# CONNECTIONS ON ASSOCIATED VECTOR BUNDLES

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Quick summary on $G$-torsors, associated vector bundles and associated connections. Discussion on direct definition of $\nabla^A$ in terms of local gauges; independence of local gauge. Lastly, $R^{\nabla^A}$ versus $F_A$.

## 1 $G$-torsors

Let's quickly introduce the language of $G$-torsors, as it will be useful for the discussion later.

**Definition 1**

A set $X$ is called a (left) $G$-torsor if it is equipped with a free and transitive action $G \circlearrowright X$. Equivalently, it is a $G$-set for which the enriched map

$$G \times X \ni (x, g) \mapsto (x, g \cdot x) \in X \times X$$

is an isomorphism.

**Remark.** Right $G$-torsors are defined on a similar way and the theory is unchanged, as a left $G$-torsor can be changed into a right $G$-torsor, and vice-versa. Affine spaces with translation vector space $V$ are nothing more than right $V$-torsors.

Given $x', x'' \in X$, there is a unique element $g \in G$ such that $x'' = g \cdot x'$. We will denote it by $x''/x'$. So, on a $G$-torsor, one can "multiply" elements of $G$ by elements of $X$, but one cannot multiply two elements of $X$. What one can do, instead, is to "divide" elements of $X$ to obtain elements in $G$. The division notation is precise, because all algebraic manipulations you think should hold, will hold.

**Proposition 1**

Let $X$ be a $G$-torsor, $x', x'', x''' \in X$ and $g \in G$. Then:

(a) $\dfrac{x'''}{x''} \cdot \dfrac{x''}{x'} = \dfrac{x'''}{x'}$.

(b) $\dfrac{x'}{x'} = e$.

(c) $\left(\frac{x''}{x'}\right)^{-1} = \frac{x'}{x''}$.

(d) $\frac{g \cdot x''}{x'} = g \cdot \frac{x''}{x'}$.

(e) $\frac{x''}{g \cdot x'} = \frac{x''}{x'} \cdot g^{-1}$.

**Proof:**

(a) Compute

$$\left(\frac{x'''}{x''} \cdot \frac{x''}{x'}\right) \cdot x' = \frac{x'''}{x''} \cdot \left(\frac{x''}{x'} \cdot x'\right) = \frac{x'''}{x''} \cdot x'' = x'''.$$

(b) Obvious.

(c) Compute

$$\left(\frac{x''}{x'}\right)^{-1} \cdot x'' = \left(\frac{x''}{x'}\right)^{-1} \cdot \left(\frac{x''}{x'} \cdot x'\right) = \left(\left(\frac{x''}{x'}\right)^{-1} \cdot \left(\frac{x''}{x'}\right)\right) \cdot x' = e \cdot x' = x'.$$

(d) Compute

$$\left(g \cdot \frac{x''}{x'}\right) \cdot x' = g \cdot \left(\frac{x''}{x'} \cdot x'\right) = g \cdot x''.$$

(e) Use the previous items:

$$\frac{x''}{g \cdot x'} = \left(\frac{g \cdot x'}{x''}\right)^{-1} = \left(g \cdot \frac{x'}{x''}\right)^{-1} = \frac{x''}{x'} \cdot g^{-1}.$$

□

## 2 Review

We work on the smooth category. Let $\pi\colon P \to M$ be a principal $G$-bundle, and $\rho\colon G \to \mathrm{GL}(V)$ be a representation of the Lie group $G$ on a vector space $V$. Given $g \in G$ and $v \in V$, we'll write $gv$ for $\rho(g)v$. Then we have a right action $(P \times V) \circlearrowleft G$ given by $(p, v) \cdot g = (p \cdot g, g^{-1}v)$. The quotient $E(P, \rho) = P \times_\rho V \doteq (P \times V)/G$ (written as $E$, for short) turns out to be a manifold. Elements of $E$ are equivalence classes $[p, v]$, subject to the rule $[p \cdot g, v] = [p, gv]$. Since the action $P \circlearrowleft G$ is fiber-preserving, the projection $P \times V \ni (p, v) \mapsto \pi(p) \in M$ induces a projection $\pi_E\colon E \to M$. The fibers $\pi_E^{-1}(x)$ will be vector spaces, all isomorphic to $V$, by using that each $P_x$ is a $G$-torsor. Namely, we write

$$[p', v'] + [p'', v''] \doteq \left[p', v' + \frac{p''}{p'}v''\right] \quad \text{and} \quad \lambda[p, v] \doteq [p, \lambda v].$$

The scalar multiplication doesn't require explanation, but for the sum, the idea is that one cannot add $(p', v')$ and $(p'', v'')$ if $p' \neq p''$. In the quotient, if $p', p'' \in P_x$, we may write that

$$[p'', v''] = \left[p' \frac{p''}{p'}, v''\right] = \left[p', \frac{p''}{p'} v''\right],$$

and this last expression is admissible to add with $[p', v']$.

**Proposition 2**

The above operations are well-defined.

**Proof:** Let's verify that the sum is well-defined first. Replace $p'$ and $v'$ with $p' \cdot g'$ and $(g')^{-1}v'$, and similarly for $p''$ and $v''$. Then compute

$$\begin{aligned}\left[p' \cdot g', (g')^{-1}v' + \frac{p'' \cdot g''}{p' \cdot g'}(g'')^{-1}v''\right] &= \left[p' \cdot g', (g')^{-1}v' + (g')^{-1}\frac{p''}{p'}v''\right] \\ &= \left[p' \cdot g', (g')^{-1}\left(v' + \frac{p''}{p'}v''\right)\right] \\ &= \left[p', v' + \frac{p''}{p'}v''\right].\end{aligned}$$

For the scalar multiplication, take $[p, v] \in \pi_E^{-1}(x)$, $\lambda \in \mathbb{R}$, replace $p$ with $p \cdot g$, $v$ with $g^{-1}v$, and compute

$$[p \cdot g, \lambda g^{-1}v] = [p \cdot g, g^{-1}(\lambda v)] = [p, \lambda v],$$

since the representation $\rho$ takes values in $\mathrm{GL}(V)$. ☐

To get trivializations for $E$ in terms of trivializations for $P$, one proceeds as follows: let $(U, \Phi)$ be a principal $G$-chart, where $U \subseteq M$ is open. So $\Phi\colon \pi^{-1}[U] \to U \times G$ has the form $\Phi(p) = (\pi(p), \Phi_G(p))$, with $G$-equivariant $\Phi_G\colon \pi^{-1}[U] \to G$. We set $\Phi^E\colon \pi_E^{-1}[U] \to U \times V$ via $\Phi^E[p, v] = (\pi(p), \Phi_G(p)v)$.

**Proposition 3**

$\Phi^E$ is a well-defined VB-chart for $E$ with inverse $(\Phi^E)^{-1}\colon U \times V \to \pi_E^{-1}[U]$ given by $(\Phi^E)^{-1}(x, v) = [\Phi^{-1}(x, e), v]$.

**Proof:** First, take $[p, v] \in \pi_E^{-1}[U]$, and replace $p$ and $v$ with $p \cdot g$ and $g^{-1}v$. Now

$$(\pi(p \cdot g), \Phi_G(p \cdot g)g^{-1}v) = (\pi(p), \Phi_G(p)gg^{-1}v) = (\pi(p), \Phi_G(p)v),$$

as required, since $\Phi_G$ is $G$-equivariant. Next, we have that

$$\Phi^E[\Phi^{-1}(x, e), v] = (\pi\Phi^{-1}(x, e), \Phi_G(\Phi^{-1}(x, e))v) = (x, ev) = (x, v),$$

as well as

$$[\Phi^{-1}(\pi(p), e), \Phi_G(p)v] = [\Phi^{-1}(\pi(p), e)\Phi_G(p), v] = [\Phi^{-1}(\pi(p), \Phi_G(p)), v] = [p, v].$$

So, what we claim to be $(\Phi^E)^{-1}$, indeed is. Finally, let's show that restrictions of $\Phi^E$ to fibers of $\pi_E$ are linear isomorphisms.

- Linearity.

$$\begin{aligned}\Phi^E([p',v']+[p'',v'']) &= \Phi^E\left[p',v'+\frac{p''}{p'}v''\right] \\ &= \left(\pi(p'),\Phi_G(p')\left(v'+\frac{p''}{p'}v''\right)\right) \\ &= \left(\pi(p'),\Phi_G(p')v'+\Phi_G(p')\frac{p''}{p'}v''\right) \\ &= \left(\pi(p'),\Phi_G(p')v'+\Phi_G\left(p'\frac{p''}{p'}\right)v''\right) \\ &= (\pi(p'),\Phi_G(p')v'+\Phi_G(p'')v'') \\ &= (\pi(p'),\Phi_G(p')v')+(\pi(p''),\Phi_G(p'')v'') \\ &= \Phi^E[p',v']+\Phi^E[p'',v''],\end{aligned}$$

using that $\Phi_G$ is $G$-equivariant and $\pi(p')=\pi(p'')$. Also:

$$\begin{aligned}\Phi^E(\lambda[p,v]) = \Phi^E[p,\lambda v] &= (\pi(p),\Phi_G(p)\lambda v) \\ &= (\pi(p),\lambda\Phi_G(p)v) = \lambda(\pi(p),\Phi_G(p)v),\end{aligned}$$

using that $\rho$ takes values in $\mathrm{GL}(V)$ and that the vector space structure on the fiber $\{\pi(p)\}\times V$ is the obvious one, happening only on the $V$-factor.

- Injectivity. Assume that $\Phi^E[p,v]=(\pi(p),0)$. This means that $\Phi_G(p)v=0$. But $v\mapsto\Phi_G(p)v$ is an isomorphism (because $\rho$ takes values in $\mathrm{GL}(V)$, so this immediately gives that $v=0$.

- Surjectivity. Assume given $(\pi(p),v)\in\{\pi(p)\}\times V$. Then we clearly have that $\Phi^E[p,\Phi_G(p)^{-1}v]=(\pi(p),\Phi_G(p)\Phi_G(p)^{-1}v)=(\pi(p),v)$.

□

Proceeding, to understand local sections $s$ of $E$ properly, we'll use local gauges for $P$. Namely, on some open set $U\subseteq M$, fix a local gauge $\psi\colon U\to P$. Writing the section $s$ as $s(x)=[\psi(x),s_\psi(x)]$ (this can be arranged for since $s(x)\in E_x$ and $\psi(x)\in P_x$), we obtain a bijective correspondence between local sections $s\colon U\to E$ and functions $s_\psi\colon U\to V$ — to be regarded as matter fields. The gauge group $\mathscr{G}(P)$ acts not only on $P$ by evaluation, but also on $E$. We set $\Phi\cdot[p,v]=[\Phi(p),v]$.

**Proposition 4**

The action $\mathscr{G}(P)\circlearrowright E$ is well-defined.

**Proof:** Replace $p$ and $v$ with $p\cdot g$ and $g^{-1}v$. Then

$$[\Phi(p\cdot g),g^{-1}v]=[\Phi(p)\cdot g,g^{-1}v]=[\Phi(p),v],$$

since $\Phi\in\mathscr{G}(P)$ is $G$-equivariant. □

With this in place, $\mathscr{G}(P)$ acts on local sections of $E$ as well, pointwise. To express how this happens relative to a local gauge, recall that $\mathscr{G}(P) \cong \mathscr{C}^\infty(P,G)^G$ as follows: since $\Phi(p)$ and $p$ are on the same fiber, there is $\sigma_\Phi(p) \in G$ such that $\Phi(p) = p \cdot \sigma_\Phi(p)$. Moreover, the $G$-equivariance relation, $\Phi(p \cdot g) = \Phi(p) \cdot g$, now implies that we have $(p \cdot g) \cdot \sigma_\Phi(p \cdot g) = (p \cdot \sigma_\Phi(p)) \cdot g$, so $\sigma_\Phi(p \cdot g) = g^{-1}\sigma_\Phi(p)g$. The correspondence is $\Phi \leftrightarrow \sigma_\Phi$.

**Proposition 5**

Let $s$ be a local section of $E$, $\Phi \in \mathscr{G}(P)$, and $\psi$ be a local gauge for $P$. Then we have

$$(\Phi \cdot s)(x) = [\psi(x), \sigma_\Phi(\psi(x))s_\psi(x)].$$

**Proof:** Directly compute

$$\begin{aligned}(\Phi \cdot s)(x) &= \Phi[\psi(x), s_\psi(x)] = [\Phi(\psi(x)), s_\psi(x)] \\ &= [\psi(x) \cdot \sigma_\Phi(\psi(x)), s_\psi(x)] = [\psi(x), \sigma_\Phi(\psi(x))s_\psi(x)].\end{aligned}$$

□

## 3 Differential forms

Recall that if $Q$ is a smooth manifold and we have an action $G \circlearrowright Q$ which is free and proper (so $Q/G$ is a smooth manifold), then $\Omega^k(Q/G) \cong \Omega^k_{\rm hor}(Q)^G$, where $\Omega^k_{\rm hor}(Q)^G$ consists of all the $G$-invariant horizontal $k$-forms. Here, horizontal means that the differential form produces zero whenever one of its arguments in the kernel of the derivative of the quotient projection $Q \to Q/G$. In our setting, similar arguments work, considering $V$-valued forms instead. Since the principal $G$-bundle $P \to M$ is such that $P/G \cong M$, we have that $\Omega^k_{\rm hor}(P,V)^\rho \cong \Omega^k(M,E)$. Suggestively, each $\omega \in \Omega^k_{\rm hor}(P,V)^\rho$ satisfies $\mathrm{R}_g^*\omega = \rho(g^{-1}) \circ \omega$ and $\omega$ produces zero whenever one of its arguments is horizontal (relative to the fixed $A \in \Omega^1(P,\mathfrak{g})$.

## 4 Connections

Assume that the principal bundle $\pi\colon P \to M$ is equipped with an Ehresmann connection. That is, a 1-form $A \in \Omega^1(P,\mathfrak{g})$ such that $A(X^\#) = X$ for all $X \in \mathfrak{g}$ (where $X^\# \in \mathfrak{X}(P)$ stands for the action field generated by $X$) and $\mathrm{R}_g^*A = \mathrm{Ad}(g^{-1}) \circ A$, for all $g \in G$, where $\mathrm{R}_g\colon P \to P$ is the right action of the element $g$. Choosing such $A$ is equivalent to choosing a horizontal distribution $\mathscr{H} \hookrightarrow TP$ with $TP = \mathscr{H} \oplus \mathscr{V}$, where $\mathscr{V}_p = \ker \mathrm{d}\pi_p$ is the natural vertical distribution of the bundle, and $\mathrm{d}(\mathrm{R}_g)_p[\mathscr{H}_p] = \mathscr{H}_{p\cdot g}$. The correspondence is $A \leftrightarrow \ker A$. The restriction of $\mathrm{d}\pi_p$ gives an isomorphism $\mathscr{H}_p \cong T_{\pi(p)}M$.

**Proposition 6**

Given $\gamma\colon [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$, for each $p \in P_x$ there is a unique horizontal lift $\gamma_p^{\mathsf{h}}\colon [0,1] \to P$ with $\gamma_p^{\mathsf{h}}(0) = p$.

**Proof:** Since $P \to M$ is a bundle and $[0,1]$ is contractible, there is a lift $\widetilde{\gamma}\colon [0,1] \to P$ of $\gamma$, which is not, in general, horizontal. So, we must correct it. Let's solve a differential equation for $g\colon [0,1] \to G$ making $\alpha(t) \doteq \widetilde{\gamma}(t) \cdot g(t)$ horizontal. We have that

$$\dot{\alpha}(t) = \mathrm{d}(\mathrm{R}_{g(t)})_{\widetilde{\gamma}(t)}(\dot{\widetilde{\gamma}}(t)) + \mathrm{d}(\mathbb{O}_{\widetilde{\gamma}(t)})_{g(t)}(\dot{g}(t))$$

by the chain rule, but the second term can be simplified, by using the general relation $\mathbb{O}_{p\cdot g} = \mathbb{O}_p \circ \mathrm{L}_g$:

$$\dot{\alpha}(t) = \mathrm{d}(\mathrm{R}_{g(t)})_{\widetilde{\gamma}(t)}(\dot{\widetilde{\gamma}}(t)) + \mathrm{d}(\mathbb{O}_{\alpha(t)})_e(g(t)^{-1}\dot{g}(t)).$$

This reads

$$\dot{\alpha}(t) = \mathrm{d}(\mathrm{R}_{g(t)})_{\widetilde{\gamma}(t)}(\dot{\widetilde{\gamma}}(t)) + (g(t)^{-1}\dot{g}(t))^{\#}_{\alpha(t)}.$$

Apply $A$ to obtain

$$0 = \mathrm{Ad}(g(t)^{-1})A_{\widetilde{\gamma}(t)}(\dot{\widetilde{\gamma}}(t)) + g(t)^{-1}\dot{g}(t).$$

So, simplifying Ad, we consider the initial value problem for $g$:

$$\begin{cases} \dot{g}(t) = -A_{\widetilde{\gamma}(t)}(\dot{\widetilde{\gamma}}(t))g(t) \\ g(0) = e \end{cases}$$

This system has a unique solution defined for all $t \in [0,1]$. □

With this, we define $\Pi^A_\gamma\colon P_x \to P_y$ by $\Pi^A_\gamma(p) = \gamma_p^{\mathsf{h}}(1)$. This is called the parallel transport operator along $\gamma$, induced by $A$.

**Proposition 7**

(a) $\Pi^A_\gamma\colon P_x \to P_y$ is $G$-equivariant.

(b) $\Pi^A_{\gamma*\eta} = \Pi^A_\eta \circ \Pi^A_\gamma$, where $*$ denotes concatenation and the initial point of $\eta$ equals the terminal point of $\gamma$.

(c) $(\Pi^A_\gamma)^{-1} = \Pi^A_{\gamma^{\leftarrow}}$, where $\gamma^{\leftarrow}(t) = \gamma(1-t)$ is $\gamma$ travelled in the reverse order.

(d) If $\Phi \in \mathscr{G}(P)$, then $\gamma^{\mathsf{h},A}_{\Phi(p)}(t) = \Phi(\gamma_p^{\mathsf{h},\Phi^*A}(t))$. Hence $\Pi^{\Phi^*A}_\gamma = \Phi^{-1} \circ \Pi^A_\gamma \circ \Phi$.

**Proof:**

(a) This is a general consequence of the fact that $\gamma^{\mathsf{h}}_{p\cdot g}(t) = \gamma^{\mathsf{h}}_p(t) \cdot g$ for all $t \in [0,1]$. Indeed, for $t = 0$ we have that $\gamma^{\mathsf{h}}_p(0) \cdot g = p \cdot g$, and $t \mapsto \gamma^{\mathsf{h}}_p(t) \cdot g$ is horizontal, since $\ker A$ is $G$-invariant (so that the derivative of $\mathrm{R}_g$ takes horizontal vectors to horizontal vectors).

(b) Clear.

(c) Follows from (b).

(d) If $\mathscr{H}^A$ and $\mathscr{H}^{\Phi^*A}$ are the horizontal distributions of $A$ and $\Phi^*A$, recall that we have the relation $\mathrm{d}\Phi_p[\mathscr{H}_p^{\Phi^*A}] = \mathscr{H}^A_{\Phi(p)}$, for all $p \in P$. For $t = 0$, we have that $\Phi(\gamma_p^{\mathsf{h},\Phi^*A}(0)) = \Phi(p)$. And moreover, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Phi(\gamma_p^{\mathsf{h},\Phi^*A}(t)) = \mathrm{d}\Phi_{\gamma_p^{\mathsf{h},\Phi^*A}(t)}(\dot{\gamma}_p^{\mathsf{h},\Phi^*A}(t))$$

is $A$-horizontal. This establishes the relation between horizontal lifts. Now plug $t = 1$ to conclude that $\Pi^A_\gamma(\Phi(p)) = \Phi(\Pi_\gamma^{\Phi^*A}(p))$, as required.

□

Keeping this notation, every parallel transport operator also acts on $E$. Namely, we define $\Pi_\gamma^{E,A}\colon E_x \to E_y$ by $\Pi_\gamma^{E,A}[p,v] = [\Pi^A_\gamma(p), v]$.

**Proposition 8**

$\Pi_\gamma^{E,A}$ is well-defined.

**Proof:** Replace $p$ with $p \cdot g$ and $v$ with $g^{-1}v$. Then

$$[\Pi^A_\gamma(p \cdot g), g^{-1}v] = [\Pi^A_\gamma(p)\cdot g, g^{-1}v] = [\Pi^A_\gamma(p), v],$$

as required. □

To explore things further, we'll use the expressions for $A$ relative to a local gauge $\psi\colon U \to P$. The pull-back $\psi^*A$ is denoted simply by $A_\psi \in \Omega^1(U, \mathfrak{g})$. Generally, we know that parallel transport operators between fibers of a vector bundle allow us to reconstruct the covariant derivative $\nabla$. We'll use the $\Pi_\gamma^{E,A}$ to define a connection $\nabla^A$ on $E$, as follows:

(1) Pick $x \in M$ and $\boldsymbol{v} \in T_xM$. Take a curve $\gamma\colon [0,1] \to M$ with $\gamma(0) = x$. For each $t \in [0,1]$, write $\gamma_t = \gamma|_{[0,t]}$

(2) Take a (local) section $s$ of $E$. Then $s(\gamma(t)) \in E_{\gamma(t)}$ for all $t \in [0,1]$. Then transport it back: $(\Pi_{\gamma_t}^{E,A})^{-1}(s(\gamma(t)) \in E_x$.

(3) Take the derivative:
$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\Pi_{\gamma_t}^{E,A})^{-1}(s(\gamma(t))).$$

**Proposition 9**

Relative to $\psi$, if $\dot\gamma(0) = \boldsymbol{v}$, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Pi^{E,A}_{\gamma_t})^{-1}(s(\gamma(t))) = [\psi(x), \mathrm{d}(s_\psi)_x(\boldsymbol{v}) + \rho_*(A_\psi(\boldsymbol{v}))s_\psi(x)].$$

**Proof:** We clearly have

$$(\Pi^{E,A}_{\gamma_t})^{-1}s(\gamma(t)) = (\Pi^{E,A}_{\gamma_t})^{-1}[\psi(\gamma(t)), s_\psi(\gamma(t))] = \left[(\Pi^{A}_{\gamma_t})^{-1}\psi(\gamma(t)), s_\psi(\gamma(t))\right],$$

but since $(\Pi^{A}_{\gamma_t})^{-1}\psi(\gamma(t)) \in P_x$ for all $t$, there is $g\colon [0,1] \to G$ such that

$$(\Pi^{A}_{\gamma_t})^{-1}\psi(\gamma(t)) = \psi(x) \cdot g(t)$$

for all $t \in [0,1]$. This immediately gives that

$$(\Pi^{E,A}_{\gamma_t})^{-1}s(\gamma(t)) = [\psi(x), g(t)s_\psi(\gamma(t))].$$

Now, $\gamma_0$ is the constant curve $x$, meaning that the corresponding parallel transport operator is the identity, and thus $g(0) = e$. We will also need to find $\dot g(0) \in \mathfrak{g}$, since

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Pi^{E,A}_{\gamma_t})^{-1}s(\gamma(t)) = [\psi(x), \mathrm{d}(s_\psi)_x(\boldsymbol{v}) + \rho_*(\dot g(0))s_\psi(x)]$$

by the product rule (and recalling that the juxtaposition $g(t)s_\psi(\gamma(t))$ was a shorthand for $\rho(g(t))s_\psi(\gamma(t))$). Taking derivatives at 0, we immediately see that

$$\psi(x)\cdot g(t) = (\Pi^{A}_{\gamma_t})^{-1}\psi(\gamma(t)) \implies \dot g(0)^{\#}_{\psi(x)} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Pi^{A}_{\gamma_t})^{-1}\psi(\gamma(t)),$$

but differentiating this last expression on the right requires the little usual trick: define

$$F(s,t) = \Pi^{A}_{\gamma_s}((\Pi^{A}_{\gamma_t})^{-1}\psi(\gamma(t)))$$

and note that $\psi(\gamma(t)) = F(t,t)$, and compute

$$\mathrm{d}\psi_x(\boldsymbol{v}) = \frac{\partial F}{\partial t}(0,0) + \frac{\partial F}{\partial s}(0,0) = \dot g(0)^{\#}_{\psi(x)} + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Pi^{A}_{\gamma_t}(\psi(x)).$$

Since the curve $t \mapsto \Pi^{A}_{\gamma_t}(\psi(x))$ is horizontal, applying $A$ to everything gives that $A_\psi(\boldsymbol{v}) = \dot g(0)$, as required. □

In particular, such expression depends on $\gamma(0)$ and $\dot\gamma(0)$, but not on $\gamma$ itself. So if, again, $\dot\gamma(0) = \boldsymbol{v} \in T_xM$, we define

$$\nabla^A_{\boldsymbol{v}} s = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Pi^{E,A}_{\gamma_t})^{-1}(s(\gamma(t))),$$

and if $\boldsymbol{X} \in \mathfrak{X}(M)$, we also define $\nabla^A_{\boldsymbol{X}} s$. Relative to $\psi$, we write $\nabla^A_{\boldsymbol{X}} s = [\psi, \nabla^A_{\boldsymbol{X}} s_\psi]$, where

$$\nabla^A_{\boldsymbol{X}} s_\psi = \mathrm{d}(s_\psi)(\boldsymbol{X}) + \rho_*(A_\psi(\boldsymbol{X}))s_\psi.$$

So $\nabla^A = \mathrm{d} + \rho_* A_\psi$. For this reason, $A_\psi$ is called the Christoffel form of $A$ relative to $\psi$.

**Proposition 10**

$\nabla^A$ is a Koszul connection on $E$.

**Proof:** All the properties are local, so we may verify them with a local gauge $\psi$, as usual. The expression for $\nabla^A_{\boldsymbol{X}} s_\psi$ is clearly additive in $\boldsymbol{X}$ and $s_\psi$, and $\mathscr{C}^\infty(M)$-linear in the variable $\boldsymbol{X}$. Let's verify the Leibniz rule. Let $f \in \mathscr{C}^\infty(M)$. Clearly $(fs)_\psi = fs_\psi$, so

$$\begin{aligned}\nabla^A_{\boldsymbol{X}}(fs_\psi) &= \mathrm{d}(fs_\psi)(\boldsymbol{X}) + \rho_*(A_\psi(\boldsymbol{X}))(fs_\psi) \\ &= \boldsymbol{X}(f)s_\psi + f\mathrm{d}(s_\psi)(\boldsymbol{X}) + f\rho_*(A_\psi(\boldsymbol{X}))s_\psi \\ &= \boldsymbol{X}(f)s_\psi + f\nabla^A_{\boldsymbol{X}} s_\psi,\end{aligned}$$

□

**Remark.** A shorter argument: $\nabla^A = \mathrm{d} + \rho_* A_\psi$ equals a connection (d) plus a tensor ($\rho_* A_\psi$), so it is a connection.

Last three remarks:

- If $\mathscr{F}$ is a smooth endofunctor of the category of finite-dimensional real vector spaces and linear maps (smooth means that the action on the level of morphisms is smooth), then for each $\rho\colon G \to \mathrm{GL}(V)$ we get $\mathscr{F}\rho\colon G \to \mathrm{GL}(\mathscr{F}V)$, and so we can form the associated bundle $P \times_{\mathscr{F}\rho} \mathscr{F}V$. But $\mathscr{F}$ also acts fiberwise on the associated vector bundle $P \times_\rho V$, producing $\mathscr{F}(P \times_\rho V)$. These two bundles are isomorphic simply because they are described by the same cocycles (relative to trivializations induced by principal $G$-charts for $P$). If $\tau_{\alpha\beta}$ is an element of the cocycle, then $\rho \circ \tau_{\alpha\beta}$ is an element of the cocycle defining $P \times_\rho V$. And the associativity law $\mathscr{F} \circ (\rho \circ \tau_{\alpha\beta}) = (\mathscr{F} \circ \rho) \circ \tau_{\alpha\beta}$ holds. If $\mathscr{F}$ is a multivariable smooth functor, a similar argument applies. So, for example, the associated vector bundle to $P$ under the dual representation of $\rho$ is in fact the dual of the associated vector bundle to $P$ under $\rho$. And so on.

- Generally, if $V$ carries a linear $G$-structure, then we have an induced $G$-structure on $\mathrm{Fr}(E)$, which is parallel relative to $\nabla^A$. Here's one concrete example: if we take a covariant $k$-tensor $T \in (V^*)^{\otimes k}$ on $V$ which is $T$ invariant, then we have $T^E \in \Gamma((E^*)^{\otimes k})$ defined by $T^E_x([p, v_1], \ldots, [p, v_k]) = T(v_1, \ldots, v_k)$ (it is well-defined). Now, suppressing $\rho$ and $\rho_*$, differentiating the $G$-invariance relation $T(gv_1, \ldots, gv_k) = T(v_1, \ldots, v_k)$ relative to the variable $g$ and evaluating at $X \in \mathfrak{g}$, we obtain
$$\sum_{i=1}^n T(v_1, \ldots, Xv_i, \ldots, v_k) = 0.$$
This means that, choosing a local gauge $\psi$ for $P$, we have
$$\begin{aligned}\sum_{i=1}^n T^E(s_1, \ldots, \nabla^A_{\boldsymbol{X}} s_i, \ldots, s_k) &= \sum_{i=1}^n T(s_{1,\psi}, \ldots, \nabla^A_{\boldsymbol{X}} s_{i,\psi}, \ldots, s_{k,\psi}) \\ &= \sum_{i=1}^n T(s_{1,\psi}, \ldots, \mathrm{d}(s_{i,\psi})(\boldsymbol{X}) + \rho_*(A_\psi(\boldsymbol{X}))s_{i,\psi}, \ldots, s_{k,\psi})\end{aligned}$$

$$= \sum_{i=1}^{n} T(s_{1,\psi}, \ldots, \mathrm{d}(s_{i,\psi})(\boldsymbol{X}), \ldots, s_{k,\psi})$$
$$= \boldsymbol{X}(T(s_{1,\psi}, \ldots, s_{k,\psi})$$
$$= \boldsymbol{X}(T^E(s_1, \ldots, s_k)),$$

for all $\boldsymbol{X} \in \mathfrak{X}(M)$ and (local) sections $s_1, \ldots, s_k$ of $E$. This means that $\nabla^A T^E = 0$. In particular, a $G$-invariant inner product on $V$ (which always exists when $G$ is compact, by Weyl's unitary trick) induces a parallel fiber metric on $E$.

- If we denote $\nabla^A$ by $\nabla^{A,\rho}$, and noting that if $E$ has a connection, then $\mathscr{F}E$ also inherits one (usually by requiring some structure to be parallel — e.g., connections in hom-bundles are characterized by making the evaluation map parallel), then it turns out that $\mathscr{F}(\nabla^{A,\rho}) = \nabla^{A,\mathscr{F}\rho}$ by default, so there is no ambiguity when writing things like $\nabla^A T^E$ as in the previous item.

## 5 Gauge independence of direct definition of $\nabla^A$

Often, one defines $\nabla^A$ on $E$ by choosing a local gauge $\psi\colon U \to P$ and declaring

$$\nabla^A_{\boldsymbol{X}} s = [\psi, \mathrm{d}(s_\psi)(\boldsymbol{X}) + \rho_*(A_\psi(\boldsymbol{X}))s_\psi].$$

Then it is necessary to check that this definition is independent of $\psi$. So, let's make a change of gauge $\psi \mapsto \psi' = \psi \cdot g$, where $g\colon U \to G$ is a physical gauge transformation. More precisely, $\psi'(x) = \psi(x) \cdot g(x)$ for all $x \in U$.

**Proposition 11**

(a) $s_{\psi\cdot g} = g^{-1} s_\psi$.

(b) $\mathrm{d}(s_{\psi\cdot g})_x(\boldsymbol{v}) = -\rho_*((g^*\Theta)_x(\boldsymbol{v}))\rho(g(x)^{-1})s_\psi(x) + \rho(g(x)^{-1})\mathrm{d}(s_\psi)_x(\boldsymbol{v})$.

(c) $\mathrm{d}(\psi \cdot g)_x(\boldsymbol{v}) = \mathrm{d}(R_{g(x)})_{\psi(x)}(\mathrm{d}\psi_x(\boldsymbol{v})) + (g^*\Theta)_x(\boldsymbol{v})^\#_{\psi(x)\cdot g(x)}$, for all $x \in U$ and $\boldsymbol{v} \in T_x M$.

(d) $A_{\psi\cdot g} = \mathrm{Ad}(g^{-1}) \circ A_\psi + g^*\Theta$.

**Remark.** Above, $\Theta \in \Omega^1(G, \mathfrak{g})$ is the left-invariant Maurer-Cartan form on $G$, given by $\Theta_a(\boldsymbol{w}) = \mathrm{d}(\mathrm{L}_{a^{-1}})_a \boldsymbol{w}$. Occasionally, we'll just write $a^{-1}\boldsymbol{w}$.

**Proof:**

(a) $s = [\psi \cdot g, s_{\psi\cdot g}] = [\psi, g s_{\psi\cdot g}]$ implies that $s_\psi = g s_{\psi\cdot g}$, and the conclusion follows.

(b) The usual trick of separating variables works: define $F(x,y) = \rho(g(x)^{-1})s_\psi(y)$ and note that $s_{\psi\cdot g}(x) = F(x,x)$, so

$$\mathrm{d}(s_{\psi\cdot g})_x(\boldsymbol{v}) = (\partial_1 F)_{(x,x)}(\boldsymbol{v}) + (\partial_2 F)_{(x,x)}(\boldsymbol{v}).$$

But

$$(\partial_2 F)_{(x,x)}(v) = \rho(g(x)^{-1})\mathrm{d}(s_\psi)_x(v),$$

and

$$\begin{aligned}(\partial_1 F)_{(x,x)}(v) &= \mathrm{d}\rho_{g(x)^{-1}}(-g(x)^{-1}\mathrm{d}g_x(v)g(x)^{-1})s_\psi(x)\\ &= -\mathrm{d}\rho_{g(x)^{-1}}((g^*\Theta)_x(v)g(x)^{-1})s_\psi(x)\\ &= -\rho_*((g^*\Theta)_x(v))\rho(g(x)^{-1})s_\psi(x).\end{aligned}$$

We're using the standard formulas for the derivative of the inversion in any Lie group, the chain rule to differentiate $\rho \circ \mathrm{R}_{g(x)^{-1}} = \mathrm{R}_{\rho(g(x)^{-1})} \circ \rho$ (because $\rho$ is a homomorphism) at the identity $e \in G$, and that multiplication in $\mathrm{GL}(V)$ is the restriction of a linear map $\mathfrak{gl}(V) \to \mathfrak{gl}(V)$ (so its derivative is itself).

(c) The usual trick works again: define $F(x,y) = \psi(x) \cdot g(y)$, note that $\psi'(x) = F(x,x)$, so

$$\mathrm{d}(\psi')_x(v) = (\partial_1 F)_{(x,x)}(v) + (\partial_2 F)_{(x,x)}(v).$$

But

$$(\partial_1 F)_{(x,x)}(v) = \mathrm{d}(R_{g(x)})_{\psi(x)}(\mathrm{d}\psi_x(v)),$$

and

$$\begin{aligned}(\partial_2 F)_{(x,x)}(v) &= \mathrm{d}(\mathcal{O}_{\psi(x)})_{g(x)}(\mathrm{d}g_x(v))\\ &= \mathrm{d}(\mathcal{O}_{\psi(x)\cdot g(x)})_e((\mathrm{d}(\mathrm{L}_{g(x)})^{-1}_{g(x)}\mathrm{d}g_x(v))))\\ &= \mathrm{d}(\mathcal{O}_{\psi(x)\cdot g(x)})_e((g^*\Theta)_x(v))\\ &= (g^*\Theta)_x(v)^\#_{\psi(x)\cdot g(x)}.\end{aligned}$$

(d) From (c), we have that

$$\begin{aligned}(A_{\psi\cdot g})_x(v) &= A_{\psi(x)\cdot g(x)}(\mathrm{d}(\psi \cdot g)_x(v))\\ &= A_{\psi(x)\cdot g(x)}(\mathrm{d}(R_{g(x)})_{\psi(x)}(\mathrm{d}\psi_x(v)) + (g^*\Theta)_x(v)^\#_{\psi(x)\cdot g(x)})\\ &= A_{\psi(x)\cdot g(x)}(\mathrm{d}(R_{g(x)})_{\psi(x)}(\mathrm{d}\psi_x(v))) + (g^*\Theta)_x(v)\\ &= (\mathrm{R}^*_{g(x)}A)_{\psi(x)}(\mathrm{d}\psi_x(v)) + (g^*\Theta)_x(v)\\ &= \mathrm{Ad}(g(x)^{-1})(A_{\psi(x)}(\mathrm{d}\psi_x(v)) + (g^*\Theta)_x(v)\\ &= \mathrm{Ad}(g(x)^{-1})((A_\psi)_x(v)) + (g^*\Theta)_x(v).\end{aligned}$$

□

Now everything is in place. Since

$$[\psi \cdot g, \mathrm{d}(s_{\psi\cdot g}) + \rho_*(A_{\psi\cdot g})s_{\psi\cdot g}] = [\psi, g(\mathrm{d}(s_{\psi\cdot g}) + \rho_*(A_{\psi\cdot g})s_{\psi\cdot g})],$$

there is only one computation left to do. Let's carry $\rho$, $x$ and $v$ in full detail.

$$\rho(g(x))\left(\mathrm{d}(s_{\psi\cdot g})_x(v) + \rho_*((A_{\psi\cdot g})_x(v))s_{\psi\cdot g}(x)\right) =$$

$$
\begin{aligned}
&= \rho(g(x))\Big( -\rho_*((g^*\Theta)_x(v))\rho(g(x)^{-1})s_\psi(x) + \rho(g(x)^{-1})\mathrm{d}(s_\psi)_x(v) \\
&\qquad + \rho_*\big(\mathrm{Ad}_{g(x)^{-1}}(A_\psi)_x(v) + (g^*\Theta)_x(v)\big)\rho(g(x)^{-1})s_\psi(x)\Big) \\
&\overset{(\dagger)}{=} \rho(g(x))\Big(\rho(g(x)^{-1})\mathrm{d}(s_\psi)_x(v) + \rho_*(\mathrm{Ad}_{g(x)^{-1}}(A_\psi)_x(v))\rho(g(x)^{-1})s_\psi(x)\Big) \\
&\overset{(\ddagger)}{=} \mathrm{d}(s_\psi)_x(v) + \rho_*((A_\psi)_x(v))s_\psi(x),
\end{aligned}
$$

where in ($\dagger$) we cancel all the terms with $\Theta$, and in ($\ddagger$) we use that $\rho$ is a homomorphism and $g(x)\mathrm{Ad}_{g(x)^{-1}}((A_\psi)_x(v))g(x)^{-1} = (A_\psi)_x(v)$.

# 6 Curvature

The curvature of $A \in \Omega^1(P,\mathfrak{g})$ is $F_A \in \Omega^2(P,\mathfrak{g})$ given by

$$F_A = \mathrm{d}A + \frac{1}{2}[A,A]$$

or, more explictly, $F_A(X,Y) = \mathrm{d}A(X,Y) + [A(X),A(Y)]$, for all $X,Y \in \mathfrak{X}(P)$. If we have coordinates $(x^\mu)$ for the base manifold, and a local gauge $\psi$, all on some open set $U \subseteq M$, then we have $F_{A,\psi} = \psi^*(F_A) \in \Omega^2(U,\mathfrak{g})$, and we set $F_{\mu\nu} = F_{A,\psi}(\partial_\mu,\partial_\nu)$. Then, we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

where $A_\mu = (A_\psi)(\partial_\mu)$. Note that $A_\mu, F_{\mu\nu}$ are smooth functions on $U$, valued on $\mathfrak{g}$. When $G$ is abelian, $[A_\mu,A_\nu] = 0$. It remains to establish what is the relation between $R^{\nabla^A}$ and $F_A$. We do this using $\psi$.

**Proposition 12**

$R^{\nabla^A}(\partial_\mu,\partial_\nu)s_\psi = \rho_*(F_{\mu\nu})s_\psi$.

**Proof:** It's a direct computation:

$$
\begin{aligned}
R^{\nabla^A}(\partial_\mu,\partial_\nu)s_\psi &= \nabla^A_{\partial_\mu}\nabla^A_{\partial_\nu}s_\psi - \nabla^A_{\partial_\nu}\nabla^A_{\partial_\mu}s_\psi \\
&= \nabla^A_{\partial_\mu}(\partial_\nu s_\psi + \rho_*(A_\nu)s_\psi) - \nabla^A_{\partial_\nu}(\partial_\mu s_\psi + \rho_*(A_\mu)s_\psi) \\
&= \partial_\mu\partial_\nu s_\psi + \rho_*(A_\mu)\partial_\nu s_\psi + \partial_\mu[\rho_*(A_\nu)s_\psi] + \rho_*(A_\mu)\rho_*(A_\nu)s_\psi \\
&\qquad - \partial_\nu\partial_\mu s_\psi + \rho_*(A_\nu)\partial_\mu s_\psi - \partial_\nu[\rho_*(A_\mu)s_\psi] - \rho_*(A_\nu)\rho_*(A_\mu)s_\psi \\
&= \rho_*(A_\mu)\partial_\nu s_\psi + \rho_*(\partial_\mu A_\nu)s_\psi + \rho_*(A_\nu)\partial_\mu s_\psi \\
&\qquad - \rho_*(A_\nu)\partial_\mu s_\psi - \rho_*(\partial_\nu A_\mu)s_\psi - \rho_*(A_\mu)\partial_\nu s_\psi + \rho_*([A_\mu,A_\nu])s_\psi \\
&= \rho_*(\partial_\mu A_\nu)s_\psi - \rho_*(\partial_\nu A_\mu)s_\psi + \rho_*([A_\mu,A_\nu])s_\psi \\
&= \rho_*(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu,A_\nu])s_\psi \\
&= \rho_*(F_{\mu\nu})s_\psi,
\end{aligned}
$$

as required. □

**Remark.** It's not clear how to write such an expression without relying on a local gauge. If $X \in \mathfrak{g}$, trying to define $\rho_*(X)[p,v]$ as $[p,\rho_*(X)v]$ doesn't work, as replacing $p$ and $v$ with $p \cdot g$ and $\rho(g^{-1})v$ leads to $[p,\rho_*(\mathrm{Ad}_g(X))v]$ instead.

# 7 Arbitrary associated fiber bundles

Essentially everything that happened here can be done replacing $\rho$ and $V$ with a manifold $F$ and an action $G \circlearrowright F$. We have that $(P \times F) \circlearrowleft G$ via $(p,y)g = (pg, g^{-1}y)$, and $P \times_G F = (P \times F)/G$ is a manifold whose elements are classes $[p,y]$. This is a locally trivial fiber bundle with typical fiber $F$, and local trivializations $(U,\widetilde{\Phi})$ are constructed from principal $G$-charts $(U,\Phi)$ for $P$, via $\widetilde{\Phi}[p,y] = (\pi(p), \Phi_G(p)y)$, as before (it is well-defined). Inverses are $\widetilde{\Phi}^{-1}(x,y) = [\Phi^{-1}(x,e),y]$. Restrictions to fibers are diffeomorphisms onto $F$. One can locally define horizontal lifts (but the domains pay the price: given $x \in M$ and $\gamma[0,1] \to M$ with $\gamma(0) = x$, the map $y \mapsto \gamma^{\mathrm{h}}_y(t)$ is not necessarily defined for all $y \in (P \times_G F)_y$ and/or $t \in [0,1]$). And so on.

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