Formulas with the covariant exterior derivative

Ivo Terek*

Fix throughout the text a smooth vector bundle $E \to M$ over a smooth manifold. Here we will discuss some basics about *exterior covariant derivatives* for vector bundle-valued forms and register some general formulas. Hopefully this will build some intuition for the object, and justify why people don't bother computing higher order covariant exterior derivatives *ad infinitum* – we will see that it is unnecessary, as everything ends up in terms of the usual curvature tensor.

Let $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ be a Koszul connection on E. An E-valued differential k-form is a section of the exterior bundle $(\bigwedge^k(T^*M)) \otimes E$. We will denote the space of E-valued differential k-forms by $\Omega^k(M;E)$. In particular, if $E = M \times \mathbb{R}$ is the trivial line bundle, the $(M \times \mathbb{R})$-valued differential k-forms are identified with the usual differential k-forms from calculus courses, and we set $\Omega^k(M) \doteq \Omega^k(M; M \times \mathbb{R})$. The exterior derivative operator $\mathrm{d}\colon \Omega^k(M) \to \Omega^{k+1}(M)$, given in a coordinate-free way by the formula

$$\begin{aligned}\mathrm{d}\omega(\boldsymbol{X}_0,\ldots,\boldsymbol{X}_k) = \sum_{i=0}^{k}(-1)^i \boldsymbol{X}_i(\omega(\boldsymbol{X}_0,\ldots,\widehat{\boldsymbol{X}_i},\ldots,\boldsymbol{X}_k)) \\ + \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([\boldsymbol{X}_i,\boldsymbol{X}_j],\ldots,\widehat{\boldsymbol{X}_i},\ldots,\widehat{\boldsymbol{X}_j},\ldots,\boldsymbol{X}_k),\end{aligned}$$

can be extended to the *covariant exterior derivative* $\mathrm{d}^\nabla\colon \Omega^k(M;E) \to \Omega^{k+1}(M;E)$ with the aid of ∇, and we define

$$\begin{aligned}(\mathrm{d}^\nabla\omega)(\boldsymbol{X}_0,\ldots,\boldsymbol{X}_k) \doteq \sum_{i=0}^{k}(-1)^i \nabla_{\boldsymbol{X}_i}(\omega(\boldsymbol{X}_0,\ldots,\widehat{\boldsymbol{X}_i},\ldots,\boldsymbol{X}_k)) \\ + \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([\boldsymbol{X}_i,\boldsymbol{X}_j],\ldots,\widehat{\boldsymbol{X}_i},\ldots,\widehat{\boldsymbol{X}_j},\ldots,\boldsymbol{X}_k).\end{aligned}$$

Choosing a torsion-free connection in TM to form covariant derivatives of ω, we may rewrite the above as

$$(\mathrm{d}^\nabla\omega)(\boldsymbol{X}_0,\ldots,\boldsymbol{X}_k) = \sum_{i=0}^{k}(-1)^i(\nabla_{\boldsymbol{X}_i}\omega)(\boldsymbol{X}_0,\ldots,\widehat{\boldsymbol{X}_i},\ldots,\boldsymbol{X}_k)$$

We have that $\mathrm{d}^2 = 0$, since the standard connection in $M \times \mathbb{R}$ is flat, but this is no longer the case for arbitrary connections in vector bundles, and the curvature tensor R^∇ of ∇, defined by

$$R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi = \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\psi - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\psi - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\psi,$$

*terekcouto.1@osu.edu

plays an important role. Let's illustrate that with the:

Proposition. *Given $\psi \in \Gamma(E) = \Omega^0(M;E)$, we have:*

(i) $(\mathrm{d}^\nabla \psi)(\boldsymbol{X}) = \nabla_{\boldsymbol{X}}\psi$.

(ii) $\big((\mathrm{d}^\nabla)^2\psi\big)(\boldsymbol{X},\boldsymbol{Y}) = R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi$.

(iii) $\big((\mathrm{d}^\nabla)^3\psi\big)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) = R^\nabla(\boldsymbol{X},\boldsymbol{Y})\nabla_{\boldsymbol{Z}}\psi + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\nabla_{\boldsymbol{X}}\psi + R^\nabla(\boldsymbol{Z},\boldsymbol{X})\nabla_{\boldsymbol{Y}}\psi$.

(iv)
$$\begin{aligned}\big((\mathrm{d}^\nabla)^4\psi\big)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) &= R^\nabla(\boldsymbol{X},\boldsymbol{Y})R^\nabla(\boldsymbol{Z},\boldsymbol{W})\psi + R^\nabla(\boldsymbol{Z},\boldsymbol{X})R^\nabla(\boldsymbol{Y},\boldsymbol{W})\psi \\ &\quad + R^\nabla(\boldsymbol{X},\boldsymbol{W})R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\psi + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})R^\nabla(\boldsymbol{X},\boldsymbol{W})\psi \\ &\quad + R^\nabla(\boldsymbol{W},\boldsymbol{Y})R^\nabla(\boldsymbol{X},\boldsymbol{Z})\psi + R^\nabla(\boldsymbol{Z},\boldsymbol{W})R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi.\end{aligned}$$

Proof:

(i) Obvious.

(ii) We have that

$$\begin{aligned}\big((\mathrm{d}^\nabla)^2\psi\big)(\boldsymbol{X},\boldsymbol{Y}) &= \big(\mathrm{d}^\nabla\mathrm{d}^\nabla\psi\big)(\boldsymbol{X},\boldsymbol{Y}) \\ &= \nabla_{\boldsymbol{X}}((\mathrm{d}^\nabla\psi)(\boldsymbol{Y})) - \nabla_{\boldsymbol{Y}}((\mathrm{d}^\nabla\psi)(\boldsymbol{X})) - (\mathrm{d}^\nabla\psi)([\boldsymbol{X},\boldsymbol{Y}]) \\ &= \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\psi - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\psi - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\psi \\ &= R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi.\end{aligned}$$

(iii) This time, we have:

$$\begin{aligned}&\big((\mathrm{d}^\nabla)^3\psi\big)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) = \big(\mathrm{d}^\nabla((\mathrm{d}^\nabla)^2\psi)\big)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) \\ &= \nabla_{\boldsymbol{X}}\big((\mathrm{d}^\nabla)^2\psi\big)(\boldsymbol{Y},\boldsymbol{Z}) - \nabla_{\boldsymbol{Y}}\big((\mathrm{d}^\nabla)^2\psi\big)(\boldsymbol{X},\boldsymbol{Z}) + \nabla_{\boldsymbol{Z}}\big((\mathrm{d}^\nabla)^2\psi\big)(\boldsymbol{X},\boldsymbol{Y}) \\ &\quad - \big((\mathrm{d}^\nabla)^2\psi\big)([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}) + \big((\mathrm{d}^\nabla)^2\psi\big)([\boldsymbol{X},\boldsymbol{Z}],\boldsymbol{Y}) - \big((\mathrm{d}^\nabla)^2\psi\big)([\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X}) \\ &= \nabla_{\boldsymbol{X}}R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\psi - \nabla_{\boldsymbol{Y}}R^\nabla(\boldsymbol{X},\boldsymbol{Z})\psi + \nabla_{\boldsymbol{Z}}R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi \\ &\quad - R([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z})\psi + R([\boldsymbol{X},\boldsymbol{Z}],\boldsymbol{Y})\psi - R^\nabla([\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X})\psi \\ &= \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{Z}}\psi - \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{Y}}\psi - \nabla_{\boldsymbol{X}}\nabla_{[\boldsymbol{Y},\boldsymbol{Z}]}\psi - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Z}}\psi + \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{X}}\psi \\ &\quad + \nabla_{\boldsymbol{Y}}\nabla_{[\boldsymbol{X},\boldsymbol{Z}]}\psi + \nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\psi - \nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\psi - \nabla_{\boldsymbol{Z}}\nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\psi - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\nabla_{\boldsymbol{Z}}\psi \\ &\quad + \nabla_{\boldsymbol{Z}}\nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\psi + \nabla_{[[\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}]}\psi + \nabla_{[\boldsymbol{X},\boldsymbol{Z}]}\nabla_{\boldsymbol{Y}}\psi - \nabla_{\boldsymbol{Y}}\nabla_{[\boldsymbol{X},\boldsymbol{Z}]}\psi - \nabla_{[[\boldsymbol{X},\boldsymbol{Z}],\boldsymbol{Y}]}\psi \\ &\quad - \nabla_{[\boldsymbol{Y},\boldsymbol{Z}]}\nabla_{\boldsymbol{X}}\psi + \nabla_{\boldsymbol{X}}\nabla_{[\boldsymbol{Y},\boldsymbol{Z}]}\psi + \nabla_{[[\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X}]}\psi \\ &= R^\nabla(\boldsymbol{X},\boldsymbol{Y})\nabla_{\boldsymbol{Z}}\psi + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\nabla_{\boldsymbol{X}}\psi + R^\nabla(\boldsymbol{Z},\boldsymbol{X})\nabla_{\boldsymbol{Y}}\psi,\end{aligned}$$

since the $\nabla.\nabla_{[\cdot,\cdot]}$ terms cancel in pairs and the three $\nabla_{[[\cdot,\cdot],\cdot]}$ terms add to zero in view of the Jacobi identity for the Lie bracket.

(iv) One can explore symmetries to deal efficiently with what would be a total of 90 terms, but you can check the black box in the end of the text for the gory details.

□

Remark.

- One can make reasonable guesses for what happens to $(\mathrm{d}^\nabla)^k\psi$ with $k \geq 5$. It is probably possible to write a neat summation in terms of certain elements of the symmetric group S_k. I will not pursue this further, given that everything will be expressed in terms of R^∇.

- If we take coordinates (x^i) for M and local trivializing sections (e_a) for E, we see that in the same way that $\mathrm{d}f = (\partial_i f)\,\mathrm{d}x^i = f_{;i}\,\mathrm{d}x^i$, we have that

$$(\mathrm{d}^\nabla\psi)(\boldsymbol{X}) = \nabla_{\boldsymbol{X}}\psi = X^i\nabla_{\partial_i}\psi = X^i\psi^a{}_{;i}e_a = \psi^a{}_{;i}\,\mathrm{d}x^i(\boldsymbol{X})e_a,$$

so if $\psi = \psi^a e_a$, then $\mathrm{d}^\nabla\psi = \psi^a{}_{;i}\,\mathrm{d}x^i \otimes e_a$. Also, item (ii) becomes

$$(\mathrm{d}^\nabla)^2\psi = R_{jka}{}^b\psi^a\,\mathrm{d}x^j \otimes \mathrm{d}x^k \otimes e_b.$$

Compare the relation between the coordinate expressions for $\mathrm{d}^\nabla\psi$ and $(\mathrm{d}^\nabla)^2\psi$ with the coordinate Ricci identity $R_{jka}{}^b\psi^a = \psi^b{}_{;jk} - \psi^b{}_{;kj}$, obtained with the aid of a torsion-free connection in TM. The coordinate-free version of this identity is $R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi = (\nabla_{\boldsymbol{X}}(\nabla\psi))\boldsymbol{Y} - (\nabla_{\boldsymbol{Y}}(\nabla\psi))\boldsymbol{X}$, and the second order covariant derivatives are defined by $(\nabla_{\partial_j}(\nabla\psi))\partial_k = \psi^a{}_{;jk}e_a$.

So we have seen how d^∇ acts over $\Omega^0(M;E)$. What about $\Omega^1(M;E)$?

Proposition. *Let $\alpha \in \Omega^1(M)$ and $\psi \in \Gamma(E)$, and consider $\alpha\otimes\psi \in \Omega^1(M;E)$. Then*

$$\mathrm{d}^\nabla(\alpha\otimes\psi) = \mathrm{d}\alpha\otimes\psi - \alpha\wedge\nabla\psi,$$

where the wedge product is naturally extended as a map $\Omega^k(M)\times\Omega^\ell(M;E) \to \Omega^{k+\ell}(M;E)$, for any $k,\ell \geq 0$.

Proof: Take $\boldsymbol{X},\boldsymbol{Y} \in \mathfrak{X}(M)$ and compute

$$\begin{aligned}\mathrm{d}^\nabla(\alpha\otimes\psi)(\boldsymbol{X},\boldsymbol{Y}) &= \nabla_{\boldsymbol{X}}(\alpha(\boldsymbol{Y})\psi) - \nabla_{\boldsymbol{Y}}(\alpha(\boldsymbol{X})\psi) - \alpha([\boldsymbol{X},\boldsymbol{Y}])\psi \\ &= \boldsymbol{X}(\alpha(\boldsymbol{Y}))\psi + \alpha(\boldsymbol{Y})\nabla_{\boldsymbol{X}}\psi - \boldsymbol{Y}(\alpha(\boldsymbol{X}))\psi - \alpha(\boldsymbol{X})\nabla_{\boldsymbol{Y}}\psi - \alpha([\boldsymbol{X},\boldsymbol{Y}])\psi \\ &= \mathrm{d}\alpha(\boldsymbol{X},\boldsymbol{Y})\psi - \alpha(\boldsymbol{X})\nabla_{\boldsymbol{Y}}\psi + \alpha(\boldsymbol{Y})\nabla_{\boldsymbol{X}}\psi,\end{aligned}$$

as wanted. □

This is a particular instance of a more general phenomenon about "simple" forms:

Proposition. *Let $\alpha \in \Omega^k(M)$ and $\psi \in \Gamma(E)$, and consider $\alpha\otimes\psi \in \Omega^k(M;E)$. Then $\mathrm{d}^\nabla(\alpha\otimes\psi) \in \Omega^{k+1}(M;E)$ is given by the formula*

$$\mathrm{d}^\nabla(\alpha\otimes\psi) = \mathrm{d}\alpha\otimes\psi + (-1)^k\alpha\wedge\nabla\psi.$$

Proof: Take $X_0, \ldots, X_k \in \mathfrak{X}(M)$ and compute

$$\begin{aligned} \mathrm{d}^\nabla(\alpha \otimes \psi)(X_0, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i \nabla_{X_i}\big((\alpha \otimes \psi)(X_0, \ldots, \widehat{X_i}, \ldots, X_k)\big) \\ &\qquad + \sum_{0 \le i < j \le k} (-1)^{i+j} (\alpha \otimes \psi)([X_i, X_j], \cdots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k) \\ &= \sum_{i=0}^{k} (-1)^i \nabla_{X_i}\big(\alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k)\psi\big) \\ &\qquad + \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([X_i, X_j], \cdots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k)\psi \\ &= \sum_{i=0}^{k} (-1)^k X_i(\alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k))\psi \\ &\quad + \sum_{i=0}^{k} (-1)^k \alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k) \nabla_{X_i}\psi \\ &\quad + \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([X_i, X_j], \cdots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k)\psi \\ &= \mathrm{d}\alpha(X_0, \ldots, X_k)\psi + (-1)^k (\alpha \wedge \nabla\psi)(X_0, \ldots, X_k). \end{aligned}$$

□

The curvature R^∇ itself may be regarded as a $\mathrm{End}(E)$-valued 2-form. That is, we have $R^\nabla \in \Omega^2(M; \mathrm{End}(E))$. Since the connection ∇ in E induces a connection in $\mathrm{End}(E)$ via Leibniz rule, it makes sense to talk about $\mathrm{d}^\nabla R^\nabla \in \Omega^3(M; \mathrm{End}(E))$. Beware of the abuse of notation in this last d^∇, as it refers to the connection in $\mathrm{End}(E)$, also denoted by ∇. We have the:

Proposition. $\mathrm{d}^\nabla R^\nabla = 0$.

Proof: We start looking at the endomorphism level

$$\begin{aligned} (\mathrm{d}^\nabla R^\nabla)(X, Y, Z) &= \nabla_X R^\nabla(Y, Z) - \nabla_Y R^\nabla(X, Z) + \nabla_Z R^\nabla(X, Y) \\ &\qquad - R^\nabla([X, Y], Z) + R^\nabla([X, Z], Y) - R^\nabla([Y, Z], X). \end{aligned}$$

Now let's probe the above expression with a section $\psi \in \Gamma(E)$ and use the definition of the connection in $\operatorname{End}(E)$:

$$
\begin{aligned}
(\mathrm{d}^\nabla R^\nabla)(X,Y,Z)\psi &= \nabla_X R^\nabla(Y,Z)\psi - R^\nabla(Y,Z)\nabla_X\psi - \nabla_Y R^\nabla(X,Z)\psi \\
&\quad + R^\nabla(X,Z)\nabla_Y\psi + \nabla_Z R^\nabla(X,Y)\psi - R^\nabla(X,Y)\nabla_Z\psi \\
&\quad - R^\nabla([X,Y],Z)\psi + R^\nabla([X,Z],Y)\psi - R^\nabla([Y,Z],X)\psi \\
&= \nabla_X\nabla_Y\nabla_Z\psi - \nabla_X\nabla_Z\nabla_Y\psi - \nabla_X\nabla_{[Y,Z]}\psi - \nabla_Y\nabla_Z\nabla_X\psi \\
&\quad + \nabla_Z\nabla_Y\nabla_X\psi + \nabla_{[Y,Z]}\nabla_X\psi - \nabla_Y\nabla_X\nabla_Z\psi + \nabla_Y\nabla_Z\nabla_X\psi \\
&\quad + \nabla_Y\nabla_{[X,Z]}\psi + \nabla_X\nabla_Z\nabla_Y\psi - \nabla_Z\nabla_X\nabla_Y\psi - \nabla_{[X,Z]}\nabla_Y\psi \\
&\quad + \nabla_Z\nabla_X\nabla_Y\psi - \nabla_Z\nabla_Y\nabla_X\psi - \nabla_Z\nabla_{[X,Y]}\psi - \nabla_X\nabla_Y\nabla_Z\psi \\
&\quad + \nabla_Y\nabla_X\nabla_Z\psi + \nabla_{[X,Y]}\nabla_Z\psi - \nabla_{[X,Y]}\nabla_Z\psi + \nabla_Z\nabla_{[X,Y]}\psi \\
&\quad + \nabla_{[[X,Y],Z]}\psi + \nabla_{[X,Z]}\nabla_Y\psi - \nabla_Y\nabla_{[X,Z]}\psi - \nabla_{[[X,Z],Y]}\psi \\
&\quad - \nabla_{[Y,Z]}\nabla_X\psi + \nabla_X\nabla_{[Y,Z]}\psi + \nabla_{[[Y,Z],X]}\psi \\
&= 0,
\end{aligned}
$$

since terms in the same color group cancel in pairs, and the $\nabla_{[[\cdot,\cdot],\cdot]}\psi$ terms add to zero in view of the Jacobi identity for Lie brackets. □

Remark. In the above proof, note that we do not form $\nabla_X R^\nabla$ at any moment – as this would require a choice of connection in TM.

Alongside $\mathrm{d}^\nabla R^\nabla$, we also could talk about ∇R^∇. But a priori this can have more than one meaning. Let's clarify this with the:

Proposition. *Assume that we have a non-degenerate fiber metric $\langle\cdot,\cdot\rangle \in \Gamma(E^*\otimes E^*)$ which is ∇-parallel, i.e., $\nabla\langle\cdot,\cdot\rangle = 0$. Fix any auxiliary connection in TM, to form covariant derivatives of the curvature. Let's adopt here different notations for different guises of the curvature of ∇:*

$$
\begin{aligned}
&R^{1,3}\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E),\\
&R^{0,4}\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\times\Gamma(E)\times\Gamma(E)\to\mathscr{C}^\infty(M)\\
&R^{0,2}\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\to\operatorname{End}(E).
\end{aligned}
$$

Then we have:

(i) $(\nabla_X R^{0,4})(Y,Z,\psi,\phi) = \langle(\nabla_X R^{1,3})(Y,Z,\psi),\phi\rangle$.

(ii) $(\nabla_X R^{0,2})(Y,Z)\psi = (\nabla_X R^{1,3})(Y,Z,\psi)$.

In particular, $\nabla R^{1,3} = 0$ if and only if $\nabla R^{0,4} = 0$, if and only if $\nabla R^{0,2} = 0$. So $\nabla R = 0$ has only one possible meaning.

Proof: Unwind the definitions. There is no tricky term gathering this time. □

Corollary (Second Bianchi Identity). *In the same setting as the previous proposition, we have the following (equivalent) versions of the second Bianchi identity:*

(i) $(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$;

(ii) $(\nabla_{\boldsymbol{X}} R)(\boldsymbol{Y}, \boldsymbol{Z}, \psi) + (\nabla_{\boldsymbol{Y}} R)(\boldsymbol{Z}, \boldsymbol{X}, \psi) + (\nabla_{\boldsymbol{Z}} R)(\boldsymbol{X}, \boldsymbol{Y}, \psi) = 0$;

(iii) $(\nabla_{\boldsymbol{X}} R)(\boldsymbol{Y}, \boldsymbol{Z}, \psi, \phi) + (\nabla_{\boldsymbol{Y}} R)(\boldsymbol{Z}, \boldsymbol{X}, \psi, \phi) + (\nabla_{\boldsymbol{Z}} R)(\boldsymbol{X}, \boldsymbol{Y}, \psi, \phi) = 0$;

Here's another important application of the derivative d^∇. Any $\mathscr{C}^\infty(M)$-bilinear map $B\colon \mathfrak{X}(M) \times \Gamma(E) \to \mathscr{C}^\infty(M)$ can be seen as an element of $\Gamma(T^*M \otimes E^*)$, and also as a E^*-valued 1-form, simply by $\boldsymbol{X} \mapsto B(\boldsymbol{X}, \cdot)$. Thus $B \in \Omega^1(M; E^*)$. Since E^* also inherits a connection from E, which we'll also denote by ∇, it makes sense to talk about $\mathrm{d}^\nabla B \in \Omega^2(M; E^*)$.

Proposition. *Let $B \in \Gamma(T^*M \otimes E^*)$. If ∇ also denotes a torsion-free connection in TM, then the formula*

$$(\mathrm{d}^\nabla B)(\boldsymbol{X}, \boldsymbol{Y})\psi = (\nabla_{\boldsymbol{X}} B)(\boldsymbol{Y}, \psi) - (\nabla_{\boldsymbol{Y}} B)(\boldsymbol{X}, \psi)$$

holds for every $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(M)$ and $\psi \in \Gamma(E)$. Here, the covariant derivatives of B are formed using the van der Waerden-Bortolotti (direct sum) connection in $TM \oplus E$.

Remark. If $\mathrm{d}^\nabla B = 0$, we say that B is a *Codazzi tensor*.

Proof: We start with

$$(\mathrm{d}^\nabla B)(\boldsymbol{X}, \boldsymbol{Y}) = \nabla_{\boldsymbol{X}} B(\boldsymbol{Y}, \cdot) - \nabla_{\boldsymbol{Y}} B(\boldsymbol{X}, \cdot) - B([\boldsymbol{X}, \boldsymbol{Y}], \cdot),$$

and now we evaluate at ψ to get

$$\begin{aligned}(\mathrm{d}^\nabla B)(\boldsymbol{X}, \boldsymbol{Y})\psi &= \boldsymbol{X}(B(\boldsymbol{Y}, \psi)) - B(\boldsymbol{Y}, \nabla_{\boldsymbol{X}}\psi) - \boldsymbol{Y}(B(\boldsymbol{X}, \psi)) + B(\boldsymbol{X}, \nabla_{\boldsymbol{Y}}\psi) - B([\boldsymbol{X}, \boldsymbol{Y}], \psi) \\ &= \boldsymbol{X}(B(\boldsymbol{Y}, \psi)) - B(\nabla_{\boldsymbol{X}}\boldsymbol{Y}, \psi) - B(\boldsymbol{Y}, \nabla_{\boldsymbol{X}}\psi) - \boldsymbol{Y}(B(\boldsymbol{X}, \psi)) + B(\boldsymbol{X}, \nabla_{\boldsymbol{Y}}\psi) + B(\nabla_{\boldsymbol{Y}}\boldsymbol{X}, \psi) \\ &= (\nabla_{\boldsymbol{X}} B)(\boldsymbol{Y}, \psi) - (\nabla_{\boldsymbol{Y}} B)(\boldsymbol{X}, \psi),\end{aligned}$$

as wanted. $\square$

Remark. With respect to local coordinates (x^j) on M and local trivializing sections (e_a) for E, with duals (e^a), write $B = B_{ja}\,\mathrm{d}x^j \otimes e^a$. Then we have that

$$\mathrm{d}^\nabla B = (B_{ka;j} - B_{ja;k})\,\mathrm{d}x^j \otimes \mathrm{d}x^k \otimes e^a = \sum_{j<k}(B_{ka;j} - B_{ja;k})\,\mathrm{d}x^j \wedge \mathrm{d}x^k \otimes e^a,$$

where the sum over a is still understood and we only use the summation in the last step to indicate that the sum there is no longer taken over all values of j and k. Another way to express this is by $(\mathrm{d}^\nabla B)_{jka} = B_{ka;j} - B_{ja;k}$.

If we apply the above for a usual differential form, how will d and d^∇ relate?

Corollary. *If $\alpha \in \Omega^2(M)$ is seen as an element of $\Omega^1(M; T^*M)$ and ∇ is a torsion-free connection in TM, then*

$$\mathrm{d}\alpha(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = (\mathrm{d}^\nabla \alpha)(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z} + (\nabla_{\boldsymbol{Z}}\alpha)(\boldsymbol{X}, \boldsymbol{Y}),$$

for all $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathfrak{X}(M)$.

For $E = TM$, there is one particular TM-valued 1-form we can consider: the identity $\mathrm{Id}\colon TM \to TM$. The torsion of ∇ is also expressed in terms of the operation d^∇, as

$$\tau^\nabla(\boldsymbol{X},\boldsymbol{Y}) = (\mathrm{d}^\nabla \mathrm{Id})(\boldsymbol{X},\boldsymbol{Y}) = \nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X} - [\boldsymbol{X},\boldsymbol{Y}].$$

How far will this go?

Proposition.

(i) $(\mathrm{d}^\nabla\tau^\nabla)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) = R^\nabla(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\boldsymbol{X} + R^\nabla(\boldsymbol{Z},\boldsymbol{X})\boldsymbol{Y}$.

(ii)
$$\begin{aligned}((\mathrm{d}^\nabla)^2\tau^\nabla)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) = {} & R^\nabla(\boldsymbol{X},\boldsymbol{Y})\tau^\nabla(\boldsymbol{Z},\boldsymbol{W}) + R^\nabla(\boldsymbol{Z},\boldsymbol{X})\tau^\nabla(\boldsymbol{Y},\boldsymbol{W}) \\ & + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\tau^\nabla(\boldsymbol{X},\boldsymbol{W}) + R^\nabla(\boldsymbol{W},\boldsymbol{X})\tau^\nabla(\boldsymbol{Z},\boldsymbol{Y}) \\ & + R^\nabla(\boldsymbol{Z},\boldsymbol{W})\tau^\nabla(\boldsymbol{X},\boldsymbol{Y}) + R^\nabla(\boldsymbol{W},\boldsymbol{Y})\tau^\nabla(\boldsymbol{X},\boldsymbol{Z}).\end{aligned}$$

Remark. This says that the content of the first Bianchi identity

$$R^\nabla(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\boldsymbol{X} + R^\nabla(\boldsymbol{Z},\boldsymbol{X})\boldsymbol{Y} = \boldsymbol{0}$$

for torsion-free connections is just "the derivative of zero equals zero".

Proof:

(i) We're going to brute force our way through:

$$\begin{aligned}(\mathrm{d}^\nabla\tau^\nabla)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) &= \nabla_{\boldsymbol{X}}\tau^\nabla(\boldsymbol{Y},\boldsymbol{Z}) - \nabla_{\boldsymbol{Y}}\tau^\nabla(\boldsymbol{X},\boldsymbol{Z}) + \nabla_{\boldsymbol{Z}}\tau^\nabla(\boldsymbol{X},\boldsymbol{Y}) \\ &\qquad - \tau^\nabla([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}) + \tau^\nabla([\boldsymbol{X},\boldsymbol{Z}],\boldsymbol{Y}) - \tau^\nabla([\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X}) \\ &= \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Z}}\boldsymbol{Y} - \nabla_{\boldsymbol{X}}[\boldsymbol{Y},\boldsymbol{Z}] - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} + \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{Z}}\boldsymbol{X} \\ &\quad + \nabla_{\boldsymbol{Y}}[\boldsymbol{X},\boldsymbol{Z}] + \nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{Y}}\boldsymbol{X} - \nabla_{\boldsymbol{Z}}[\boldsymbol{X},\boldsymbol{Y}] - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} \\ &\quad + \nabla_{\boldsymbol{Z}}[\boldsymbol{X},\boldsymbol{Y}] + [[\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}] + \nabla_{[\boldsymbol{X},\boldsymbol{Z}]}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}[\boldsymbol{X},\boldsymbol{Z}] - [[\boldsymbol{X},\boldsymbol{Z}],\boldsymbol{Y}] \\ &\quad - \nabla_{[\boldsymbol{Y},\boldsymbol{Z}]}\boldsymbol{X} + \nabla_{\boldsymbol{X}}[\boldsymbol{Y},\boldsymbol{Z}] + [[\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X}] \\ &= R^\nabla(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\boldsymbol{X} + R^\nabla(\boldsymbol{Z},\boldsymbol{X})\boldsymbol{Y},\end{aligned}$$

where the $\nabla_{[\cdot,\cdot]}$ terms cancel in pairs, the Jacobi identity for Lie brackets takes care of the three terms without ∇, and the remaining ones gather into curvatures according to the colors above.

(ii) Black box #2.

□

Black box #1:

Let's compute $(\mathrm{d}^\nabla)^4\psi$ for $\psi \in \Gamma(E)$.

$$
\begin{aligned}
((\mathrm{d}^\nabla)^4\psi)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) &= \\
&= \nabla_{\boldsymbol{X}}((\mathrm{d}^\nabla)^3\psi)(\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) - \nabla_{\boldsymbol{Y}}((\mathrm{d}^\nabla)^3\psi)(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{W}) \\
&\quad + \nabla_{\boldsymbol{Z}}((\mathrm{d}^\nabla)^3\psi)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{W}) - \nabla_{\boldsymbol{W}}((\mathrm{d}^\nabla)^3\psi)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) \\
&\quad - ((\mathrm{d}^\nabla)^3\psi)([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z},\boldsymbol{W}) + ((\mathrm{d}^\nabla)^3\psi)([\boldsymbol{X},\boldsymbol{Z}],\boldsymbol{Y},\boldsymbol{W}) \\
&\quad - ((\mathrm{d}^\nabla)^3\psi)([\boldsymbol{X},\boldsymbol{W}],\boldsymbol{Y},\boldsymbol{Z}) - ((\mathrm{d}^\nabla)^3\psi)([\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X},\boldsymbol{W}) \\
&\quad + ((\mathrm{d}^\nabla)^3\psi)([\boldsymbol{Y},\boldsymbol{W}],\boldsymbol{X},\boldsymbol{Z}) - ((\mathrm{d}^\nabla)^3\psi)([\boldsymbol{Z},\boldsymbol{W}],\boldsymbol{X},\boldsymbol{Y}) \\
&= \nabla_{\boldsymbol{X}}R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\nabla_{\boldsymbol{W}}\psi + \nabla_{\boldsymbol{X}}R^\nabla(\boldsymbol{Z},\boldsymbol{W})\nabla_{\boldsymbol{Y}}\psi + \nabla_{\boldsymbol{X}}R^\nabla(\boldsymbol{W},\boldsymbol{Y})\nabla_{\boldsymbol{Z}}\psi \\
&\quad -\nabla_{\boldsymbol{Y}}R^\nabla(\boldsymbol{X},\boldsymbol{Z})\nabla_{\boldsymbol{W}}\psi - \nabla_{\boldsymbol{Y}}R^\nabla(\boldsymbol{Z},\boldsymbol{W})\nabla_{\boldsymbol{X}}\psi - \nabla_{\boldsymbol{Y}}R^\nabla(\boldsymbol{W},\boldsymbol{X})\nabla_{\boldsymbol{Z}}\psi \\
&\quad +\nabla_{\boldsymbol{Z}}R^\nabla(\boldsymbol{X},\boldsymbol{Y})\nabla_{\boldsymbol{W}}\psi + \nabla_{\boldsymbol{Z}}R^\nabla(\boldsymbol{Y},\boldsymbol{W})\nabla_{\boldsymbol{X}}\psi + \nabla_{\boldsymbol{Z}}R^\nabla(\boldsymbol{W},\boldsymbol{X})\nabla_{\boldsymbol{Y}}\psi \\
&\quad -\nabla_{\boldsymbol{W}}R^\nabla(\boldsymbol{X},\boldsymbol{Y})\nabla_{\boldsymbol{Z}}\psi - \nabla_{\boldsymbol{W}}R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\nabla_{\boldsymbol{X}}\psi - \nabla_{\boldsymbol{W}}R^\nabla(\boldsymbol{Z},\boldsymbol{X})\nabla_{\boldsymbol{Y}}\psi \\
&\quad -R^\nabla([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z})\nabla_{\boldsymbol{W}}\psi - R^\nabla(\boldsymbol{Z},\boldsymbol{W})\nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\psi - R^\nabla(\boldsymbol{W},[\boldsymbol{X},\boldsymbol{Y}])\nabla_{\boldsymbol{Z}}\psi \\
&\quad +R^\nabla([\boldsymbol{X},\boldsymbol{Z}],\boldsymbol{Y})\nabla_{\boldsymbol{W}}\psi + R^\nabla(\boldsymbol{Y},\boldsymbol{W})\nabla_{[\boldsymbol{X},\boldsymbol{Z}]}\psi + R^\nabla(\boldsymbol{W},[\boldsymbol{X},\boldsymbol{Z}])\nabla_{\boldsymbol{Y}}\psi \\
&\quad -R^\nabla([\boldsymbol{X},\boldsymbol{W}],\boldsymbol{Y})\nabla_{\boldsymbol{Z}}\psi - R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\nabla_{[\boldsymbol{X},\boldsymbol{W}]}\psi - R^\nabla(\boldsymbol{Z},[\boldsymbol{X},\boldsymbol{W}])\nabla_{\boldsymbol{Y}}\psi \\
&\quad -R^\nabla([\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X})\nabla_{\boldsymbol{W}}\psi - R^\nabla(\boldsymbol{X},\boldsymbol{W})\nabla_{[\boldsymbol{Y},\boldsymbol{Z}]}\psi - R^\nabla(\boldsymbol{W},[\boldsymbol{Y},\boldsymbol{Z}])\nabla_{\boldsymbol{X}}\psi \\
&\quad +R^\nabla([\boldsymbol{Y},\boldsymbol{W}],\boldsymbol{X})\nabla_{\boldsymbol{Z}}\psi + R^\nabla(\boldsymbol{X},\boldsymbol{Z})\nabla_{[\boldsymbol{Y},\boldsymbol{W}]}\psi + R^\nabla(\boldsymbol{Z},[\boldsymbol{Y},\boldsymbol{W}])\nabla_{\boldsymbol{X}}\psi \\
&\quad -R^\nabla([\boldsymbol{Z},\boldsymbol{W}],\boldsymbol{X})\nabla_{\boldsymbol{Y}}\psi - R^\nabla(\boldsymbol{X},\boldsymbol{Y})\nabla_{[\boldsymbol{Z},\boldsymbol{W}]}\psi - R^\nabla(\boldsymbol{Y},[\boldsymbol{Z},\boldsymbol{W}])\nabla_{\boldsymbol{X}}\psi
\end{aligned}
$$

Since all the colored groups are (up to sign) permutations of the sum in red, it suffices to analyze the combination in red. We have that

$$
\begin{aligned}
\text{sum of terms in red} &= \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{W}}\psi - \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{W}}\psi - \nabla_{\boldsymbol{X}}\nabla_{[\boldsymbol{Y},\boldsymbol{Z}]}\nabla_{\boldsymbol{W}}\psi \\
&\quad - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{W}}\psi + \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{W}}\psi + \nabla_{\boldsymbol{Y}}\nabla_{[\boldsymbol{X},\boldsymbol{Z}]}\nabla_{\boldsymbol{W}}\psi \\
&\quad + \nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{W}}\psi - \nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{W}}\psi - \nabla_{\boldsymbol{Z}}\nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\nabla_{\boldsymbol{W}}\psi \\
&\quad - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{W}}\psi + \nabla_{\boldsymbol{Z}}\nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\nabla_{\boldsymbol{W}}\psi + \nabla_{[[\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}]}\nabla_{\boldsymbol{W}}\psi \\
&\quad + \nabla_{[\boldsymbol{X},\boldsymbol{Z}]}\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{W}}\psi - \nabla_{\boldsymbol{Y}}\nabla_{[\boldsymbol{X},\boldsymbol{Z}]}\nabla_{\boldsymbol{W}}\psi - \nabla_{[[\boldsymbol{X},\boldsymbol{Z}],\boldsymbol{Y}]}\nabla_{\boldsymbol{W}}\psi \\
&\quad - \nabla_{[\boldsymbol{Y},\boldsymbol{Z}]}\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{W}}\psi + \nabla_{\boldsymbol{X}}\nabla_{[\boldsymbol{Y},\boldsymbol{Z}]}\nabla_{\boldsymbol{W}}\psi + \nabla_{[[\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X}]}\nabla_{\boldsymbol{W}}\psi \\
&= R^\nabla(\boldsymbol{X},\boldsymbol{Y})\nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{W}}\psi + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{W}}\psi + R^\nabla(\boldsymbol{Z},\boldsymbol{X})\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{W}}\psi,
\end{aligned}
$$

since the $\nabla_\cdot\nabla_{[\cdot,\cdot]}\nabla_{\boldsymbol{W}}\psi$ terms cancel in pairs, the $\nabla_{[[\cdot,\cdot],\cdot]}\nabla_{\boldsymbol{W}}\psi$ terms add to zero in view of the Jacobi identity for the Lie bracket, and the remaining terms gather into curvatures. Now, we have the symmetries:

- the sum in blue is minus the sum in red with $\boldsymbol{Z}$ and $\boldsymbol{W}$ switched;
- the sum in teal is minus the sum in red with $\boldsymbol{Y}$ and $\boldsymbol{W}$ switched;
- the sum in purple is minus the sum in red with $\boldsymbol{X}$ and $\boldsymbol{W}$ switched.

Thus, organizing the entries of $R^\nabla(\cdot,\cdot)$, we have:

$$\begin{aligned}
\text{sum of terms in red} &= R^\nabla(\boldsymbol{X},\boldsymbol{Y})\nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{W}}\psi + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{W}}\psi + R^\nabla(\boldsymbol{Z},\boldsymbol{X})\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{W}}\psi \\
\text{sum of terms in blue} &= -R^\nabla(\boldsymbol{X},\boldsymbol{Y})\nabla_{\boldsymbol{W}}\nabla_{\boldsymbol{Z}}\psi + R^\nabla(\boldsymbol{W},\boldsymbol{Y})\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Z}}\psi + R^\nabla(\boldsymbol{X},\boldsymbol{W})\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{Z}}\psi \\
\text{sum of terms in teal} &= -R^\nabla(\boldsymbol{X},\boldsymbol{W})\nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{Y}}\psi + R^\nabla(\boldsymbol{Z},\boldsymbol{W})\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\psi - R^\nabla(\boldsymbol{Z},\boldsymbol{X})\nabla_{\boldsymbol{W}}\nabla_{\boldsymbol{Y}}\psi \\
\text{sum of terms in purple} &= -R^\nabla(\boldsymbol{W},\boldsymbol{Y})\nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{X}}\psi - R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\nabla_{\boldsymbol{W}}\nabla_{\boldsymbol{X}}\psi - R^\nabla(\boldsymbol{Z},\boldsymbol{W})\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\psi
\end{aligned}$$

Putting all of that together and recalling the six surviving $R^\nabla(\cdot,\cdot)\nabla_{[\cdot,\cdot]}\psi$ terms, we get:

$$\begin{aligned}
((\mathrm{d}^\nabla)^4\psi)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) = {}& R^\nabla(\boldsymbol{X},\boldsymbol{Y})R^\nabla(\boldsymbol{Z},\boldsymbol{W})\psi + R^\nabla(\boldsymbol{Z},\boldsymbol{X})R^\nabla(\boldsymbol{Y},\boldsymbol{W})\psi \\
&+ R^\nabla(\boldsymbol{X},\boldsymbol{W})R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\psi + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})R^\nabla(\boldsymbol{X},\boldsymbol{W})\psi \\
&+ R^\nabla(\boldsymbol{W},\boldsymbol{Y})R^\nabla(\boldsymbol{X},\boldsymbol{Z})\psi + R^\nabla(\boldsymbol{Z},\boldsymbol{W})R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi,
\end{aligned}$$

as wanted.

Black box #2:

Let's compute $(\mathrm{d}^\nabla)^2\tau^\nabla$.

$$\begin{aligned}((\mathrm{d}^\nabla)^2\tau^\nabla)(X,Y,Z,W) = \\ &= \nabla_X(\mathrm{d}^\nabla\tau^\nabla)(Y,Z,W) - \nabla_Y(\mathrm{d}^\nabla\tau^\nabla)(X,Z,W) \\ &\quad + \nabla_Z(\mathrm{d}^\nabla\tau^\nabla)(X,Y,W) - \nabla_W(\mathrm{d}^\nabla\tau^\nabla)(X,Y,Z) \\ &\quad - (\mathrm{d}^\nabla\tau^\nabla)([X,Y],Z,W) + (\mathrm{d}^\nabla\tau^\nabla)([X,Z],Y,W) \\ &\quad - (\mathrm{d}^\nabla\tau^\nabla)([X,W],Y,Z) - (\mathrm{d}^\nabla\tau^\nabla)([Y,Z],X,W) \\ &\quad + (\mathrm{d}^\nabla\tau^\nabla)([Y,W],X,Z) - (\mathrm{d}^\nabla\tau^\nabla)([Z,W],X,Y) \\ &= \nabla_X R^\nabla(Y,Z)W + \nabla_X R^\nabla(Z,W)Y + \nabla_X R^\nabla(W,Y)Z \\ &\quad -\nabla_Y R^\nabla(X,Z)W - \nabla_Y R^\nabla(Z,W)X - \nabla_Y R^\nabla(W,X)Z \\ &\quad +\nabla_Z R^\nabla(X,Y)W + \nabla_Z R^\nabla(Y,W)X + \nabla_Z R^\nabla(W,X)Y \\ &\quad -\nabla_W R^\nabla(X,Y)Z - \nabla_W R^\nabla(Y,Z)X - \nabla_W R^\nabla(Z,X)Y \\ &\quad -R^\nabla([X,Y],Z)W - R^\nabla(Z,W)[X,Y] - R^\nabla(W,[X,Y])Z \\ &\quad +R^\nabla([X,Z],Y)W + R^\nabla(Y,W)[X,Z] + R^\nabla(W,[X,Z])Y \\ &\quad -R^\nabla([X,W],Y)Z - R^\nabla(Y,Z)[X,W] - R^\nabla(Z,[X,W])Y \\ &\quad -R^\nabla([Y,Z],X)W - R^\nabla(X,W)[Y,Z] - R^\nabla(W,[Y,Z])X \\ &\quad +R^\nabla([Y,W],X)Z + R^\nabla(X,Z)[Y,W] + R^\nabla(Z,[Y,W])X \\ &\quad -R^\nabla([Z,W],X)Y - R^\nabla(X,Y)[Z,W] - R^\nabla(Y,[Z,W])X\end{aligned}$$

We use the same strategy as before and gather the above terms in convenient groups, sorted by colors.

$$\begin{aligned}\text{sum of terms in red} &= \nabla_X\nabla_Y\nabla_Z W - \nabla_X\nabla_Z\nabla_Y W - \nabla_X\nabla_{[Y,Z]}W \\ &\quad - \nabla_Y\nabla_X\nabla_Z W + \nabla_Y\nabla_Z\nabla_X W + \nabla_Y\nabla_{[X,Z]}W \\ &\quad + \nabla_Z\nabla_X\nabla_Y W - \nabla_Z\nabla_Y\nabla_X W - \nabla_Z\nabla_{[X,Y]}W \\ &\quad - \nabla_{[X,Y]}\nabla_Z W + \nabla_Z\nabla_{[X,Y]}W + \nabla_{[[X,Y],Z]}W \\ &\quad + \nabla_{[X,Z]}\nabla_Y W - \nabla_Y\nabla_{[X,Z]}W - \nabla_{[[X,Z],Y]}W \\ &\quad - \nabla_{[Y,Z]}\nabla_X W + \nabla_X\nabla_{[Y,Z]}W + \nabla_{[[Y,Z],X]}W \\ &= R^\nabla(X,Y)\nabla_Z W + R^\nabla(Z,X)\nabla_Y W + R^\nabla(Y,Z)\nabla_X W\end{aligned}$$

since the $\nabla_\cdot\nabla_{[\cdot,\cdot]}W$ terms cancel in pairs the the $\nabla_{[[\cdot,\cdot],\cdot]}W$ terms add to zero in view of the Jacobi identity for the Lie bracket. The symmetries here are:

- the sum in blue is minus the sum in red with Y and W switched;
- the sum in teal is minus the sum in red with Z and W switched;
- the sum in purple is minus the sum in red with X and W switched.

So:

$$\begin{aligned}
\text{sum of terms in red} &= R^\nabla(\boldsymbol{X},\boldsymbol{Y})\nabla_{\boldsymbol{Z}}\boldsymbol{W} + R^\nabla(\boldsymbol{Z},\boldsymbol{X})\nabla_{\boldsymbol{Y}}\boldsymbol{W} + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\nabla_{\boldsymbol{X}}\boldsymbol{W} \\
\text{sum of terms in blue} &= -R^\nabla(\boldsymbol{X},\boldsymbol{W})\nabla_{\boldsymbol{Z}}\boldsymbol{Y} - R^\nabla(\boldsymbol{Z},\boldsymbol{X})\nabla_{\boldsymbol{W}}\boldsymbol{Y} - R^\nabla(\boldsymbol{W},\boldsymbol{Z})\nabla_{\boldsymbol{X}}\boldsymbol{Y} \\
\text{sum of terms in teal} &= -R^\nabla(\boldsymbol{X},\boldsymbol{Y})\nabla_{\boldsymbol{W}}\boldsymbol{Z} - R^\nabla(\boldsymbol{W},\boldsymbol{X})\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - R^\nabla(\boldsymbol{Y},\boldsymbol{W})\nabla_{\boldsymbol{X}}\boldsymbol{Z} \\
\text{sum of terms in purple} &= -R^\nabla(\boldsymbol{W},\boldsymbol{Y})\nabla_{\boldsymbol{Z}}\boldsymbol{X} - R^\nabla(\boldsymbol{Z},\boldsymbol{W})\nabla_{\boldsymbol{Y}}\boldsymbol{X} - R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\nabla_{\boldsymbol{W}}\boldsymbol{X}
\end{aligned}$$

Together with the last six remaining terms, it follows that

$$\begin{aligned}
\big((\mathrm{d}^\nabla)^2\tau^\nabla\big)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) &= R^\nabla(\boldsymbol{X},\boldsymbol{Y})\tau^\nabla(\boldsymbol{Z},\boldsymbol{W}) + R^\nabla(\boldsymbol{Z},\boldsymbol{X})\tau^\nabla(\boldsymbol{Y},\boldsymbol{W}) \\
&\quad + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\tau^\nabla(\boldsymbol{X},\boldsymbol{W}) + R^\nabla(\boldsymbol{W},\boldsymbol{X})\tau^\nabla(\boldsymbol{Z},\boldsymbol{Y}) \\
&\quad + R^\nabla(\boldsymbol{Z},\boldsymbol{W})\tau^\nabla(\boldsymbol{X},\boldsymbol{Y}) + R^\nabla(\boldsymbol{W},\boldsymbol{Y})\tau^\nabla(\boldsymbol{X},\boldsymbol{Z}),
\end{aligned}$$

as wanted.