On rigidity of 0-isotropic submanifolds of Lorentzian space forms

Author: Ivo Terek Couto

The Ohio State University

terekcouto.1@osu.edu

Abstract

A non-degenerate submanifold of a pseudo-Riemannian manifold is called *marginally trapped* when its mean curvature vector is lightlike at all points. This is particularly relevant in General Relativity when the ambient manifold is a spacetime. Related geometric conditions include, among others, having lightlike second fundamental form, pseudo-umbilicity, and λ-isotropy. Building up on previous work from Cabrerizo, Fernández and Gómez [1], and adapting some preliminary results by Anciaux [2], we characterize (under reasonable assumptions) codimension 2 0-isotropic submanifolds of pseudo-Riemannian space forms now of arbitrary dimension and index, which turn out to be marginally trapped.

Why care about marginally trapped submanifolds?

When $(M, \langle\cdot,\cdot\rangle)$ is a spacetime, that is, a four-dimensional Lorentzian manifold equipped with a time orientation (a non-vanishing timelike vector field), we call a spacelike surface $S \subseteq M$ *future-trapped* if all the future-outgoing light rays from S converge — geometrically, this means that the mean curvature vector of S if timelike and future-directed. This concept was introduced by Penrose in 1965, and its relation with gravitational collapse and formation of black holes ultimately led to him being awarded a Nobel Prize in 2020. Marginally trapped surfaces are the ones that separate trapped surfaces from non-trapped surfaces, can be used to try and detect event horizons of black holed, and have intrinsic geometric interest.

Definitions

Definition 1. *Let $(\overline{M}^{n+2}, \langle\cdot,\cdot\rangle)$ be a pseudo-Riemannian manifold and M be a non-degenerate submanifold of $\overline{M}$, with second fundamental form II and mean curvature vector $\boldsymbol{H}$. We'll say that M is:*

- critical *if $\boldsymbol{H} = \boldsymbol{0}$;*
- marginally trapped *if $\boldsymbol{H}$ is lightlike at all points;*
- pseudo-umbilic *if there is $\rho \in \mathcal{C}^\infty(M)$ such that $A_{\boldsymbol{H}} = \rho\,\mathrm{Id}$, where $A_{\boldsymbol{H}}$ is the Weingarten operator of $\boldsymbol{H}$ — in this case, $\rho = \langle \boldsymbol{H}, \boldsymbol{H}\rangle$;*
- λ-isotropic, *where $\lambda \in \mathcal{C}^\infty(M)$, if for all vector fields $\boldsymbol{X} \in \mathfrak{X}(M)$ we have $\langle \mathrm{II}(\boldsymbol{X}), \mathrm{II}(\boldsymbol{X})\rangle = \lambda$ — we say just* isotropic *when λ is not specified.*

Let's write

$$\mathbb{M}^n_\nu(c) = \begin{cases} \mathbb{R}^n_\nu, \text{ if } c = 0; \\ \mathbb{S}^n_\nu, \text{ if } c = 1, \text{ and;} \\ \mathbb{H}^n_\nu, \text{ if } c = -1, \end{cases}$$

for $n \geq 3$ and $0 \leq \nu \leq n$. We will only consider $c \in \{-1, 0, 1\}$. As usual, $\mathbb{R}^n_\nu$ is the space $\mathbb{R}^n$ equipped with the pseudo-Euclidean scalar product whose matrix relative to the standard basis is $\mathrm{Id}_{n-\nu} \oplus (-\mathrm{Id}_\nu)$, and

$$\mathbb{S}^n_\nu = \{\boldsymbol{p} \in \mathbb{R}^{n+1}_\nu \mid \langle \boldsymbol{p}, \boldsymbol{p}\rangle = 1\},$$
$$\mathbb{H}^n_\nu = \{\boldsymbol{p} \in \mathbb{R}^{n+1}_{\nu+1} \mid \langle \boldsymbol{p}, \boldsymbol{p}\rangle = -1\}.$$

If $M^n \subseteq \mathbb{M}^{n+2}_\nu(c)$ is a non-degenerate submanifold with Lorentzian normal spaces, we'll call a pair $(\boldsymbol{L}_+, \boldsymbol{L}_-)$ a *Penrose frame* (or null frame, asymptotic frame, etc.) if both $\boldsymbol{L}_+$ and $\boldsymbol{L}_-$ are normal and lightlike vector fields normal to M, whose product is a non-zero constant — here, we'll take it to be 2. In Minkowski space, here's a picture:

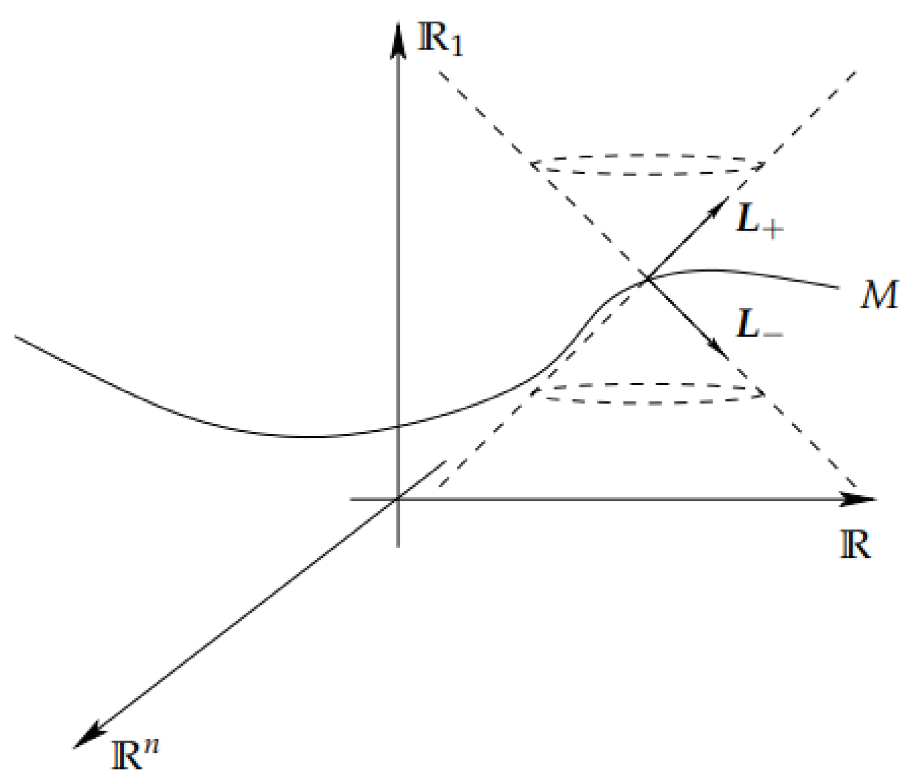

Figure: A Penrose frame for $M^n \subseteq \mathbb{R}^{n+2}_1$.

Basic lemmas and the null II case

We start with a useful computational lemma:

Lemma 2. *Let $M^n \subseteq \mathbb{M}^{n+2}_\nu(c)$ be a non-degenerate submanifold with Lorentzian normal spaces, $(\boldsymbol{L}_+, \boldsymbol{L}_-)$ a Penrose frame, and $(\boldsymbol{E}_i)_{i=1}^n$ a (local) tangent frame for M. Writing*

$$A_{\boldsymbol{L}_\pm}(\boldsymbol{E}_j) = \sum_{i=1}^n h_\pm{}^i{}_j \boldsymbol{E}_i \quad \text{and} \quad \mathrm{II}(\boldsymbol{E}_i, \boldsymbol{E}_j) = h^+_{ij}\boldsymbol{L}_+ + h^-_{ij}\boldsymbol{L}_-,$$

we have that

(i) $\langle \mathrm{II}(\boldsymbol{E}_i, \boldsymbol{E}_j, \boldsymbol{L}_\pm\rangle = 2h^\mp_{ij}$

(ii) $\mathrm{d}(\boldsymbol{L}_\pm)(\boldsymbol{E}_j) = -\sum_{i=1}^n h_\pm{}^i{}_j \boldsymbol{E}_i + \frac{1}{2}\langle \mathrm{d}(\boldsymbol{L}_\pm)(\boldsymbol{E}_j), \boldsymbol{L}_\mp\rangle \boldsymbol{L}_\pm$

(iii) $2n\boldsymbol{H} = \mathrm{tr}(A_{\boldsymbol{L}_-})\boldsymbol{L}_+ + \mathrm{tr}(A_{\boldsymbol{L}_+})\boldsymbol{L}_-$.

With this in place, we adapt and correct a result from [1]:

Lemma 3. *Let $M^n \subseteq \mathbb{M}^{n+2}_\nu(c)$ be a non-degenerate submanifold with Lorentzian normal spaces, $(\boldsymbol{L}_+, \boldsymbol{L}_-)$ a Penrose frame. If II is always proportional to $\boldsymbol{L}_+$, then so is $\boldsymbol{H}$ and the rank of $\mathrm{d}(\boldsymbol{L}_+)$ is at most 1. If $M^n \subseteq \mathbb{M}^{n+2}_1(c)$ is* spacelike, *the converse holds.*

This leads to:

Theorem 4. *Let $\boldsymbol{L}_0 \in \mathbb{R}^{n+2+|c|}_{\nu+(c^2+c)/2}$ be a lightlike vector and $M^n \subseteq \mathbb{M}^{n+2}_\nu(c)$ be a non-degenerate submanifold such that $M^n \subseteq \boldsymbol{L}_0^\perp$. Then M has null II, constant sectional curvature c, and flat normal bundle. Conversely, if M^n has lightlike II, every point in M has a neighborhood contained in a lightlike hyperplane.*

A small variation of the proof gives the:

Theorem 5. *Let $M^n \subseteq \mathbb{M}^{n+2}_\nu(c)$ be a (simply connected) marginally trapped submanifold with flat normal bundle. Then M^n is contained in a lightlike hyperplane (and hence has lightlike II and constant sectional curvature c).*

Fundamental equivalences for marginally trapped surfaces in 4-dim ambient spaces

Now following [2], we obtain the following lemma (originally stated only for spacelike surfaces in Lorentzian ambient spaces:

Lemma 6. *Let $(\overline{M}^n, \langle\cdot,\cdot\rangle)$ be a pseudo-Riemannian manifold and $M^2 \subseteq \overline{M}^n$ be a non-degenerate λ-isotropic surface. Then M is pseudo-umbilic and*

$$3\langle \boldsymbol{H}, \boldsymbol{H}\rangle = 2\lambda + K_{\mathrm{ext}},$$

where K_{ext} is the extrinsic curvature of M.

Thus, we have:

Theorem 7. *Let $(\overline{M}^4, \langle\cdot,\cdot\rangle)$ be a pseudo-Riemannian manifold and M^2 be a non-degenerate and marginally trapped surface in $\overline{M}^4$. The following are equivalent:*

(i) M is pseudo-umbilic;

(ii) M is 0-isotropic;

(iii) M is isotropic.

Rigidity in space-forms $\mathbb{M}^{n+2}_\nu(c)$

Theorem 4 here, together about generalities about pseudo-Riemannian coverings (see [3]) give our last result here:

Theorem 8. *Fix $c \in \{-1, 0, 1\}$. Let $M^n \subseteq \mathbb{M}^{n+2}_\nu(c)$ be a non-degenerate submanifold. Assume that M is complete, simply connected, 0-isotropic and free of flat points (i.e., where $\mathrm{II} = 0$). Then:*

(i) M is isometric to $\mathbb{M}^n_{\nu-1}(c)$;

(ii) M is congruent *to the image of a map $\boldsymbol{x}\colon \mathbb{M}^n_{\nu-1}(c) \to \mathbb{M}^{n+2}_\nu(c)$ given by*

$$\boldsymbol{x}(\boldsymbol{u}) = (\tau(\boldsymbol{u}), \boldsymbol{u}, \tau(\boldsymbol{u})) \in \mathbb{R} \times \mathbb{M}^n_{\nu-1}(c) \times \mathbb{R}_1,$$

where $\tau \in \mathcal{C}^\infty(\mathbb{M}^n_{\nu-1}(c))$, whose mean curvature vector is given by

$$\boldsymbol{H} \circ \boldsymbol{x} = \frac{\triangle\tau + nc\tau}{n}(1, \boldsymbol{0}, 1).$$

Remark. *Provided the function τ is* not *an eigenfunction of the spherical Laplacian (in $\mathbb{S}^n$), this is a source of compact marginally trapped submanifolds in the de Sitter space $\mathbb{S}^{n+1}_1$.*

References

[1] Anciaux, H.; *Marginally Trapped Submanifolds in Space Forms with Arbitrary Signature*, Pacific Journal of Mathematics 272, n° 2, p. 257-274, 2014.

[2] Cabrerizo, J. L.; Fernández, M.; Gómez, J. S.; *Isotropy and marginally trapped surfaces in a spacetime*, Classical and Quantum Gravity 27, IOP Publishing, 2010.

[3] O'Neill, B. *Semi-Riemannian Geometry With Applications to Relativity*, Academic Press, 2006.

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