Compact locally homogeneous manifolds with parallel Weyl curvature

(joint work with Andrzej Derdzinski)

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These slides can also be found at

https://www.asc.ohio-state.edu/terekcouto.1/texts/SGSF_slides_february2024.pdf

The Weyl curvature tensor

We will start by recalling the definition of the Weyl curvature tensor W of a pseudo-Riemannian manifold (M, g).

The curvature tensor of $\mathbb{S}^n$ equipped with its round metric is given by

$$R(X,Y,Z,V) = \mathrm{g}(Y,Z)\mathrm{g}(X,V) - \mathrm{g}(X,Z)\mathrm{g}(Y,V)$$

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This is a quadratic expression in g. Polarize!

$$\begin{aligned} 2(T \owedge S)(X,Y,Z,V) \doteq{} & T(Y,Z)S(X,V) - T(X,Z)S(Y,V) \\ & + S(Y,Z)T(X,V) - S(X,Z)T(Y,V) \end{aligned}$$

The $\owedge$-multiplication between symmetric type $(0,2)$ tensor fields is always a type $(0,4)$ tensor field with the "symmetries of a curvature".

In any pseudo-Riemannian manifold (M^n, g), we may $\owedge$-divide R by g:

$$R = \mathrm{g} \owedge P + W, \qquad W = \text{Weyl curvature tensor of } (M, \mathrm{g}).$$

Here are the main facts about W:

- W is the remainder of the $\owedge$-division of R by g.
- W is the "Ricci-traceless" part of R.
- W is the part of R not constrained by Einstein's field equations.
- R has $n^2(n^2-1)/12$ independent components, while Ric has $n(n+1)/2$: the remaining ones all come from W.
- $W = 0$ whenever $\dim M \leq 3$.
- If $\dim M \geq 4$, (M, g) is conformally flat if and only if $W = 0$.

The condition we are interested in is $\nabla W = 0$.

Definition (ECS manifold)

A pseudo-Riemannian manifold (M, g) is called *essentially conformally symmetric* if $\nabla W = 0$ but neither $W = 0$ nor $\nabla R = 0$.

The metric signature

ECS manifolds are objects of strictly indefinite nature:

Theorem (Roter, 1977)

For a Riemannian manifold (M, g)*:* $\nabla W = 0 \iff W = 0$ *or* $\nabla R = 0$.

Roter has also shown that ECS manifolds exist in all dimensions starting from 4, and realizing all possible indefinite metric signatures.

Every ECS manifold carries a distinguished null parallel distribution, which helps control its geometry:

Definition

The Olszak distribution of an ECS manifold (M, g) is $\mathcal{D} \hookrightarrow TM$ given by

$$\mathcal{D}_x = \{v \in T_xM \mid \mathrm{g}_x(v, \cdot) \wedge W_x(v', v'', \cdot, \cdot) = 0, \text{ for all } v', v'' \in T_xM\},$$

for every $x \in M$.

More on the Olszak distribution

The Olszak distribution was originally introduced for the more general study of conformally recurrent manifolds, and in this setting it is already true that $\mathcal{D}$ is indeed smooth, parallel and null.

In the ECS case, the rank of $\mathcal{D}$ is always equal to 1 or 2. For this reason, we speak of rank-one/rank-two ECS manifolds.

Theorem (Derdzinski-Roter, 2009)

Let (M, g) be an ECS manifold, and $\mathcal{D}$ be its Olszak distribution. Then:

- (i) *The Ricci endomorphism of (M, g) is $\mathcal{D}$-valued.*
- (ii) *The connection induced in the quotient bundle $\mathcal{D}^\perp/\mathcal{D}$ over M is flat.*
- (iii) *The connection induced in $\mathcal{D}$ itself is flat when (M, g) is of rank one.*

The local structure of ECS manifolds has been determined by Derdzinski and Roter in 2009.

A rank-one example

Example (Conformally symmetric pp-wave manifolds)

Let $(V, \langle\cdot,\cdot\rangle)$ be a pseudo-Euclidean vector space of dimension $n-2 \geq 2$, $A \in \mathfrak{sl}(V)$ be self-adjoint, $I \subseteq \mathbb{R}$ be an open interval and $f\colon I \to \mathbb{R}$ be a smooth function. Consider

$$(\widehat{M}, \widehat{\mathrm{g}}) = (I \times \mathbb{R} \times V, \kappa\,\mathrm{d}t^2 + \mathrm{d}t\,\mathrm{d}s + \langle\cdot,\cdot\rangle),$$

where $\kappa\colon \widehat{M} \to \mathbb{R}$ is given by $\kappa(t,s,v) = f(t)\langle v,v\rangle + \langle Av, v\rangle$.

Then $(\widehat{M}, \widehat{\mathrm{g}})$ has $\nabla W = 0$, with:

- $W = 0 \iff A = 0$;
- $\nabla R = 0 \iff f$ is constant.

In the ECS case, the Olszak distribution $\mathcal{D}$ is spanned by the null parallel coordinate vector field ∂_s, and $(V, \langle\cdot,\cdot\rangle)$ is isometrically identified with the vector space of parallel sections of $\mathcal{D}^\perp/\mathcal{D}$.

About compact ECS manifolds

With the local structure of ECS manifolds being fully understood, the next step is to address global aspects. The first question is whether compact ECS manifolds exist.

Theorem (Derdzinski-Roter, 2010)

In every dimension $n = 3j + 2$, $j = 1, 2, 3, \ldots$, there exists a compact Ricci-recurrent ECS manifold (M, g) of any prescribed indefinite metric signature, which is diffeomorphic to a torus bundle over $\mathbb{S}^1$, but not homeomorphic to (or even covered by) a torus.

These examples are all of the form $M = \widehat{M}/\Gamma$, where Γ is some subgroup of $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ acting freely and properly discontinuously on $\widehat{M}$.

The strange dimensions $n = 3j + 2$ were a particularity of their construction, which obtained a 5-dimensional example with $\dim V = 3$, but turned out to be "compatible" with taking cartesian powers of $(V, \langle\cdot,\cdot\rangle)$, leading also to dimensions 8, 11, 14, etc..

The isometry group of $(\widehat{M}, \widehat{\mathrm{g}})$

Again: $(V, \langle\cdot,\cdot\rangle)$ has $\dim V = n-2$, $A \in \mathfrak{sl}(V) \smallsetminus \{0\}$ is self-adjoint, f is nonconstant on an open interval $I \subseteq \mathbb{R}$, and our "rank-one ECS model" is $(\widehat{M}, \widehat{\mathrm{g}}) = (I \times \mathbb{R} \times V, \kappa\, \mathrm{d}t^2 + \mathrm{d}t\, \mathrm{d}s + \langle\cdot,\cdot\rangle)$.

1. S is the group of the triples $\sigma = (q, p, C) \in \mathrm{Aff}(\mathbb{R}) \times \mathrm{O}(V)$ with $CAC^{-1} = q^2 A$ and $q^2 f(qt+p) = f(t)$.
2. $(\mathcal{E}, \Omega)$ is the symplectic vector space of solutions $u\colon I \to V$ of $\ddot{u}(t) = f(t)u(t) + Au(t)$, with $\Omega(u, \hat{u}) = \langle \dot{u}, \hat{u}\rangle - \langle u, \dot{\hat{u}}\rangle$.

Note: $\mathrm{S} \circlearrowright \mathcal{E}, I, \mathbb{R}$ via $(\sigma u)(t) = Cu(q^{-1}(t-p))$, $\sigma t = qt + p$, $\sigma s = q^{-1}s$.

3. The Heisenberg group $\mathrm{H} = \mathbb{R} \times \mathcal{E}$ associated with $(\mathcal{E}, \Omega)$, with operation given by $(r, u)(\widehat{r}, \widehat{u}) = (r + \widehat{r} - \Omega(u, \widehat{u}), u + \widehat{u})$.

Theorem

$\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ *is isomorphic to a semidirect product* $\mathrm{S} \ltimes \mathrm{H}$.

- $(\sigma, r, u)(\widehat{\sigma}, \widehat{r}, \widehat{u}) = (\sigma\widehat{\sigma}, r + q^{-1}\widehat{r} - \Omega(u, \sigma\widehat{u}), u + \sigma\widehat{u})$
- $(\sigma, r, u)(t, s, v) = (\sigma t, -\langle \dot{u}(\sigma t), 2\sigma v + u(\sigma t)\rangle + q^{-1}s + r, \sigma v + u(\sigma t))$

The groups $\mathrm{G}(\sigma)$

S: group of all $\sigma = (q, p, C) \in \mathrm{Aff}(\mathbb{R}) \times \mathrm{O}(V, \langle\cdot,\cdot\rangle)$ respecting f and A.

As we have seen, the group $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}}) = \mathrm{S} \ltimes \mathrm{H}$ can be difficult to deal with.

We restrict our search for compact-quotient subgroups Γ of $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ to specific groups $\mathrm{G}(\sigma)$, with $\sigma \in \mathrm{S}$.

More precisely: $\mathrm{G}(\sigma) = \{(\sigma^k, r, u) \mid k \in \mathbb{Z} \text{ and } (r, u) \in \mathrm{H}\} \cong \mathbb{Z} \ltimes \mathrm{H}$.

The formulas for the group operation in $\mathrm{G}(\sigma)$ and its action on $\widehat{M}$ become simplified versions of what we had in the previous page.

The element $\sigma \in \mathrm{S}$ is always chosen according to two situations:

1. *translational*: $I = \mathbb{R}$ and $\sigma = (1, p, C)$ for some "period" $p > 0$.
2. *dilational*: $I = (0, \infty)$ and $\sigma = (q, 0, C)$ for some $q \in (0, \infty) \smallsetminus \{1\}$.

(In both cases, $C \in \mathrm{O}(V, \langle\cdot,\cdot\rangle)$.)

The translational-dilational dichotomy

The reason for the names "translational" and "dilational" goes beyond the meaning suggested by the actions of the elements $(1, p), (q, 0) \in \mathrm{Aff}(\mathbb{R})$.

In general, we say that an abstract ECS manifold (M, g) is translational or dilational according to whether the holonomy group of the natural flat connection induced in $\mathcal{D}$ is finite or infinite.

If $(\widetilde{M}, \widetilde{\mathrm{g}})$ is the universal covering of (M, g), with $M = \widetilde{M}/\Gamma$ for some $\Gamma \cong \pi_1(M)$, and $t \colon \widetilde{M} \to \mathbb{R}$ is a function whose (parallel) gradient spans $\widetilde{\mathcal{D}}$, then for every $\gamma \in \Gamma$ there is $(q, p) \in \mathrm{Aff}(\mathbb{R})$ such that $t \circ \gamma = qt + p$.

This gives us two homomorphisms

$$\Gamma \ni \gamma \mapsto (q, p) \in \mathrm{Aff}(\mathbb{R}) \quad \text{and} \quad \Gamma \ni \gamma \mapsto q \in \mathbb{R} \smallsetminus \{0\},$$

and it turns out that the holonomy group of the connection induced in $\mathcal{D}$ equals the image of the second homomorphism.

First-order subspaces

Recall: any rank-one ECS model $(\widehat{M}, \widehat{\mathrm{g}})$ gives rise to the symplectic vector space $(\mathcal{E}, \Omega)$ of solutions $u\colon I \to V$ of the ODE $\ddot{u}(t) = f(t)u(t) + Au(t)$.

For each $t \in I$, we have the corresponding evaluation mapping $\delta_t\colon \mathcal{E} \to V$, given by $\delta_t(u) = u(t)$. (They're obviously surjective.)

Definition

A vector subspace $\mathcal{L} \subseteq \mathcal{E}$ is called a first-order subspace of $(\mathcal{E}, \Omega)$ if, for every $t \in I$, the restriction $\delta_t|_{\mathcal{L}}\colon \mathcal{L} \to V$ is an isomorphism.

First-order subspaces of $(\mathcal{E}, \Omega)$ are in one-to-one correspondence with curves $B\colon I \to \mathrm{End}(V)$ satisfying $\dot{B} + B^2 = f + A$, via

$$\mathcal{L} = \{u \in \mathcal{E} \mid \dot{u}(t) = B(t)u(t) \text{ for all } t \in I\}.$$

Here:

1. $\mathcal{L}$ is Lagrangian if and only if each $B(t)$ is self-adjoint.
2. $\mathcal{L}$ is σ-invariant if and only if $B(\sigma t) = q^{-1}CB(t)C^{-1}$.

A criterion for the $\exists$ of cpct.-quot. subgps. of $\mathrm{G}(\sigma)$

Theorem

For a rank-one ECS model manifold $(\widehat{M}, \widehat{\mathrm{g}})$, and an isometry $\gamma = (\sigma, b, w)$ with $\sigma \in \mathrm{S}$ chosen as before, the following conditions are equivalent:

- (a) *There is a discrete subgroup Γ of $\mathrm{G}(\sigma)$ acting freely and properly discontinuously on $\widehat{M}$ with a compact quotient $M = \widehat{M}/\Gamma$.*
- (b) *There is a σ-invariant first-order subspace $\mathcal{L}$ of $(\mathcal{E}, \Omega)$, a lattice $\Sigma \subseteq \mathbb{R} \times \mathcal{L}$ with $\mathrm{C}_\gamma[\Sigma] = \Sigma$, and $\theta \geq 0$ such that $\Sigma \cap (\mathbb{R} \times \{0\}) = \mathbb{Z}\theta \times \{0\}$ and $\Omega(u, \hat{u}) \in \mathbb{Z}\theta$ for all $u, \hat{u} \in \Lambda$, where Λ is the image of Σ under the projection $\mathbb{R} \times \mathcal{L} \to \mathcal{L}$.*

If (b) holds, Γ in (a) can be taken to be the group generated by γ and Σ and there is a locally trivial fibration $M \to \mathbb{S}^1$ whose fibers, all diffeomorphic to a torus or to a 2-step nilmanifold according to whether $\mathcal{L}$ is Lagrangian or not, are the leaves of $\mathcal{D}^\perp$. Finally, M equipped with its natural quotient metric is translational and complete, or dilational and incomplete, according to whether $\sigma = (1, p, C)$ or $\sigma = (q, 0, C)$.

Very brief sketch of (b) implies (a)

First, we show that the quotient $N = (\mathbb{R} \times \mathcal{L})/\Sigma$ is compact, where the lattice Σ acts on $\mathbb{R} \times \mathcal{L}$ by *Heisenberg* left-translations.

Then, if ε is 0 or 1 (depending on whether σ is translational or dilational), we let $\widetilde{w} \in \mathcal{L}$ be the unique element with $\widetilde{w}(\sigma\varepsilon) = w(\sigma\varepsilon)$, and let $\widetilde{b}$ be given by $\widetilde{b} = b - \langle \dot{w}(\sigma\varepsilon) - B(\sigma\varepsilon)w(\sigma\varepsilon), w(\sigma\varepsilon)\rangle$.

Then $\phi\colon \mathbb{R} \times \mathcal{L} \to \mathbb{R} \times \mathcal{L}$ given by

$$\phi(r, u) = \left(q^{-1}r + \widetilde{b} - \Omega(\widetilde{w} - 2w, \sigma u), \sigma u + \widetilde{w}\right)$$

is Σ-equivariant, and hence passes to the quotient $\Phi\colon N \to N$.

Finally, we set $M = (I \times N)/\mathbb{Z}$, where $k \cdot (t, \Sigma(r, u)) = (\sigma^k t, \Phi^k \Sigma(r, u))$.

This works.

Theorem (Derdzinski-T., 2022)

There exist compact rank-one translational ECS manifolds of all dimensions $n \geq 5$ and all indefinite metric signatures, forming the total space of a nontrivial torus bundle over $\mathbb{S}^1$ with its fibers being the leaves of $\mathcal{D}^\perp$, all geodesically complete, and none locally homogeneous. In each fixed dimension and metric signature, there is an infinite-dimensional moduli space of local-isometry types.

Theorem (Derdzinski-T., 2023)

There exist compact rank-one dilational ECS manifolds of all odd dimensions $n \geq 5$ and with semi-neutral metric signature, including locally homogeneous ones, forming the total space of a nontrivial torus bundle over $\mathbb{S}^1$ with its fibers being the leaves of $\mathcal{D}^\perp$, all of them geodesically incomplete. In each fixed odd dimension, there is an infinite-dimensional moduli space of local-isometry types.

The dilational construction

Fix $\sigma = (q, 0, C)$, with $q \in (0, \infty) \smallsetminus \{1\}$ and C to be chosen later.

Based on the criterion for the existence of cocompact subgroups of $\mathrm{G}(\sigma)$, with $\theta = 0$, our goal is to find: a first-order σ-invariant Lagrangian subspace $\mathcal{L}$ of $(\mathcal{E}, \Omega)$ and a conjugation-invariant lattice $\Sigma \subseteq \mathbb{R} \times \mathcal{L}$.

At the same time, we must find a smooth function $f\colon (0, \infty) \to \mathbb{R}$ and a self-adjoint $A \in \mathfrak{sl}(V) \smallsetminus \{0\}$ with the correct spectral properties to be used as model data.

Obtaining such f and A, in this case, is simple, and it is the existence of $\mathcal{L}$ and Σ which pose a challenge. It ultimately relies on the combinatorial structure we will discuss next.

$\mathbb{Z}$-spectral systems

Definition

A *$\mathbb{Z}$-spectral system* is a quadruple (m, k, E, J) consisting of two integers $m, k \geq 2$, an injective function $E\colon \mathcal{V} \to \mathbb{Z} \smallsetminus \{-1\}$, where $\mathcal{V} = \{1, \ldots, 2m\}$, and a function $J\colon \mathcal{V} \to \{0, 1\}$, satisfying for every $i, i' \in \mathcal{V}$ that:

- (a) $k + 1 = 2E(1)$ (and so k must be odd).
- (b) $E(i) + E(i') = -1$ and $J(i) + J(i') = 1$ whenever $i + i' = 2m + 1$.
- (c) $E(i) - E(i') = k$ and $J(i) + J(i') = 1$ whenever $i' = i + 1$ is even.
- (d) The set $Y = \{-1\} \cup \{E(i) \mid i \in \mathcal{V} \text{ and } J(i) = 1\}$ is symmetric about zero.

The spectral selector $S = J^{-1}(1)$ is simultaneously a selector for both two-element subset families

$$\{\{i, i'\} \mid i + i' = 2m + 1\} \quad \text{and} \quad \{\{i, i'\} \mid i' = i + 1 \text{ is even}\}.$$

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a. $k + 1 = 2E(1)$ (and so k must be odd).
b. $E(i) + E(i') = -1$ and $J(i) + J(i') = 1$ whenever $i + i' = 2m + 1$.
c. $E(i) - E(i') = k$ and $J(i) + J(i') = 1$ whenever $i' = i + 1$ is even.
d. The set $Y = \{-1\} \cup \{E(i) \mid i \in \mathcal{V} \text{ and } J(i) = 1\}$ is symmetric about zero.

The reason we care about this is that for any $\mathbb{Z}$-spectral system (m, k, E, J) and $q \in (0, \infty) \setminus \{1\}$ such that $q + q^{-1} \in \mathbb{Z}$, the $(m+1)$-element set $\{q^a \mid a \in Y\}$ is the spectrum of some matrix in $\mathrm{GL}(m+1, \mathbb{Z})$.

“Odd-dimensional” systems...

Example

For every odd integer $m \geq 3$, there is a $\mathbb{Z}$-spectral system $(m, m+2, E, J)$. Writing $m = 2r-3$ with $r \geq 3$, and $(i, i') = (2j-1, 2j)$ whenever $i, i' \in \mathcal{V}$ and $i' = i+1$ is even, we define the function E by

$$(E(2j-1), E(2j)) = \begin{cases} (r, -r+1) & \text{if } j = 1, \\ (j-1, -2r+j) & \text{if } 1 < j < r-1 \text{ and } r \text{ is even}, \\ (2r+j-2, j-1) & \text{if } 1 < j < r-1 \text{ and } r \text{ is odd}, \\ (r-1, -r) & \text{if } j = r-1, \\ (j-2r+2, j-4r+3) & \text{if } r-1 < j < m \text{ and } r \text{ is odd}, \\ (j+1, j-2r+2) & \text{if } r-1 < j < m \text{ and } r \text{ is even}, \\ (r-2, -r-1) & \text{if } j = m, \end{cases}$$

and let the function J be given by $J(i) = E(i) \bmod 2$, so that

$$Y = \{-1\} \cup (\mathbb{Z}_{\mathrm{odd}} \cap E[\mathcal{V}]), \quad \text{where } \mathbb{Z}_{\mathrm{odd}} = \mathbb{Z} \setminus 2\mathbb{Z}.$$

... and no "even-dimensional" ones.

Proposition

There are no $\mathbb{Z}$-spectral systems (m, k, E, J) with even m.

Proof idea: Let (m, k, E, J) be a $\mathbb{Z}$-spectral system with even m, written as $m = 2s$ for some $s \in \mathbb{Z}$. The "exponent vector" $\mathbf{E} \in \mathbb{Z}^{4s}$ has the form

$$\mathbf{E} = (a_1, a_1 - k, \ldots, a_s, a_s - k, -1 - a_s + k, -1 - a_s, \ldots, -1 - a_1 + k, -1 - a_1)$$

for some $a_1, \ldots, a_s \in \mathbb{Z}$. Now, let ε_j be 1 or -1 according to whether $\{2j-1, 2m-2j+1\} \subseteq S$ or $\{2j, 2m-2j+2\} \subseteq S$. As the set $Y = \{-1\} \cup E[S]$ is symmetric about zero,

$$1 = \sum_{i \in S} E(i) = \sum_{j=1}^{s} (-1 + \varepsilon_j k),$$

and so $\left(\sum_{j=1}^{s} \varepsilon_j\right) k = s + 1$. For ℓ negative ε_j's, we obtain the relation $(s - 2\ell)k = s + 1$. Both sides have different parities, a contradiction.

Defining dilational ECS data

Let $n \geq 5$ be odd and set $m = n - 2$.

Fix a $\mathbb{Z}$-spectral system (m, n, E, J), a m-dimensional pseudo-Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$ with semi-neutral signature, and $q \in (0, \infty) \smallsetminus \{1\}$ with $q + q^{-1} \in \mathbb{Z}$.

Defining A and C: let $(e_1, \ldots, e_m)$ be a basis of $(V, \langle\cdot,\cdot\rangle)$ on which

$$\langle\cdot,\cdot\rangle \sim \begin{bmatrix} 0 & 0 & \ldots & 0 & \varepsilon \\ 0 & 0 & \ldots & \varepsilon & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \varepsilon & \ldots & 0 & 0 \\ \varepsilon & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad \varepsilon \in \{1, -1\},$$

and define

$$a(j) = E(2j-1) + \frac{1-n}{2} = E(2j) + \frac{1+n}{2}, \quad j = 1, \ldots, m.$$

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and define $a(j) = E(2j-1) + \dfrac{1-n}{2} = E(2j) + \dfrac{1+n}{2}, \quad j = 1, \ldots, m.$
Now set:

- $Ae_m = e_1$, and $Ae_j = 0$ for $j = 1, \ldots, m-1$.
- $Ce_j = q^{a(j)} e_j$ for $j = 1, \ldots, m$.

Then $A \in \mathfrak{sl}(V) \setminus \{0\}$ is self-adjoint, $C \in \mathrm{O}(V, \langle \cdot, \cdot \rangle)$, and $CAC^{-1} = q^2 A$.

Defining dilational ECS data

The function f: here, we consider the "scalar version" of $(\mathcal{E}, \Omega)$, that is, the space $\mathcal{W}$ of solutions $y\colon (0,\infty) \to \mathbb{R}$ of $\ddot{y}(t) = f(t)y(t)$.

The operator $T\colon \mathcal{W} \to \mathcal{W}$ given by $(Ty)(t) = y(t/q)$ is indeed $\mathcal{W}$-valued whenever f has the property $q^2 f(qt) = f(t)$.

Its spectrum μ^+, μ^- satisfies $\mu^+\mu^- = q^{-1}$, as $T^*\alpha = \alpha$ for the (symplectic) area form $\alpha(y,z) = \dot{y}(t)z(t) - y(t)\dot{z}(t)$.

The spectrum of $\sigma\colon \mathcal{E} \to \mathcal{E}$ then becomes

$$(\mu^+ q^{a(1)}, \mu^- q^{a(1)}, \ldots, \mu^+ q^{a(m)}, \mu^- q^{a(m)}). \qquad (*)$$

Choosing f so that $\mu^+ = q^{(-1-n)/2}$ and $\mu^- = q^{(-1+n)/2}$, such as

$$f(t) = \frac{n^2-1}{4t^2},$$

the spectrum $(*)$ becomes precisely

$$(q^{E(1)}, q^{E(2)}, \ldots, q^{E(2m-1)}, q^{E(m)}).$$

Defining dilational ECS data

So far: f, A, and $\sigma = (q, 0, C)$ are in place, and the spectrum of $\sigma\colon \mathcal{E} \to \mathcal{E}$ is $(q^{E(1)}, q^{E(2)}, \ldots, q^{E(2m-1)}, q^{E(m)})$, for our spectral system $(n-2, n, E, J)$.

The space $\mathcal{L}$: using more linear algebra, we obtain a basis

$$(u_1, u_2, \ldots, u_{2m-1}, u_{2m}) = (u_1^+, u_1^-, \ldots, u_m^+, u_m^-)$$

of $\mathcal{E}$, of eigenvectors of σ associated with $(q^{E(1)}, q^{E(2)}, \ldots, q^{E(2m-1)}, q^{E(m)})$.

This basis satisfies that $\Omega(u_i, u_j) = 0$, whenever $i, j \in \{1, \ldots, 2m\}$ have $i + j \neq 2m + 1$. Hence, if $S = J^{-1}(1)$ is the spectral selector of the $\mathbb{Z}$-spectral system $(n-2, n, E, J)$,

the direct sum $\mathcal{L} = \bigoplus_{i \in S} \mathbb{R}u_i$ is a first-order σ-invariant Lagrangian subspace of $(\mathcal{E}, \Omega)$.

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Now, σ-invariance of $\mathcal{L}$ makes $\mathbb{R} \times \mathcal{L}$ C_γ-invariant for any $\gamma \in \mathrm{G}(\sigma)$.

The spectrum of the restriction $\mathrm{C}_\gamma|_{\mathbb{R}\times\mathcal{L}}$ is given by $\{q^a \mid a \in Y\}$, for $Y = \{-1\} \cup E[S]$ arising from $(n-2, n, E, J)$.

This means that a C_γ-invariant lattice $\Sigma \subseteq \mathbb{R} \times \mathcal{L}$ exists.

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Thank you for your attention!

(scan here for more
on my research)