On compact rank-one ECS manifolds

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These slides can also be found at

https://www.asc.ohio-state.edu/terekcouto.1/texts/KSU_slides_march2023.pdf

Setup

The curvature tensor of a pseudo-Riemannian manifold (M, g), $n = \dim M$, admits the orthogonal decomposition

$$R = \frac{\mathrm{s}}{n(n-1)}\mathrm{g} \owedge \mathrm{g} + \frac{2}{n-2}\mathrm{g} \owedge \left(\mathrm{Ric} - \frac{\mathrm{s}}{n}\mathrm{g}\right) + W,$$

where Ric, s and W are the Ricci, scalar, and Weyl curvatures of (M, g).

We will assume throughout this talk that $n \geq 4$, in which case (M, g) is *conformally flat* if and only if $W = 0$.

The condition we are interested in is $\nabla W = 0$.

Definition (ECS manifold)

A pseudo-Riemannian manifold (M, g) is called *essentially conformally symmetric* if $\nabla W = 0$ but neither $W = 0$ nor $\nabla R = 0$.

What is known

ECS manifolds are objects of strictly indefinite nature:

Theorem (Roter, 1977)

For a Riemannian manifold (M, g)*:* $\nabla W = 0 \iff W = 0$ *or* $\nabla R = 0$*.*

Other important facts:

- The local structure of ECS manifolds has been completely described by Derdzinski and Roter in 2009.
- Every ECS manifold carries a distinguished null parallel distribution $\mathcal{D}$, whose rank equals 1 or 2. We call $\mathcal{D}$ the *Olszak distribution* of (M, g) and refer to the rank of $\mathcal{D}$ as the *rank of* (M, g).
- There are **compact** ECS manifolds of all dimensions of the form $3j + 2$, $j \geq 1$, realizing all indefinite metric signatures (Derdzinski-Roter, 2010).

A rank-one example

Example (Conformally symmetric "pp-waves")

Let $(V, \langle\cdot,\cdot\rangle)$ be a pseudo-Euclidean vector space of dimension $n-2 \geq 2$, $A \in \mathfrak{sl}(V)$ be self-adjoint, $I \subseteq \mathbb{R}$ be an open interval and $f\colon I \to \mathbb{R}$ be a smooth function. Consider

$$(\widehat{M}, \widehat{\mathrm{g}}) = (I \times \mathbb{R} \times V, \kappa\,\mathrm{d}t^2 + \mathrm{d}t\,\mathrm{d}s + \langle\cdot,\cdot\rangle),$$

where $\kappa\colon \widehat{M} \to \mathbb{R}$ is given by $\kappa(t,s,v) = f(t)\langle v, v\rangle + \langle Av, v\rangle$.

Then $(\widehat{M}, \widehat{\mathrm{g}})$ has $\nabla W = 0$, with:

- $W = 0 \iff A = 0$;
- $\nabla R = 0 \iff f$ is constant.

In the ECS case, the Olszak distribution $\mathcal{D}$ is spanned by the null parallel coordinate vector field ∂_s, and $(V, \langle\cdot,\cdot\rangle)$ is isometrically identified with the vector space of parallel sections of $\mathcal{D}^\perp/\mathcal{D}$.

$(\widehat{M}, \widehat{\mathrm{g}})$ is complete if and only if $I = \mathbb{R}$ (which we'll assume from now on).

About compact examples

The 2010 compact ECS examples all have rank one, and were obtained by finding suitable subgroups $\Gamma \subseteq \mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ acting freely and properly discontinuously on $(\widehat{M}, \widehat{\mathrm{g}})$ with compact quotient $M = \widehat{M}/\Gamma$.

The previously mentioned dimensions of the form $3j + 2$ were a particularity of the construction performed then: a 5-dimensional example was obtained with $\dim V = 3$, but the construction was "compatible" with taking cartesian powers of $(V, \langle \cdot, \cdot \rangle)$, leading also to dimensions 8, 11, 14, etc.

Theorem (Derdzinski-T., 2022)

There exist compact rank-one ECS manifolds of ***all dimensions n*** $\boldsymbol{\geq}$ **5** *and all indefinite metric signatures, diffeomorphic to nontrivial torus bundles over the circle, geodesically complete, and not locally homogeneous. Moreover, in each fixed dimension and metric signature, there is an infinite-dimensional moduli space of local isometry types of such manifolds.*

Outline of proof (1/4): searching for Γ inside $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$

Fixing a period $p > 0$, we will look for subgroups $\Gamma \leq \mathrm{G}(p) \leq \mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ producing a compact quotient $\widehat{M}/\Gamma$, for suitable choices of f and A.

To first describe $\mathrm{G}(p)$, we need the symplectic vector space $(\mathcal{E}, \Omega)$ of solutions $u\colon \mathbb{R} \to V$ to the second-order differential equation

$$\ddot{u}(t) = f(t)u(t) + Au(t), \quad \text{with} \quad \Omega(u, \hat{u}) = \langle \dot{u}, \hat{u} \rangle - \langle u, \dot{\hat{u}} \rangle,$$

and the translation operator $T\colon \mathcal{E} \to \mathcal{E}$ given by $(Tu)(t) = u(t - p)$.

Then $\mathrm{G}(p) = \mathbb{Z} \times \mathbb{R} \times \mathcal{E}$ acts isometrically on $(\widehat{M}, \widehat{\mathrm{g}})$ by

$$(k, q, u) \cdot (t, s, v) = \big(t + kp, s + q - \langle \dot{u}(t), 2v + u(t) \rangle, v + u(t)\big),$$

and has its group operation given by

$$(k, q, u) \cdot (\ell, r, w) = (k + \ell, q + r - \Omega(u, T^{\ell} w), T^{-\ell} u + w).$$

Outline of proof (2/4): first-order subspaces of $(\mathcal{E}, \Omega)$

Such a subgroup Γ would give rise to a "lattice" Λ inside a T-invariant *first-order* subspace $\mathcal{L}$ of $(\mathcal{E}, \Omega)$. The goal here is to reverse-engineer Γ from the spectrum of $T|_{\mathcal{L}}$ while at the same time finding f and A.

But what is a first-order subspace? It is a subspace $\mathcal{L} \leq \mathcal{E}$ such that for every $t \in \mathbb{R}$, the evaluation map $\delta_t \colon \mathcal{L} \to V$ is an isomorphism.

$$\{\mathcal{L} \mid \mathcal{L} \text{ is a first-order subspace}\} \leftrightharpoons \{B \colon \mathbb{R} \to \mathrm{End}(V) \mid \dot{B} + B^2 = f + A\}$$
$$\mathcal{L} = \{u \in \mathcal{E} \mid \dot{u}(t) = B(t)u(t) \text{ for all } t \in \mathbb{R}\}$$

In this correspondence: $\mathcal{L}$ is Lagrangian $\iff$ each $B(t)$ is self-adjoint.

The projection $\mathrm{G}(p) \to \mathbb{Z}$ restricts to a homomorphism $\Gamma \to \mathbb{Z}$ whose kernel Σ projects to a subset $\Lambda \subseteq \mathcal{E}$, which spans a first-order subspace $\mathcal{L}$.

Either $\mathcal{L}$ is Lagrangian and Σ is a lattice in $\mathbb{R} \times \mathcal{L}$ which projects isomorphically onto Λ, or Λ itself is a lattice in $\mathcal{L}$.

Outline of proof (3/4): reverse-engineering the spectrum

Next:

- Step 1: Choose $\lambda_1, \ldots, \lambda_{n-2}$ to be positive, mutually distinct, and not equal to 1, so that $\{\lambda_1, \ldots, \lambda_{n-2}\}$ is not of the form $\{\lambda\}$ or $\{\lambda, \lambda^{-1}\}$ for any $\lambda > 0$.
- Step 2: As $n \geq 5$, $\lambda_1, \ldots, \lambda_{n-2}$ are the roots of the characteristic polynomial P of a matrix in $\mathrm{GL}(n-2, \mathbb{Z})$.
- Step 3: Using the Implicit Function Theorem, we may obtain an infinite-dimensional space of p-periodic functions f for which there are a diagonal traceless nonzero matrix A and a curve $t \mapsto B(t)$ of diagonal matrices such that $\dot{B} + B^2 = f + A$ and

$$\mathrm{diag}(\log \lambda_1, \ldots, \log \lambda_{n-2}) = -\int_0^p B(t)\,\mathrm{d}t.$$

Outline of proof (4/4): reconstructing Γ

Step 4: Let $\mathcal{L}$ correspond to B obtained in Step 3, and note that the condition given there means that the spectrum of $T|_{\mathcal{L}}$ is $\lambda_1, \ldots, \lambda_{n-2}$ and its characteristic polynomial is P, so that $T[\Lambda] = \Lambda$ for some lattice $\Lambda \subseteq \mathcal{L}$.

Step 5: As $\mathcal{L}$ is Lagrangian, the action of Λ on $\mathcal{L}$ by vector space translations coincides with its action by left-translations with the group operation induced from $\mathbb{R} \times \mathcal{E} \hookrightarrow \mathrm{G}(p)$;

Step 6: Fixing any $\theta > 0$, we let Γ be the group generated by $\{0\} \times \mathbb{Z}\theta \times \Lambda$ and the element $(1, 0, 0) \in \mathrm{G}(p)$. This works.

For instance, one possible compact fundamental domain for the action of Γ on $\widehat{M}$ is $K = \{(t, s, v) \in \widehat{M} \mid s \in [0, \theta] \text{ and } (s, v) \in K'\}$, where K' is the image under the diffeomorphism

$$\mathbb{R} \times \mathcal{L} \ni (t, w) \mapsto (t, w(t)) \in \mathbb{R} \times V$$

of $[0, p] \times \hat{K}''$, K'' being a compact fundamental domain for $\Lambda \circlearrowright \mathcal{L}$.

Final considerations

The other details stated in the Theorem are consequences of the construction. For example:

- Γ not being virtually Abelian precludes coverings of M by tori;
- The map $\widehat{M} \ni (t, s, v) \mapsto t/p \in \mathbb{R}$ is Γ-equivariant and induces a fibration $M \to \mathbb{S}^1$.
- The fibers $(\{t\} \times \mathbb{R} \times V)/(\{0\} \times \mathbb{Z}\theta \times \Lambda)$ are tori, as they are diffeomorphic to $(\mathbb{R} \times \mathcal{L})/(\mathbb{Z}\theta \times \Lambda)$.
- etc.

This bundle structure is not an accident:

Theorem (Derdzinski-T., 2022)

Every non-locally homogeneous compact rank-one ECS manifold is (up to a double isometric covering) diffeomorphic to a bundle over $\mathbb{S}^1$ in such a way that $\mathcal{D}^\perp$ becomes the vertical distribution.

Thank you for your attention!

References

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