AFFINE MODIFICATIONS ON CURVATURE

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If M is a smooth manifold, then the space $\mathrm{Conn}(TM)$ of affine connections on the tangent bundle of M is an affine space, whose associated translation space is the space of type $(1,2)$-tensor fields on M, i.e., sections of the vector bundle $[T^*M]^{\otimes 2} \otimes TM$ over M. For any $\nabla \in \mathrm{Conn}(TM)$, we have τ^∇, R^∇, and Ric^∇. Let's see how each of them behaves when we add a tensor field A to ∇. Write:

- $\widetilde{A}$ for (twice) the skew-symmetrization of A, i.e., $\widetilde{A}_X Y = A_X Y - A_Y X$.
- A_{τ^∇} for the type $(1,3)$-tensor field given by $(X,Y,Z) \mapsto A_{\tau^\nabla(X,Y)} Z$.
- $\mathrm{tr}_1^1(A)$ for the 1-form defined by $\mathrm{tr}_1^1(A)(Y) = \mathrm{tr}(AY)$, as well as $\mathrm{tr}_1^1(A_{\tau^\nabla})$ for the type $(0,2)$-tensor field given by $\mathrm{tr}_1^1(A_{\tau^\nabla})(Y,Z) = \mathrm{tr}\big(X \mapsto A_{\tau^\nabla(X,Y)} Z\big)$.
- $[A,A]$ for the commutator given by $[A,A](X,Y) = 2[A_X, A_Y]$.
- $\mathrm{d}^\nabla A$ for the exterior derivative (computed with the aid of ∇) of the tensor A regarded a $\mathrm{End}(TM)$-valued 1-form.
- div^∇ for the ∇-divergence of A, i.e., the $(0,2)$-tensor field defined by the trace $\mathrm{div}^\nabla(A)(X,Y) = \mathrm{tr}\big(Z \mapsto (\nabla_Z A)(X,Y)\big)$.
- $\mathrm{tr}(A \circ A)$ for the map $(Y,Z) \mapsto \mathrm{tr}(AZ \circ A_Y)$.
- $\nabla\omega$ is given by $(X,Y) \mapsto [\nabla_X \omega]Y$ (in this order of arguments), for each 1-form ω on M.

Theorem

(i) $\tau^{\nabla + A} = \tau^\nabla + \widetilde{A}$.

(ii) $R^{\nabla + A} = R^\nabla + \mathrm{d}^\nabla A + [A,A]/2$.

(iii) $\mathrm{Ric}^{\nabla+A} = \mathrm{Ric}^\nabla + \mathrm{div}^\nabla A - \nabla \mathrm{tr}_1^1(A) + \mathrm{tr}_1^1(A) \circ A - \mathrm{tr}(A \circ A) + \mathrm{tr}_1^1(A_{\tau^\nabla})$.

Proof:

(i) This is the easiest one to establish, as expected:

$$\begin{aligned}
\tau^{\nabla+A}(X,Y) &= (\nabla + A)_X Y - (\nabla + A)_Y X - [X,Y] \\
&= \nabla_X Y + A_X Y - \nabla_Y X - A_Y X - [X,Y] \\
&= \tau^\nabla(X,Y) + \widetilde{A}_X Y.
\end{aligned}$$

(ii) This is a slightly longer brute-force computation. First we compute

$$(\nabla + A)_X(\nabla + A)_Y Z = \nabla_X \nabla_Y Z + \nabla_X A_Y Z + A_X \nabla_Y Z + A_X A_Y Z$$

as well as

$$(\nabla + A)_{[X,Y]} Z = \nabla_{[X,Y]} Z + A_{[X,Y]} Z,$$

so that

$$R^{\nabla + A}(X,Y)Z = R^{\nabla}(X,Y)Z + [A_X, A_Y]Z + (*),$$

where

$$(*) = \nabla_X A_Y Z + A_X \nabla_Y Z - \nabla_Y A_X Z - A_Y \nabla_X Z - A_{[X,Y]} Z.$$

It remains to see that the above expression indeed equals $(\mathrm{d}^{\nabla} A)(X,Y)Z$. But

$$\begin{aligned}(\mathrm{d}^{\nabla} A)(X,Y)Z &= (\nabla_X(A_Y) - \nabla_Y(A_X) - A_{[X,Y]})Z \\ &= \nabla_X A_Y Z - A_Y \nabla_X Z - \nabla_Y A_X Z + A_X \nabla_Y Z - A_{[X,Y]} Z,\end{aligned}$$

as required, by definition of exterior derivative.

(iii) It suffices to compute traces in the first variable of $\mathrm{d}^{\nabla} A$ and $[A,A]/2$. Let's do a coordinate computation[1]. Fix $Y = \partial_r$ and $Z = \partial_s$. First, we have that

$$\begin{aligned}[A_{\partial_j}, A_{\partial_r}]\partial_s &= A_{\partial_j} A_{\partial_r} \partial_s - A_{\partial_r} A_{\partial_j} \partial_s \\ &= A_{\partial_j}(A^k_{rs}\partial_k) - A_{\partial_r}(A^k_{js}\partial_k) \\ &= (A^k_{rs} A^i_{jk} - A^k_{js} A^i_{rk})\partial_i,\end{aligned}$$

so that contracting $j = i$ yields

$$\mathrm{tr}\big(X \mapsto [A_X, A_{\partial_r}]\partial_s\big) = A^k_{rs} A^i_{ik} - A^k_{is} A^i_{rk} = (\mathrm{tr}^1_1(A))_k A^k_{rs} - A^k_{is} A^i_{rk}.$$

Hence $\mathrm{tr}\big(X \mapsto [A_X, A_Y]Z\big) = \mathrm{tr}^1_1(A) A_Y Z - \mathrm{tr}(AZ \circ A_Y)$ for general Y and Z, by tensoriality. Lastly, we focus on the exterior derivative of A. Namely, recall that

$$(\mathrm{d}^{\nabla} A)_{jrs}{}^i = A^i_{rs;j} - A^i_{js;r} + \tau^k_{jr} A^i_{ks},$$

where τ^k_{jr} are the coordinate components of τ^{∇}. Now, contract $j = i$ to obtain

$$(\mathrm{d}^{\nabla} A)_{irs}{}^i = (\mathrm{div}^{\nabla} A)_{rs} - \mathrm{tr}^1_1(A)_{s;r} + \tau^k_{ir} A^i_{ks},$$

as required.

□

Remark. The torsion $\tau\colon \mathrm{Conn}(TM) \to \Gamma([T^*M]^{\wedge 2} \otimes TM)$ is an affine map, whose associated linear part is (twice) skew-symmetrization. The Riemann map $\nabla \mapsto R^{\nabla}$ and the Ricci map $\nabla \mapsto \mathrm{Ric}^{\nabla}$, in turn, are not affine (as we have obtained terms which are manifestly quadratic in the variable A, in the expressions for the differences $R^{\nabla+A} - R^{\nabla}$ and $\mathrm{Ric}^{\nabla+A} - \mathrm{Ric}^{\nabla}$).

[1] If you see a quick(er) way to do this without coordinates, let me know.