Compact essentially conformally symmetric (ECS) manifolds

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Introduction and general background

The curvature tensor of the Levi-Civita connection of a pseudo-Riemannian manifold (M, g), *always assumed to have dimension $n \geq 4$ (unless otherwise stated)*, admits the orthogonal decomposition

$$R = \frac{\mathrm{s}}{n(n-1)}\, \mathrm{g} \owedge \mathrm{g} + \frac{2}{n-2}\mathrm{g} \owedge \left(\mathrm{Ric} - \frac{\mathrm{s}}{n}\mathrm{g}\right) + \mathrm{W},$$

where Ric, s and W stand, respectively, for the Ricci tensor, scalar curvature, and Weyl tensor of (M, g), and $\owedge$ denotes the Kulkarni-Nomizu product[1] between tensor fields of type $(0, 2)$ on M. It is natural to consider what happens when some of those factors vanish. For example, we say that:

- (M, g) is Einstein, when $\mathrm{Ric} = \lambda \mathrm{g}$ for some constant $\lambda \in \mathbb{R}$, which is necessarily equal to s/n (by tracing) — and those metrics appear as critical points of the Einstein-Hilbert functional $\mathscr{E} \colon \mathrm{Met}(M) \to \mathbb{R}$ given by

$$\mathscr{E}[\mathrm{g}] = \frac{1}{\mathrm{vol}(M, \mathrm{g})^{(n-2)/n}} \int_M \mathrm{s}[\mathrm{g}]\, \mathrm{d}\nu_{\mathrm{g}},$$

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[1] In coordinates, defined by $2(T \owedge S)_{ijk\ell} = T_{jk}S_{i\ell} - T_{ik}S_{j\ell} + S_{jk}T_{i\ell} - S_{ik}T_{j\ell}$. The factor 2 is a natural consequence of a certain polarization process.

where $\mathrm{Met}(M)$ denotes the space of all pseudo-Riemannian metrics on M and $\mathrm{d}\nu_{\mathrm{g}}$ stands for the volume form of (M, g). We note that this is a Riemannian functional in the sense that if $f\colon M \to M$ is any diffeomorphism, then we have that $\mathscr{E}[f^*\mathrm{g}] = \mathscr{E}[\mathrm{g}]$. The loose intuition here is that Einstein metrics are the ones for which the scalar curvature is more "evenly distributed" throughout M.

- (M, g) is conformally flat when $\mathrm{W} = 0$, and such condition is equivalent to saying that around each point in M there is a neighborhood U and a smooth function $\varphi\colon U \to \mathbb{R}$ such that $\mathrm{e}^{2\varphi}\mathrm{g}$ is a flat metric on U — this is called the Weyl-Schouten theorem.

The next step would be to look for weaker conditions, involving only covariant derivatives and divergences. So we recall that (M, g) is called locally symmetric if $\nabla R = 0$, and this condition is equivalent to the existence, around each point in M, of local geodesic symmetries[2] which are also local isometries of (M, g) — this is called Cartan's theorem. This condition is still something strong to require, as it implies several other restrictions on the geometry of (M, g). For instance, it immediately implies that Ric and W are also parallel tensors, and that s is constant. Narrowing it down further, if instead we only assume that $\nabla\mathrm{Ric} = 0$, it again follows that s is constant, and while we cannot conclude that R and W are parallel tensors, we do obtain that those are divergence-free. To summarize the conclusions in this paragraph (all formally established, e.g., by standard coordinate computations), it is also convenient to introduce the Schouten tensor Sch of (M, g), which is defined by

$$\mathrm{Sch} = \mathrm{Ric} - \frac{\mathrm{s}}{2(n-1)}\mathrm{g}.$$

We also note that Sch is very useful to express the relation between R and W in a more concise way. Namely, we have that

$$R = \frac{2}{n-2}\mathrm{g} \owedge \mathrm{Sch} + \mathrm{W}.$$

Thus:

Proposition 1

Let (M, g) be a pseudo-Riemannian manifold. If δ and d^∇ stand, respectively, for the divergence operator and the covariant exterior derivative of type $(0, 2)$ tensor fields on M regarded as T^*M-valued 1-forms, then we have:

(i) $\delta R = \mathrm{d}^\nabla \mathrm{Ric}$;

(ii) $\delta \mathrm{Ric} = \dfrac{\mathrm{d}s}{2}$;

[2] A local geodesic symmetry of an affine manifold (M, ∇) around a point x is a map φ from a neighborhood of x onto itself such that $\varphi(x) = x$ and φ reverses geodesics, i.e., if γ is any geodesic defined on some neighborhood of 0 starting at x, then $\varphi(\gamma(t)) = \gamma(-t)$ (this obviously implies that $\mathrm{d}\varphi_x = -\mathrm{Id}_{T_xM}$). Such maps do not need to be connection-preserving in general.

(iii) $\delta \mathrm{Sch} = \dfrac{n-2}{2(n-1)} \mathrm{d}s$;

(iv) $\delta \mathrm{W} = \dfrac{n-3}{n-2} \mathrm{d}^{\nabla} \mathrm{Sch}$.

Corollary 1

A pseudo-Riemannian manifold has harmonic curvature if and only if it has harmonic Weyl curvature[a] and constant scalar curvature.

[a] (M, g) has harmonic curvature if $\delta R = 0$, and harmonic Weyl curvature if $\delta \mathrm{W} = 0$.

With this in place, there is still one condition left to consider: $\nabla \mathrm{W} = 0$. It would stand to reason to call manifolds with parallel Weyl tensor "conformally symmetric", but this can be misleading, as this condition is not equivalent to (M, g) being (locally) conformally equivalent to a locally symmetric manifold. Necessary and sufficient conditions for this conformal equivalence, depending on whether $\nabla \mathrm{W}$ vanishes or not, were discussed in [21].

Theorem 1 (Roter, [6])

If a Riemannian manifold has parallel Weyl tensor, then it is locally symmetric or conformally flat.

This means that if we wish to consider only "non-trivial" cases of manifolds with parallel Weyl tensor, this becomes a problem of (strictly indefinite) pseudo-Riemannian nature. We will assume from here on that this is always the case. Notwithstanding the possible ambiguity with the terminology "conformally symmetric", we register the:

Definition 1 (ECS manifold)

A pseudo-Riemannian manifold is called *essentially conformally symmetric* if it has parallel Weyl tensor, but it is not conformally flat nor locally symmetric.

We will investigate ECS manifolds.

Generalities and local types

ECS manifolds come in two local types. To understand them, we start at the vector space level.

Definition 2

Let $(V, \langle \cdot, \cdot \rangle)$ be a pseudo-Euclidean vector space with indefinite signature, with dimension $\dim V \geq 2$, and W be a Weyl curvaturelike tensor on V. The *Olszak space* of W is defined as $\mathscr{D} = \{\xi \in V^* \mid \xi \wedge \Omega = 0, \text{ for all } \Omega \in \mathrm{Im}(\mathrm{W})\}$, where we regard W as a self-adjoint (traceless) endomorphism of $(V^*)^{\wedge 2}$. We'll always write $d = \dim \mathscr{D}$.

Let's gather all the immediate relevant information about $\mathscr{D}$ (which will be identified with a subspace of V via $\langle \cdot, \cdot \rangle$, without further comments) in the following:

Lemma 1 ([11])

With the setup of Definition 2 above, we have that:

(i) $d \in \{0, 1, 2, n\}$, and $d = n$ if and only if $\mathrm{W} = 0$.

(ii) if $d < n$, then $\mathscr{D}$ is a null subspace of V.

(iii) $d = 2$ if and only if the rank of the curvature operator W is equal to 1, in which case we have that $\mathrm{W} = \varepsilon\,\omega \otimes \omega$ for some $\varepsilon \in \{\pm 1\}$ and $\omega \in (V^*)^{\wedge 2}$ (identified with a skew-symmetric endomorphism of V, with the aid of $\langle \cdot, \cdot \rangle$), as well as $\mathscr{D} = \mathrm{Im}(\omega)$.

So, given a pseudo-Riemannian manifold (M, g), applying Definition 2 to each point in M, we have its *Olszak distribution* $\mathscr{D} \hookrightarrow TM$. It is a null distribution, parallel because W is, and whose rank d is necessarily equal 1 or 2 if (M, g) is ECS.

Certain arguments of the curvature tensor annihilate null parallel distributions, and this is sometimes useful to simplify computations. Let's state this phenomenon precisely:

Lemma 2 ([8])

Let (M, g) be a pseudo-Riemannian manifold and $\mathscr{P} \hookrightarrow TM$ be a null parallel distribution. Then $\mathscr{P}^\perp$ is also a parallel distribution, $\mathscr{P} \subseteq \mathscr{P}^\perp$, and we have

(i) $R(\mathscr{P}, \mathscr{P}, \cdot, \cdot) = 0$;

(ii) $R(\mathscr{P}, \mathscr{P}^\perp, \cdot, \cdot) = 0$;

(iii) $R(\mathscr{P}^\perp, \mathscr{P}^\perp, \mathscr{P}, \cdot) = 0$.

In addition, when (M, g) is ECS and the Olszak distribution $\mathscr{D}$ has $d = 2$, we also have

(iv) $R(\mathscr{D}, \cdot, \mathscr{D}^\perp, \cdot) = 0$;

(v) $R(\mathscr{D}^\perp, \mathscr{D}^\perp, \cdot, \cdot) = 0$;

(vi) Ric, ∇Ric, ω (as in Lemma 1) and W all annihilate $\mathscr{D}^\perp$.

Remark. Note that condition (v) is an improvement of condition (iii).

With this, let's move on to examples of conformally symmetric manifolds:

Example 1 (Conformally symmetric manifolds, $d = 1$)

Let $(V, \langle \cdot, \cdot \rangle)$ be a pseudo-Euclidean vector space of dimension $n \geq 3$, $I \subseteq \mathbb{R}$ an open interval, $f \colon I \to \mathbb{R}$ a smooth function, and a self-adjoint $A \in \mathfrak{sl}(V) \setminus \{0\}$. Define a smooth map $\kappa \colon I \times \mathbb{R} \times V \to \mathbb{R}$ by

$$\kappa(t, s, v) = f(t)\langle v, v \rangle + \langle Av, v \rangle,$$

and consider the pseudo-Riemannian manifold

$$(M^{n+2} \doteq I \times \mathbb{R} \times V, \mathrm{g} \doteq \kappa\, \mathrm{d}t^2 + \mathrm{d}t\, \mathrm{d}s + \langle \cdot, \cdot \rangle).$$

Above, the tensor fields $\mathrm{d}t$, $\mathrm{d}s$ and $\langle \cdot, \cdot \rangle$ on the factors I, $\mathbb{R}$ and V are identified with their pull-backs to M. This inofensive abuse of notation will be done again in the future without further comment. Some facts about this manifold (ranging from trivial to just established by straightforward computations) are:

- The coordinate field ∂_s is null, parallel and spans the Olszak distribution.
- $\mathrm{Ric} = -nf\, \mathrm{d}t \otimes \mathrm{d}t$.
- $\nabla R = 0 \iff f' = 0$.
- The fully covariant Weyl tensor is given by $\mathrm{W} = -2(\mathrm{d}t \otimes \mathrm{d}t) \owedge A$ — hence $\nabla \mathrm{W} = 0$ because $\mathrm{d}t$ is a parallel 1-form (it corresponds to ∂_s via the metric) and A is parallel.
- $\mathrm{s} = \langle R, R \rangle = \langle \mathrm{Ric}, \mathrm{Ric} \rangle = \langle \mathrm{W}, \mathrm{W} \rangle = 0$.

Choosing f to be non-constant and A non-zero, we have that (M, g) is ECS. For suitable choices of $\langle \cdot, \cdot \rangle$, we see that these examples realize all indefinite metric signatures. In particular, we see from the expression for Ric that (M, g) is Ricci-recurrent[a].

[a] A tensor field T on a manifold M equipped with a connection ∇ is called *recurrent* (or ∇-*recurrent*) if $\nabla_X T$ and T are linearly dependent at all points, for every vector field $X \in \mathfrak{X}(M)$. This is equivalent to requiring the existence of a (unique) 1-form α, defined on the open set where $T \neq 0$, such that $\nabla T = \alpha \otimes T$.

Remark. The fact that ECS manifolds with $d = 1$ are Ricci-recurrent doesn't really rely on the explicit expression for the Ricci tensor above. An altenative argument uses the following general fact: if (M, ∇) is a manifold equipped with an arbitrary connection, $\mathscr{D} \hookrightarrow T^*M$ is a rank 1 parallel subbundle, and $T \in \Gamma(T^*M^{\odot 2})$ is such that $\mathrm{Im}(T \colon TM \to T^*M) \subseteq \mathscr{D}$, then T is recurrent. Indeed, it suffices to work locally on some set where $T \neq 0$ — then we may take a local recurrent (as $\mathscr{D}$ is parallel) 1-form ξ spanning $\mathscr{D}$, write $T = \beta \otimes \xi$ for some local 1-form β (since $\mathrm{Im}(T) \subseteq \mathscr{D}$), and use that T is symmetric to conclude that $\beta = h\xi$ for some smooth function h. Then $T = h\, \xi \otimes \xi$ is recurrent as the tensor product of recurrent tensors is again recurrent. With this in place, that every ECS manifold with $d = 1$ is Ricci-recurrent follows from the fact (proved in [11]) that the Olszak distribution $\mathscr{D}$ contains the image of the Ricci tensor (regarded, with the aid of the metric, as a self-adjoint endomorphism of the tangent bundle).

Describing examples of ECS manifolds when $d = 2$ is more challenging, and requires two brief concepts.

Definition 3 (Patterson-Walker extensions, [22])

Let Σ be a smooth manifold equipped with a linear connection ∇ on its cotangent bundle $T^*\Sigma$. The *Patterson-Walker extension of* ∇ is the pseudo-Riemannian metric $\mathtt{h}^\nabla$ on $T^*\Sigma$ defined by declaring the ∇-horizontal spaces $\mathscr{H}^\nabla_{(x,p)}$ to be null, and by setting $\mathtt{h}^\nabla_{(x,p)}(\boldsymbol{v},\boldsymbol{w}) = \boldsymbol{v}(\mathrm{d}\pi_{(x,p)}(\boldsymbol{w}))$, for all vertical $\boldsymbol{v} \in \mathscr{V}_{(x,p)} \cong T^*_x\Sigma$ and $\boldsymbol{w} \in T_{(x,p)}(T^*\Sigma)$, where π stands for the bundle projection $T^*\Sigma \to \Sigma$.

Remark.

- If $\pi\colon (E,\nabla) \to M$ is a vector bundle with connection over a manifold M, there is a natural decomposition $TE = \mathscr{H}^\nabla \oplus \mathscr{V}$, where $\mathscr{V} = \ker(\mathrm{d}\pi\colon TE \to TM)$. Namely, given $x \in M$, $\phi \in E_x$ and $\boldsymbol{v} \in T_xM$, we define the *horizontal lift* $\boldsymbol{v}^{\mathsf{h}}_{(x,\phi)} \in T_{(x,\phi)}E$ by $\boldsymbol{v}^{\mathsf{h}}_{(x,\phi)} = \mathrm{d}\psi_x(\boldsymbol{v})$, where ψ is any local section around x with $(\psi_x, (\nabla\psi)_x) = (\phi, 0)$, and this is independent on the choice of ψ. With this, the ∇*-horizontal space* is defined as $\mathscr{H}^\nabla_{(x,\phi)} = \{\boldsymbol{v}^{\mathsf{h}}_{(x,\phi)} \mid \boldsymbol{v} \in T_xM\}$. The derivative of the bundle projection restricts to an isomorphism $\mathscr{H}^\nabla_{(x,\phi)} \cong T_xM$ and the distribution $\mathscr{H}^\nabla \hookrightarrow TE$ is integrable if and only if ∇ is flat.
- By definition of $\mathtt{h}^\nabla$, each vertical space $\mathscr{V}_{(x,p)}$ is automatically null as well.
- If ∇^{PW} stands for the Levi-Civita connection of $\mathtt{h}^\nabla$ and we choose local cotangent coordinates (q^k, p_k) for $T^*\Sigma$, a straightforward computation shows that the relations $\nabla^{\mathrm{PW}}_{\partial_{q^i}}\partial_{p_j} = -\Gamma^j_{(ik)}\partial_{p_k}$ and $\nabla^{\mathrm{PW}}_{\partial_{p_i}}\partial_{p_j} = \mathbf{0}$ hold, meaning that the vertical distribution $\mathscr{V}$ on $T^*\Sigma$ is ∇^{PW}-parallel and Lemma 2 applies. The distribution $\mathscr{H}^\nabla$ in turn, is not ∇^{PW}-parallel in general.

Definition 4 (Projectively flat connections)

Let Σ be a smooth manifold. A connection D on Σ is called *projectively flat* if Σ can be covered with coordinate systems for which the D-geodesics appear as straight lines.

Here's one general family of examples: let V be a vector space and $\Sigma \subseteq V$ a hypersurface which is transverse to all rays emanating from the origin of V. This allows us to write that $TV = T\Sigma \oplus \mathbb{R}\boldsymbol{N}$, where the radial field $\boldsymbol{N}\colon \Sigma \to V$ is just given by $\boldsymbol{N}(x) = x$. The connection D on $T\Sigma$ obtained from projecting the standard flat connection $\mathrm{D}^{\mathrm{std}}$ of V is called a *centroaffine connection*, and it is projectively flat and torsion-free. The D-geodesics are intersections of the form $\Sigma \cap \Pi$, where $\Pi \subseteq V$ is a plane passing through the origin, and if we write $\mathrm{D}^{\mathrm{std}}_{\boldsymbol{X}}\boldsymbol{Y} = \mathrm{D}_{\boldsymbol{X}}\boldsymbol{Y} + b(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{N}$, where $b \in \Gamma(T^*\Sigma^{\odot 2})$ is a scalar-valued second fundamental form, tracing the Gauss equation $R^{\mathrm{D}}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} = -b(\boldsymbol{Y},\boldsymbol{Z})\boldsymbol{X} + b(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z}$ says that $\mathrm{Ric}^{\mathrm{D}} = (1-\dim\Sigma)b$ is symmetric. In particular, $\mathrm{d}^{\mathrm{D}}\mathrm{Ric}^{\mathrm{D}} = 0$ (that is, $\mathrm{Ric}^{\mathrm{D}}$ is a Codazzi tensor[3]). We have the interesting "converse", due to Kurita (see [19]): every manifold equipped with a projectively flat torsionfree connection, whose holonomy preserves a volume form, is (locally) realized by the above construction. The case of interest, in the above definition, will be when Σ is a (frequently assumed to be simply connected) surface, in which case we have a more detailed correspondence result given in [8]. With this in place, we may discuss the next:

Example 2 (Conformally symmetric manifolds, $d = 2$)

Let $(V, \langle\cdot,\cdot\rangle)$ be a pseudo-Euclidean vector space with dimension $\dim V = n \geq 0$, Σ a surface, D be a projectively flat and torsion-free connection on Σ, $\alpha \in \Omega^2(\Sigma)$ be a D-parallel area form, $\mathtt{h}^{\mathrm{D}}$ be the Patterson-Walker extension metric in $T^*\Sigma$, and $T \in \Gamma(T\Sigma^{\odot 2})$ be such that $\mathrm{div}^{\mathrm{D}}(\mathrm{div}^{\mathrm{D}}T) + \big(\mathrm{Ric}^{\mathrm{D}}, T\big) = \varepsilon$, where $\varepsilon \in \{\pm 1\}$. Since Σ is a surface, we can use the area form α to identify $T\Sigma \cong T^*\Sigma$ and convert T to $\tau \in \Gamma(T^*\Sigma^{\odot 2})$. Let $\theta\colon V \to \mathbb{R}$ be the smooth function given by $\theta(\boldsymbol{v}) = \langle \boldsymbol{v}, \boldsymbol{v}\rangle$, and consider the pseudo-Riemannian manifold

$$(M^{4+n} \doteq T^*\Sigma \times V, \mathrm{g} \doteq \mathtt{h}^{\mathrm{D}} - 2\tau + \langle\cdot,\cdot\rangle - \theta\,\mathrm{Ric}^{\mathrm{D}}).$$

Some facts (see Theorem 21.1 in [8] and nearby results):

- Different choices of T (hence τ) yield isometric manifolds.
- The Olszak distribution $\mathscr{D}$ is the vertical distribution $\mathscr{V}$ of the factor $T^*\Sigma$.
- (M, g) is Ricci-recurrent if and only if D is Ricci-recurrent.
- $\nabla R = 0 \iff \mathrm{Ric}^{\mathrm{D}}$ is D-parallel.
- The description of the Weyl tensor is not as simple as in Example 1: one can show that, locally, any ECS manifold with $d = 2$ is the total space of a bundle with fibers $\mathscr{D}^{\perp}$ over a surface Σ, which inherits a projectively flat torsionfree connection D, and the parallel 2-form ω defining W (as per Lemma 1) is the pull-back under the bundle projection of a D-parallel area form α on the surface.
- Again, we have $\mathrm{s} = 0$.

Remark. In the four-dimensional case, the vector space is $V = \{0\}$, so the local model is just $(T^*\Sigma, \mathtt{h}^{\mathrm{D}} - 2\tau)$ where D is a projectively flat torsionfree connection on Σ and $\tau \in \Gamma((T^*\Sigma)^{\odot 2})$ comes from $T \in \Gamma((T\Sigma)^{\odot 2})$ satisfying the double-divergence equation $\mathrm{div}^{\mathrm{D}}(\mathrm{div}^{\mathrm{D}}T) + \big(\mathrm{Ric}^{\mathrm{D}}, T\big) = \varepsilon$ via a D-parallel area form on Σ.

[3]Extensively studied in [13] (where several examples are given), [15], [17], [20], [4], [5], to name a few.

Theorem 2 (Derdzinski-Roter, [8], [11])

Every point of a ECS manifold of dimension at least 4 has a neighborhood isometric, for a suitable choice of "initial data", to an open subset of one of the manifolds described in Example 1 or Example 2.

Remark.

- The case where the dimension of the manifold is equal to 3 has also been discussed in [3], but the definition of being ECS has to be adjusted, as W automatically vanishes. We consider instead the Cotton-York tensor $\mathrm{CY} \doteq \mathrm{d}^{\nabla}\mathrm{Sch}$, whose vanishing is equivalent to conformal flatness. Motivated by this, we'll call a 3-dimensional manifold ECS if $\nabla(\mathrm{CY}) = 0$ with $\mathrm{CY} \neq 0$. All such manifolds are necessarily Lorentzian and fit in the $d = 1$ case, and each point has a neighborhood isometric to an open subset of $(\mathbb{R}^3, \mathrm{g}_{\mathfrak{a}} \doteq \mathrm{d}t\,\mathrm{d}y + \mathrm{d}x^2 + (x^3 + \mathfrak{a}(y)x)\,\mathrm{d}y^2)$, where $\mathfrak{a}$ is an arbitrary smooth function.

- Both the situation mentioned above, as well as the local model for $d = 1$, are particular cases — in the Lorentzian case — of a very particular class of spacetimes: *pp-wave spacetimes* (short for "plane-fronted waves with parallel rays"), first introduced by Ehlers and Kundt in [14]. Here's a coordinate-free definition: a Lorentzian manifold (M, g) is called a *pp-wave spacetime* if it admits a non-zero parallel null field $\boldsymbol{L} \in \mathfrak{X}(M)$ for which the connection induced on the quotient screen bundle $\boldsymbol{L}^{\perp}/\mathbb{R}\boldsymbol{L} \to M$ is flat[4]. Geometric properties of pp-waves have been thoroughly studied (even from a strictly mathematical point of view — e.g., every pp-wave is Ricci recurrent), for example, in [16] and [18]. In fact (see [11]), not only the Olszak distribution $\mathscr{D}$ of an ECS manifold (of *either* local type) has the property that the connection induced on $\mathscr{D}^{\perp}/\mathscr{D}$ is always flat, in the case $d = 1$ we also have that the connection induced on $\mathscr{D}$ itself is flat (which is to say that locally one may always choose the spanning null field for $\mathscr{D}$ to be parallel as well).

- The appearance of pseudo-Riemannian manifolds equipped with (in some sense) natural null parallel distributions is so frequent, that such manifolds — called Walker manifolds — have been thoroughly studied. If the null parallel distribution $\mathscr{P} \hookrightarrow TM$ has rank r and $n = \dim M$, then around each point in M there are coordinates $(x^1, \ldots, x^n)$ such that

$$(g_{ij})_{i,j=1}^{n} = \begin{pmatrix} 0 & 0 & \mathrm{Id}_r \\ 0 & A & H \\ \mathrm{Id}_r & H^{\top} & B \end{pmatrix},$$

  where A and B are square matrices of order $n - 2r$ and r, H has order $(n-2r) \times r$, and A and H are independent of $(x^1, \ldots, x^r)$ — moreover, $\mathscr{P}$ is locally spanned by $(\partial_i)_{i=1}^{r}$. For a proof, see the original paper [23] by Walker, or the presentation

[4] This is equivalent to writing that $R(\boldsymbol{X}, \boldsymbol{Y})\colon \boldsymbol{L}^{\perp} \to \mathbb{R}\boldsymbol{L}$ for all $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(M)$, and it in fact implies that $R(\boldsymbol{X}, \boldsymbol{Y}) = 0$ if $\boldsymbol{X}, \boldsymbol{Y} \in \Gamma(\boldsymbol{L}^{\perp})$.

given in [2]. The choice of such Walker coordinates is highly non-unique, but an invariant formulation of the structure described by such coordinates is given in [7].

What is known so far: the compact case

We have seen examples of ECS manifolds of both types: with $d = 1$ and $d = 2$. The next question is: are there *compact* examples? The answer is "yes", although there are still open questions regarding this matter.

Namely, examples of compact ECS manifolds with $d = 1$ (and hence all Ricci-recurrent) are known in dimensions of the form $3k + 2$, $k \geq 1$, and are due to Derdzinski and Roter (see [12]). They realize all indefinite metric signatures, and the resulting manifolds are diffeomorphic to non-trivial torus bundles over a circle. The (very technical) construction is summarized in [9]. We'll reproduce the main steps of such brief summary here not only for convenience of the reader, but to explain the reason for such cryptic dimensions[5].

Step 1: Take a $d = 1$ model (as in Example 1) with $I = \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ non-constant and periodic, with period $p > 0$. Let $\mathscr{E}$ be the space of $\mathscr{C}^\infty$ solutions to the differential equation $\ddot{u}(t) = f(t)u(t) + Au(t)$. There is a certain group (more on that later) $G \subseteq \mathbb{Z} \times \mathbb{R} \times \mathscr{E}$ acting on M by isometries.

Step 2: There is a discrete and properly discontinuous subgroup $\Gamma \leq G$ admiting a compact fundamental domain (i.e., such that the quotient M/Γ is a manifold and is compact) if and only if both conditions below hold:

(i) there is a periodic curve $B\colon \mathbb{R} \to \mathrm{GL}(V)$ (with same period $p > 0$) satisfying the differential equation $\dot{B}(t) + B(t)^2 = f(t) + A$ (such curves B are in bijective correspondence with what may be called first-order subspaces of $\mathscr{E}$, while $\mathscr{E}$ itself turns out to carry a natural symplectic structure Ω);

(ii) there is $\theta \geq 0$, a lattice $\Sigma \subseteq \mathbb{R} \times \mathscr{L}$ (where $\mathscr{L} = \mathscr{L}(B)$ is the (solution) vector space $\mathscr{L} = \{u \in \mathscr{C}^\infty(\mathbb{R}, V) \mid \dot{u}(t) = B(t)u(t)\}$), and a linear functional $\varphi \in \mathscr{L}^*$, such that

- $\Sigma \cap (\mathbb{R} \times \{0\}) = \mathbb{Z}\theta \times \{0\}$,
- $\Omega(\Sigma, \Sigma) \in \mathbb{Z}\theta$,
- $\Psi_{p,\varphi}[\Sigma] = \Sigma$, where $\Psi_{p,\varphi}\colon \mathscr{L} \to \mathscr{L}$ is given by $\Psi_{p,\varphi}(r, u) = (r + \varphi(u), Tu)$ and $(Tu)(t) = u(t - p)$.

Step 3: There is a lattice Σ satisfying condition (ii) in the previous step for some linear functional $\varphi \in \mathscr{L}^*$ if and only if $|\det T| = 1$ and the matrix of T relative to some basis of $\mathscr{L}$ is in $\mathrm{GL}(n, \mathbb{Z})$.

[5]Note that the property of being ECS is not preserved under products, just like being conformally flat is not.

With the three above steps in mind, which are true for any dimension greater or equal to 5 (in dimension 3 the problem is uninteresting, as discussed before, and for failure of this construction in dimension 4, see Theorem 8.1 in [12]), we see that *the goal is to find B such that (i) in* **Step 2** *holds, and for which the corresponding T makes* **Step 3** *work as well*. The next steps describe how this can be done when $\dim V = 3$ (we in fact may take $V = \mathbb{R}^3$ equipped with a pseudo-Euclidean scalar product of arbitrary signature, which will be used to identify matrices with endomorphisms of $\mathbb{R}^3$ via orthonormal bases) — in particular, suitable f and A will also be chosen here.

Step 4: As an attempt to reverse engineer the characteristic polynomial of the translation operator T, take integers k and ℓ such that $4 < k < \ell < k^2/4$ and let $P(x) = -x^3 + kx^2 - \ell x + 1$ (given $k > 4$, there is always such ℓ — e.g., by induction — while $k^2/4 > \ell$ says that $P(x)$ has two distinct critical points). This polynomial has three distinct roots λ, μ and ν satisfying the chain of inequalities

$$\ell^{-1} < \lambda < 1 < \mu < k/2 < \nu < k.$$

Step 5: Given $p > 0$, let $\mathscr{F}_p$ be the set of all $(\alpha, \beta, \gamma, f, a, b, c)$ such that:

(i) $\alpha, \beta, \gamma, f \colon \mathbb{R} \to \mathbb{R}$ are smooth, have period p, and $\alpha > \beta > \gamma$;

(ii) $a, b, c \in \mathbb{R}$ are pairwise distinct, $a + b + c = 0$ and $b < \min\{a, c\}$.

(iii) $\dot{\alpha} + \alpha^2 = f + a$, $\dot{\beta} + \beta^2 = f + b$ and $\dot{\gamma} + \gamma^2 = f + \gamma$.

Let $\mathscr{C} \subseteq \mathscr{F}_p$ consist of the $(\alpha, \beta, \gamma, f, a, b, c)$ with constant α, β, γ and f, and define $\operatorname{spec}\colon \mathscr{F}_p \to \mathbb{R}^3$ by

$$\operatorname{spec}(\alpha, \beta, \gamma, f, a, b, c) \doteq \left(\exp\left(-\int_0^p \alpha \right), \exp\left(-\int_0^p \beta \right), \exp\left(-\int_0^p \gamma \right) \right).$$

The reason for the name spec will be clear on the next step.

Step 6: Show that $\operatorname{spec}[\mathscr{F}_p \setminus \mathscr{C}]$ equals the open subset of $\mathbb{R}^3$ consisting of the (λ, μ, ν) satisfying the inequalities $0 < \lambda < \mu < \nu$, $\lambda < 1 < \nu$, $\lambda\mu < 1 < \mu\nu$ and $\lambda\nu \neq 1$, as per **Step 4**. In fact, given such (λ, μ, ν), the inverse image $\operatorname{spec}^{-1}(\lambda, \mu, \nu)$ is not only non-empty, but infinite-dimensional.

Step 7: This is where we put all of the previous steps together. Pick $p > 0$ and take (λ, μ, ν) as in **Step 4**. By **Step 6**, pick $(\alpha, \beta, \gamma, f, a, b, c) \in \mathscr{F}_p \setminus \mathscr{C}$ that gets sent to (λ, μ, ν) via the spec map. Now define

$$B(t) = \begin{pmatrix} \alpha(t) & 0 & 0 \\ 0 & \beta(t) & 0 \\ 0 & 0 & \gamma(t) \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

By **Step 5**, $\dot{B} + B^2 = f + A$ and $A \in \mathfrak{sl}_3(\mathbb{R})$. Since all the $B(t)$ commute, the spectrum of the translation operator T is precisely (λ, μ, ν), and thus the polynomial $P(x)$ of **Step 4** is the characteristic polynomial of T. Then T satisfies the condition given in **Step 3**, and we chose f non-constant, so **Step 2** does produce a compact 5-dimensional ECS manifold.

However, the same properties just produced above for operators in $\mathbb{R}^3$ remain valid if we consider the actions of their k-th cartesian powers in $\mathbb{R}^{3k}$. Thus, we obtain compact ECS manifolds of dimensions $3k+2$, and the metric signatures remain arbitrary because they were arbitrary in each $\mathbb{R}^3$ factor to begin with. Since $\mathbb{R}^2 \times V$ is simply connected, it follows that $\Gamma = \pi_1(M/\Gamma)$ and, as a consequence of this specific construction, M/Γ is the total space of a (non-trivial) bundle over a circle, whose fibers are tori. There are also other topological restrictions for the existence of ECS metrics on a given manifold. We'll list a few of them, established in [10].

Theorem 3

Let (M, g) be an ECS manifold with $\dim M \geq 4$. Then all the real Pontryagin classes $p_k(M) \in H^{4k}_{\mathrm{dR}}(M)$, $k \geq 1$, vanish. If, in addition, M is compact, then $\chi(M) = 0$ and $\pi_1(M)$ is infinite.

Remark. Recall that we may write $\det(\mathrm{Id} + tX) = 1 + f_1(X)t + \cdots + f_n(X)t^n$ for some $\mathrm{Ad}(\mathrm{GL}_n(\mathbb{R}))$-invariant polynomials $f_1(X), \ldots, f_n(X) \in \mathbb{R}[x^i{}_j : i, j = 1, \ldots, n]$, and for a pseudo-Riemannian manifold (M, g), we define the k-th Pontryagin class $p_k(M)$ as

$$p_k(M) \doteq \left[f_{2k}\left(\frac{\mathrm{i}\Omega}{2\pi}\right)\right] \in H^{4k}_{\mathrm{dR}}(M)$$

for all k, where Ω is the curvature 2-form matrix of the Levi-Civita connection of (M, g) relative to some local frame field. This construction is independent of the choice of local frame and we have globally defined closed differential forms, so the definition makes sense. Theorem 3 is essentially a consequence of the fact (see [1]) that these classes remain unchanged if instead of the Riemann tensor R, one uses the Weyl tensor W (and the Weyl tensor of an ECS manifold has plenty of nullity — see e.g. item (vi) in Lemma 2).

Theorem 4

Any Lorentzian 4-dimensional ECS manifold is non-compact.

Remark. The gist of the proof (done by contradiction) is showing that the universal cover of such a manifold is necessarily isometric to one of the $d = 1$ models given in Example 1, and showing that in dimension 4 such Lorentzian models cannot be the universal cover of any compact pseudo-Riemannian manifold. The details rely on lemmas which are similar in spirit to the arguments needed to establish **Step 2** on the construction of compact ECS manifolds previously outlined. Thus, any 4-dimensional compact ECS manifold must have neutral signature (while, on the other hand, Lorentzian ECS metrics must be of type $d = 1$, as they cannot support a two dimensional null distribution).

Theorem 5

Let (M, g) be a compact ECS Lorentzian manifold. Then there is a two-fold covering $\overline{M}$ of M which is the total space of a smooth bundle $\overline{M} \to \mathbb{S}^1$ whose typical fiber carries a torsionfree flat connection and a non-zero parallel field. More precisely, the fibers of the bundle are the inverse images of the leaves of $\mathscr{D}^\perp$ in M under the projection $\overline{M} \to M$.

To further elaborate on the last remark, we'll register one last result (in fact used in the proof of theorems 4 and 5):

Theorem 6

The Olszak distribution $\mathscr{D}$ of a ECS Lorentz manifold is one-dimensional and, passing to a two-fold covering if necessary, we may assume that $\mathscr{D}$ is trivial as a real line bundle (and so it is spanned by a *global* parallel null field).

With this in place, here are some interesting directions to pursue:

(1) Are there compact 4-dimensional ECS manifolds?

(2) Are there compact ECS manifolds of dimensions $n \geq 5$, other than of the form $n = 3k + 2$, $k \geq 1$?

(3) Does any torus admit an ECS metric? Or, is there a compact ECS manifold with Abelian fundamental group?

(4) Must compact ECS manifolds be of type $d = 1$? Or, must they be Ricci-recurrent, even if of type $d = 2$?

(5) Can a compact ECS manifold be locally homogeneous?

(6) For fun, if time allows: are there compact 3-dimensional ECS manifolds, in the sense of [3]?

Ideas and attack strategies — the crossroads

Compactness for $d = 1$. We will first attempt to explore the case $d = 1$, starting with a description of all the metric-preserving maps between two local models with the same dimension, to try and obtain information about local isometries of an arbitrary ECS manifold in general. So, consider a manifold (M, g) given as in Example 1, and $(\widetilde{M}, \widetilde{g})$ constructed in the same way, with all the initial data carrying tildes as well, except for the inner products on V and $\widetilde{V}$, both to be denoted by $\langle \cdot, \cdot \rangle$. To reduce the amount of indices to deal with, we will write, for any smooth map $F\colon I \times \mathbb{R} \times V \to N$ (where N is any manifold), $(\partial_v F)(t, s, v)\colon V \to T_{F(t,s,v)}N$ for the partial derivative of F relative to the V-factor (it is a linear map[6]). Let $\Phi\colon I\times\mathbb{R}\times V\to\widetilde{I}\times\mathbb{R}\times\widetilde{V}$ be an isometry, and write it as

$$\Phi(t,s,v)=(\widetilde{t}(t,s,v),\widetilde{s}(t,s,v),\widetilde{v}(t,s,v)).$$

Since Φ must take $\mathscr{D}$ to $\widetilde{\mathscr{D}}$ and parallel fields to parallel fields, we immediately get the relation $\Phi_*(\partial_s)=c\partial_{\widetilde{s}}$, for some non-zero constant $c\in\mathbb{R}$ (which is to say, we also have that $\partial_s\widetilde{t}=\partial_s\widetilde{v}=0$). Using $\Phi^*\widetilde{\mathrm{g}}=\mathrm{g}$ directly now, we obtain a few relations (in order):

(i) $\kappa=(\widetilde{\kappa}\circ\Phi)(\partial_t\widetilde{t})^2+(\partial_t\widetilde{t})(\partial_t\widetilde{s})+\langle\partial_t\widetilde{v},\partial_t\widetilde{v}\rangle$;

(ii) $\partial_t\widetilde{t}=c^{-1}$;

(iii) $0=c^{-1}(\widetilde{\kappa}\circ\Phi)(\partial_v\widetilde{t})+(2c)^{-1}(\partial_t\widetilde{s})(\partial_v\widetilde{t})+\langle\partial_t\widetilde{v},(\partial_v\widetilde{v})\cdot\rangle$;

(iv) $\partial_v\widetilde{t}=0$;

(v) $\partial_v\widetilde{v}$, at each point, is a linear isometry $V\to\widetilde{V}$.

Using equation (iv), equations (i) and (iii) become

(i') $\kappa=c^{-2}(\widetilde{\kappa}\circ\Phi)+c^{-1}(\partial_t\widetilde{s})+\langle\partial_t\widetilde{v},\partial_t\widetilde{v}\rangle$;

(iii') $0=(2c)^{-1}\partial_v\widetilde{s}+\langle\partial_t\widetilde{v},(\partial_v\widetilde{v})\cdot\rangle$.

With that said, we now observe that Φ must also take $\mathscr{D}^\perp$ to $\widetilde{\mathscr{D}}^\perp$, but these distributions are integrable with totally geodesic flat leaves (namely, $t=$ cte. and $\widetilde{t}=$ cte. — this is a general phenomenon that holds for arbitrary pp-wave spacetimes), so $\widetilde{t}$, $\widetilde{s}$ and $\widetilde{v}$ are affine functions of the variables s and v. This means that we may write

$$\Phi(t,s,v)=(c^{-1}t+T,B^s(t)v+a^s(t)+cs,B(t)v+a(t)),$$

where $B(t)=(\partial_v\widetilde{v})(t,s,v)$, $B^s(t)=(\partial_v\widetilde{s})(t,s,v)$, $a^s\colon I\to\mathbb{R}$ and $a\colon I\to\widetilde{V}$ are smooth, and $T\in\mathbb{R}$. Rewriting (iii') in terms of those new objects, we have that

(iii'') $0=B^s(t)+2c\langle a'(t),B(t)\cdot\rangle+2c\langle B'(t)v,B(t)\cdot\rangle$.

Looking at v-degrees, we conclude that

$$B^s(t)=-2c\langle a'(t),B(t)\cdot\rangle\quad\text{and}\quad\langle B'(t)v,B(t)\cdot\rangle=0,$$

but since each $B(t)$ is non-singular and $\langle\cdot,\cdot\rangle$ is non-degenerate, the second relation implies that $B'(t)=0$ and so we have a single isometry $B\colon V\to\widetilde{V}$. Repeating this v-degree argument with (i') instead yields, after a short computation, that

- $f(t)=c^{-2}\widetilde{f}(\widetilde{t})$;
- $\widetilde{A}=c^2BAB^{-1}$;
- $a^s(t)=-c\langle a'(t),a(t)\rangle+r$ for some $r\in\mathbb{R}$;
- $a''(t)=f(t)a(t)+c^{-2}\widetilde{A}a(t)$.

The issue with the differential equation describing a is that it depends on the parameter c coming from Φ itself. This is corrected with the following:

[6] With the natural identification $T_vV\cong V$ already in force.

Lemma 3

Let $\mathscr{E}(\Phi)$ be the space of solutions of the ODE associated to an isometry Φ, as above. Write $\mathscr{E}$ for the space of solutions to the ODE $b''(t) = \widetilde{f}(t)b(t) + \widetilde{A}b(t)$ (say, defined on the whole real line). The reparametrization map $\Xi\colon \mathscr{E}(\Phi) \to \mathscr{E}$ given by $(\Xi a)(t) = a(c(t-T))$ is an isomorphism.

So, not only each Ξ allows us to consider a single space $\mathscr{E}$ for all isometries, the space $\mathscr{E}$ carries a natural symplectic structure Ω, defined by

$$\Omega(b_1, b_2) = \langle (b_1)', b_2\rangle - \langle b_1, (b_2)'\rangle,$$

which we may use to describe compositions of isometries more precisely. So, let $q = T$, $p = c^{-1}$, and identify a with Ξa (effectively replacing $t \mapsto a(t)$ with $t \mapsto a(pt+q)$) to write the action of Φ as

$$(p,q,B,r,a)(t,s,v) = \big(pt+q, -\langle a'(pt+q), 2Bv + a(pt+q)\rangle + p^{-1}s + r, Bv + a(pt+q)\big)$$

Thus, taking $\widetilde{M} = M$, we see that the isometry group of a single model is a subgroup G of $\mathrm{Aff}(\mathbb{R}) \times \mathrm{O}(V, \langle\cdot,\cdot\rangle) \times \mathbb{R} \times \mathscr{E}$. Let's register the composition law:

Proposition 2

Let $\widetilde{\widetilde{M}}$ be a third ECS model with $d = 1$. The composition of two isometries $(p_1, q_1, B_1, r_1, a_1)\colon M \to \widetilde{M}$ and $(p_2, q_2, B_2, r_2, a_2)\colon \widetilde{M} \to \widetilde{\widetilde{M}}$ is given (with suggestive notation — for economy of space) by

$$(p_2, q_2, B_2, r_2, a_2)(p_1, q_1, B_1, r_1, a_1) = \begin{pmatrix} p_2p_1 \\ p_2q_1 + q_2 \\ B_2B_1 \\ p_2^{-1}r_1 + r_2 - \Omega(a_2, (p_2, q_2, B_2)a_1) \\ (p_2, q_2, B_2)a_1 + a_2 \end{pmatrix},$$

where we set

$$[(p_2, q_2, B_2)a_1](t) = B_2 a_1(p_2^{-1}(t - q_2)).$$

With this in place, we can look for subgroups Γ of G for which M/Γ is smooth and compact, dealing with a slightly more general setup than what was discussed in the previous section. When $n = 4$, V is 2-dimensional and Lorentzian — in this case, there aren't many choices of initial data A. This could also eventually be useful to offer some insight in the toy-case $n = 3$.

Compactness for $d = 2$. This is a much more subtle question, with several issues to be addressed, which we'll attempt to tackle after spending some time on the $d = 1$ case. Consider a 4-dimensional type $d = 2$ ECS model $(T^*\Sigma, \mathtt{h}^{\mathrm{D}} - 2\tau)$ as in Example 2. Let's look at two types of isometries:

- For every 1-form $\xi \in \Omega^1(\Sigma)$, one may consider the fiberwise linear translation $\Phi_\xi\colon T^*\Sigma \to T^*\Sigma$ given by $\Phi_\xi(x,p) = (x, p + \xi_x)$. Recalling that τ is being identified with $\pi^*\tau$, where $\pi\colon T^*\Sigma \to \Sigma$ is the bundle projection, we have that $\Phi_\xi^*\pi^*\tau = (\pi\circ\Phi_\xi)^*\tau = \pi^*\tau$, so Φ_ξ being an isometry or not depends only on whether Φ_ξ preserves $\mathtt{h}^{\mathrm{D}}$ or not. Using cotangent coordinates, for instance, it is easy to check that $\Phi_\xi^*(\mathtt{h}^{\mathrm{D}}) = \mathtt{h}^{\mathrm{D}} + 2\xi_{i;j}\mathrm{d}q^i\mathrm{d}q^j$, so Φ_ξ is an isometry if and only if the covariant differential $\mathrm{D}\xi$ is skew-symmetric (i.e., we have Killing's equation $\xi_{i;j} + \xi_{j;i} = 0$ for all indices i and j). Let's call such a ξ a Killing 1-form. The immediate idea is to try to find two linearly independent Killing 1-forms ξ^1 and ξ^2, so that we obtain a lattice Λ acting on $T^*\Sigma$. In particular, note that the existence of a non-vanishing global 1-form on Σ implies (via the Poincaré index formula) that, when Σ is compact, that $\chi(\Sigma) = 0$ — thus, having a global coframe gives that Σ is parallelizable (hence orientable, as an area form would be $\xi^1\wedge\xi^2$), and so Σ must be homeomorphic to a torus $\mathbb{T}^2$. Note that since ξ^1 and ξ^2 parallelize $T^*\Sigma$, so $T^*\Sigma/\Lambda \cong \Sigma\times\mathbb{T}^2 \cong \mathbb{T}^4$. But finding such independent Killing 1-forms does not seem to be easy.

- If $\varphi\colon \Sigma\to\Sigma$ is any diffeomorphism, one may consider its cotangent lift, that is, the map $\widehat{\varphi}\colon T^*\Sigma \to T^*\Sigma$ defined by $\widehat{\varphi}(x,p) = (\varphi(x), p\circ \mathrm{d}\varphi_x^{-1})$. If φ is an *affine* diffeomorphism of (Σ, D) which, in addition, preserves τ, then $\widehat{\varphi}$ is an isometry. To wit, we have that $\widehat{\varphi}^*\pi^*\tau = (\pi\circ\widehat{\varphi})^*\tau = (\varphi\circ\pi)^*\tau = \pi^*\varphi^*\tau$, and since π being a surjective submersion implies that π^* is injective, we have that $\widehat{\varphi}$ preserves $\pi^*\tau$ if and only if φ preserves τ, while φ preserving D means that it will also preserve anything built from D (for instance, $\mathscr{H}^{\mathrm{D}}$ and $\mathtt{h}^{\mathrm{D}}$) — of course, a coordinate computation using that φ^{-1} is affine also does the trick. First issue: it is not even clear whether such a φ exists. Second issue: if we let a subgroup $G \leq \mathrm{Diff}(\Sigma)$ act on $T^*\Sigma$ freely and properly discontinuously and Σ is compact, then G must be finite and $T^*\Sigma/G$ cannot be compact: to wit, G must leave $\Sigma \hookrightarrow T^*\Sigma$ invariant and then act freely and properly discontinuously on it as well — the orbits must be discrete and hence finite due to compactness, and freeness allows us to inject G into an orbit, so G must be finite as well. One may then take a G-invariant fiber metric on $T^*\Sigma$, and the "norm-squared" function, which is unbounded, passes to the quotient $T^*\Sigma/G$. Compact manifolds do not admit (continuous) unbounded functions, completing the argument.

The first thing to do here would be to try and find all isometries between two copies of the $d = 2$ local model, at least in dimension 4, just like we have done for the case $d = 1$ above. If one succeeds in finding a compact ECS manifold $(M, \mathtt{g})$ with dimension 4 and $d = 2$, one attempt to get examples in higher dimensions is by exploiting warped products. Namely, one may take a compact flat manifold (V, γ) (which is the quotient of an Euclidean space under the action of a Bieberbach group — such manifolds are locally isometrically covered by flat tori), and find a function $f\colon M\to\mathbb{R}$ whose gradient is tangent to $\mathscr{D}$ and satisfying $-2\,\mathrm{Hess}\,f = f\,\mathrm{Ric}$. Then $M\times_f V$ would be the desired example (see [8]) — it is also not clear whether compactness of M forbids the existence of such f.

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