X(x) = 0 \quad \text{and} \quad T'(t) + 3\lambda T(t) = 0,$$

and the endpoint conditions on u read $X(0) = X(\pi) = 0$. From here, it follows that $\lambda = n^2$ for some natural number $n \geq 1$, and $X_n(x) = a_n \sin(nx)$ for some real number a_n. Finding the corresponding $T_n(t)$, we solve the first order ODE $T_n'(t) = -10n^2 T_n(t)$: the solution is $T_n(t) = b_n \mathrm{e}^{-10n^2 t}$ for some second real constant b_n. With this in place, setting $c_n = a_n b_n$, we have that

$$u_n(x,t) = X_n(t)T_n(t) = c_n \mathrm{e}^{-10n^2 t}\sin(nx).$$

For each n, this function u_n satisfies almost everything required to solve the heat flow problem, except for the initial distribution condition $u(x,0) = \cos x$. To achieve this, we consider the series $u = \sum_{n\geq 1} u_n$ and find the c_n's to make this work. In other words, we have that

$$\sum_{n\geq 1} c_n \mathrm{e}^{-10n^2 t}\sin(nx)\Big|_{t=0} = \sum_{n\geq 1} c_n \sin(nx) = \cos x.$$

We have already seen that

$$\cos x \sim \sum_{n\geq 1} \frac{2(1+(-1)^n)n}{\pi(n^2-1)}\sin(nx) \quad \text{on } [0,\pi],$$

so that

$$c_n = \frac{2(1+(-1)^n)n}{\pi(n^2-1)}$$

for every $n \geq 1$, and we conclude that the solution to the given heat flow problem is

$$u(x,t) = \sum_{n\geq 1} \frac{2n(1+(-1)^n)}{\pi(n^2-1)} \mathrm{e}^{-10n^2t} \sin(nx).$$

The remainder of our time together was spent discussing a couple of the homework problems.

Exercise 30

Find the eigenvalues and eigenfunctions of

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = y(2\pi), \quad y'(0) = y'(2\pi) \end{cases}$$

The question we should always be asking ourselves is: for which values of λ will we get nonzero solutions y? How this is done? We always look at three cases: $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$, and we ask ourselves if, given the form of the general solution in this case, the conditions $y(0) = y(2\pi)$ and $y'(0) = y'(2\pi)$ are strong enough to force $c_1 = c_2 = 0$. If not, we get an eigenvalue, and such conditions will say something about λ itself.

- If $\lambda < 0$, the solutions of the characteristic equation are $r = \pm\sqrt{-\lambda}$ (note that if $\lambda < 0$, then $-\lambda > 0$ being inside the root is no problem), so we have $y = c_1\mathrm{e}^{\sqrt{-\lambda}x} + c_2\mathrm{e}^{-\sqrt{-\lambda}x}$. The boundary conditions read

$$\begin{cases} c_1 + c_2 = c_1\mathrm{e}^{2\pi\sqrt{-\lambda}} + c_2\mathrm{e}^{-2\pi\sqrt{-\lambda}} \\ \sqrt{-\lambda}c_1 - \sqrt{-\lambda}c_2 = \sqrt{-\lambda}c_1\mathrm{e}^{2\pi\sqrt{-\lambda}} - \sqrt{-\lambda}c_2\mathrm{e}^{-2\pi\sqrt{-\lambda}} \end{cases}$$

  Simplifying $\sqrt{-\lambda}$ in the second equation (as $\lambda \neq 0$ in this case) and adding the equations gives us that $2c_1 = 2c_1\mathrm{e}^{2\pi\sqrt{-\lambda}}$, so $\lambda \neq 0$ says that this exponential is not equal to 1, and so $c_1 = 0$. Rinse and repeat to get $c_2 = 0$. Therefore, we have no negative eigenvalues.

- If $\lambda = 0$, the general solution is $y = c_1 + c_2x$. The boundary conditions read $c_1 = c_1 + 2\pi c_2$ and $c_2 = c_2$, so that $c_2 = 0$ and we cannot determine c_1: it is a free variable. Therefore $\lambda = 0$ is an eigenvalue, with corresponding eigenfunctions $y_0 = a_0$ (constant).

- If $\lambda > 0$, the solutions of the characteristic equation are $r = \pm\mathrm{i}\sqrt{\lambda}$, so we have that $y = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$. The boundary conditions read

$$\begin{cases} c_1 = c_1\cos(2\pi\sqrt{\lambda}) + c_2\sin(2\pi\sqrt{\lambda}) \\ \sqrt{\lambda}c_2 = -\sqrt{\lambda}c_1\sin(2\pi\sqrt{\lambda}) + \sqrt{\lambda}c_2\cos(2\pi\sqrt{\lambda}) \end{cases}$$

Simplifying $\sqrt{\lambda}$ in the second equation (as $\lambda \neq 0$ in this case), we may look at the coefficient matrix for this homogeneous linear system for c_1 and c_2. Namely, it equals

$$\begin{bmatrix} \cos(2\pi\sqrt{\lambda}) - 1 & \sin(2\pi\sqrt{\lambda}) \\ -\sin(2\pi\sqrt{\lambda}) & \cos(2\pi\sqrt{\lambda}) - 1 \end{bmatrix}$$

The only chance for us to obtain a nontrivial solution for c_1 and c_2 is if this matrix is singular (or, equivalently, its columns or rows are linearly dependent, or yet its determinant is equal to zero). Argue however you want, the conclusion is that we must have $\sin(2\pi\sqrt{\lambda}) = 0$ and $\cos(2\pi\sqrt{\lambda}) = 1$, and so we must have $2\pi\sqrt{\lambda} = 2\pi n$ for some integer n. So, $\lambda = n^2$ are the positive eigenvalues, and the corresponding eigenfunctions are given by $y_n = a_n \cos(nx) + b_n \sin(nx)$, where $a_n, b_n \in \mathbb{R}$ are completely arbitrary (namely, the coefficient matrix becomes the zero matrix for such values of λ, and so both c_1 and c_2 are free variables – hence renamed as to be indexed by n as well).

"When you feel like giving up - persevere
When faced with challenges - persevere
When you're tired and feel like no more - persevere
When others say stop - persevere.

Set your eyes on your goal and persevere.
Don't be discouraged but persevere.
The one who keeps going, who perseveres
Is the one that will reach their goals long before there peers."

CATHERINE PULSIFER — "PERSEVERE"

References

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