MWDS 2022 LIGHTNING TALK: MAGNETIC COTANGENT BUNDLES

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The correct setup to study **Hamiltonian dynamics** is **symplectic geometry**. The most prominent example is T^*Q, where Q is the configuration manifold of a mechanical system, with the following structure (described relative to cotangent coordinates (q^i, p_i) induced from coordinates (q^i) on Q):

$$\lambda = \sum_i p_i \, \mathrm{d}q^i, \quad \text{and} \quad \omega_{\text{can}} = -\mathrm{d}\lambda = \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i \sim \begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}$$

The 1-form $\lambda \in \Omega^1(T^*Q)$ is called the **tautological form** and is characterized by the relations $\sigma^*\lambda = \sigma$, for each $\sigma \in \Omega^1(Q)$. The 2-form $\omega_{\text{can}} \in \Omega^2(T^*Q)$ is called the **canonical symplectic form**; it is closed (in fact, exact) and non-degenerate (the matrix describing it has full rank).

To proceed, we will twist it with a closed 2-form $B \in \Omega^2(Q)$, by letting

$$\omega_B \doteq \omega_{\text{can}} + \pi^* B \sim \begin{bmatrix} B & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix},$$

where $\pi\colon T^*Q \to Q$ is the bundle projection. As before, ω_B is closed and non-degenerate. We call B a **magnetic field** on Q and ω_B the associated **magnetic symplectic form**. Goal: understand the motion of a particle on Q subject to the action of a magnetic field B.

Magnetic cotangent bundles (T^*Q, ω_B) have plenty of "symmetries" (i.e., symplectomorphisms):

(i) **cotangent lifts** $\hat{f}\colon T^*Q \to T^*Q$ given by $\hat{f}(x,p) = (f(x), p \circ (\mathrm{d}f_x)^{-1})$, for every diffeomorphism $f\colon Q \to Q$ such that $f^*B = B$.

(ii) **fiberwise translations** for every closed $\sigma \in \Omega^1(Q)$, namely, the diffeomorphisms $\tau_\sigma\colon T^*Q \to T^*Q$ given by $\tau_\sigma(x,p) = (x, p + \sigma_x)$. As a consequence, whenever $[B_1] = [B_2] \in H^2_{\mathrm{dR}}(Q)$, then $(T^*Q, \omega_{B_1}) \cong (T^*Q, \omega_{B_2})$, so the magnetic symplectic structure depends only on the cohomology class of the magnetic field.

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Now, assume that g is a Riemannian metric on Q, and define a Hamiltonian function $H\colon T^*Q \to \mathbb{R}$ by setting

$$H(x,p) = \frac{\|p\|_x^2}{2} = \frac{1}{2}\sum_{i,j} g^{ij}(x)p_ip_j.$$

With this in place, non-degeneracy of ω_B defines a vector field $\boldsymbol{X} \in \mathfrak{X}(T^*Q)$ by the relation $\omega_B(\boldsymbol{X},\cdot) = \mathrm{d}H$, and we obtain the **Lorentz force** as the skew-adjoint bundle morphism $\boldsymbol{F}\colon TQ \to TQ$ characterized by $\mathrm{g}_x(\boldsymbol{F}_x(v), w) = B_x(v,w)$, for every $x \in Q$ and $v, w \in T_xQ$. The reason why we care about this is because integral curves of $\boldsymbol{X}$ project down to solutions $\gamma\colon I \to Q$ of the **magnetic geodesic equation** (i.e., the Lorentz force law):

$$\frac{\mathrm{D}\dot\gamma}{\mathrm{d}t}(t) = \boldsymbol{F}_{\gamma(t)}(\dot\gamma(t)), \quad \text{for all } t \in I.$$

Assuming that Q is compact, for example, we obtain a complete flow $\phi_t\colon TQ \to TQ$ and, when $B = 0$, we recover the classical geodesic equation and the geodesic flow from Riemannian geometry. Proofs of the claims made so far are either presented or left as exercises with hints on [3].

There are, however, many other similarities. For example, whenever we have that $B = \mathrm{d}A$ for some $A \in \Omega^1(Q)$, called a **magnetic potential**, magnetic geodesics appear as critical points of the action functional

$$\mathscr{E}_A[\gamma] = \frac{1}{2}\int_I \mathrm{g}_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))\,\mathrm{d}t + \int_I A_{\gamma(t)}(\dot\gamma(t))\,\mathrm{d}t.$$

Replacing A with $A + \mathrm{d}f$ only adds a constant to $\mathscr{E}_A$, and this gauge-invariance has vague interpretations similar in spirit to the Aharonov-Bohm effect.

Lastly, we know that geodesic flows induced by metrics with negative sectional curvature are Anosov, but it has been shown in [1] that "weak" magnetic flows in negative sectional curvature are again Anosov. There, the second variation of $\mathscr{E}_A$ has been computed, suggesting the correct notion of a Jacobi field in the magnetic setting. In [2], these results have been generalized to give conditions for not only magnetic flows being Anosov in negative sectional curvature, but also for **potential flows** and **Gaussian thermostats**.

References

[1] Gouda, N.; *Magnetic flows of Anosov type*. Tohoku Math. J. (2) 49 (1997), no. 2, 165–183.

[2] Wojtkowski, M. P.; *Magnetic flows and Gaussian thermostats on manifolds of negative curvature*. Fund. Math. 163 (2000), no. 2, 177–191.

[3] Terek, I.; *A guide to symplectic geometry*. Lecture notes for the OSU Graduate Math Summer Minicourses 2021. Available at https://u.osu.edu/gmsminicourses/files/2022/05/symp_geo.pdf.