About curvaturelike tensors

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Let V be a finite-dimensional real vector space. Denote by

$$\mathcal{T}^r_s(V) = \{T\colon (V^*)^r \times V^s \to \mathbb{R} \mid T \text{ is multilinear}\}$$

the space of type (r,s)-tensors on V. Recall that $\mathcal{T}^r_s(V) \cong V^{\otimes r} \otimes (V^*)^{\otimes s}$, and note that $\mathcal{T}^0_1(V) \cong V^*$, $\mathcal{T}^1_0(V) \cong V^*$, and $\mathcal{T}^1_1(V) \cong \mathrm{Lin}(V)$, where $\mathrm{Lin}(V)$ denotes the space of all linear endomorphisms of V. The latter isomorphism is characterized by regarding a pure tensor as an endomorphism acting by $(x \otimes f)(y) = f(y)x$. We will focus on a special type of tensors here:

Definition 1. A *curvaturelike* tensor on V is a tensor $R \in \mathcal{T}^0_4(V)$ satisfying the symmetries

(i) $R(x,y,z,w) = -R(y,x,z,w) = -R(x,y,w,z)$;

(ii) $R(x,y,z,\cdot) + R(y,z,x,\cdot) + R(z,x,y,\cdot) = 0$,

for all $x,y,z \in V$. We will denote the space of curvaturelike tensors on V by $\mathcal{R}(V)$.

Remark. In other words, (i) says that R is skew on the first pair of arguments, and also skew in the second pair of arguments, while (ii) is a *"first Bianchi identity"* (familiar from Riemannian geometry). Together, (i) and (ii) imply the *pair-symmetry*:

(iii) $R(x,y,z,w) = R(z,w,x,y)$

To justify this, just compute:

$$\begin{aligned}
2R(x,y,z,w) &\overset{(i)}{=} R(x,y,z,w) + R(y,x,w,z) \\
&\overset{(ii)}{=} -R(y,z,x,w) - R(z,x,y,w) - R(x,w,y,z) - R(w,y,x,z) \\
&\overset{(i)}{=} -R(z,y,w,x) - R(x,z,w,y) - R(w,x,z,y) - R(y,w,z,x) \\
&\overset{(ii)}{=} R(w,z,y,x) + R(z,w,x,y) \\
&\overset{(i)}{=} 2R(z,w,x,y).
\end{aligned}$$

Also, *Milnor's octahedron* gives a geometric interpretation for this argument. An improved version using a tetrahedron instead may be consulted in [1].

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Let's get out of the way a very useful way to check when two curvaturelike tensors are equal:

Proposition 2. *Let $R_1, R_2 \in \mathscr{R}(V)$ be such that $R_1(x,y,y,x) = R_2(x,y,y,x)$, for all elements $x, y \in V$. Then $R_1 = R_2$.*

Proof: Assume without loss of generality that $R_2 = 0$ and denote R_1 just by R. The proof goes simply by polarizating twice, as follows: $R(x+w,y,y,x+w) = 0$ becomes

$$R(x,y,y,w) + R(w,y,y,x) = 0,$$

but also $R(w,y,y,x) \stackrel{(i)}{=} R(y,w,x,y) \stackrel{(iii)}{=} R(x,y,y,w)$, and so $R(x,y,y,w) = 0$. Now, $R(x,y+z,y+z,w) = 0$ becomes

$$R(x,y,z,w) + R(x,z,y,w) = 0,$$

meaning that R is skew in the middle pair. However, we have that

$$\begin{aligned} R(x,z,y,w) &\stackrel{(i)}{=} -R(z,x,y,w) \stackrel{(ii)}{=} R(x,y,z,w) + R(y,z,x,w) \\ &\stackrel{(*)}{=} R(x,y,z,w) - R(y,x,z,w) \stackrel{(i)}{=} 2R(x,y,z,w), \end{aligned}$$

where in $(*)$ we use the just discovered skew-symmetry of the middle pair. Thus $3R(x,y,z,w) = 0$, and so $R = 0$. $\square$

If V is also equipped with a pseudo-Euclidean scalar product[1] g, we may identify $V \cong V^*$, by $x \mapsto x_\flat = g(x,\cdot)$, and so we can regard a tensor product $x \otimes y$ either as an endomorphism via $(x \otimes y)(z) = g(y,z)x$, or instead as a $(0,2)$-tensor acting via $(x \otimes y)(z,w) = g(y,z)g(x,w)$. Skew-symmetrizing such expression allows us to see the wedge product $x \wedge y$ either as a skew-adjoint endomorphism (with respect to g) via $(x \wedge y)(z) = g(y,z)x - g(x,z)y$, or a skew-symmetric bilinear form

$$(x \wedge y)(z,w) = g(y,z)g(x,w) - g(x,z)g(y,w).$$

We now change our point of view, focusing on g instead of the vectors, and write the above as $(g \owedge g)(x,y,z,w) \doteq g(y,z)g(x,w) - g(x,z)g(y,w)$. This is our first example of a curvaturelike tensor: $g \owedge g \in \mathscr{R}(V)$. Symmetry (i) is immediate, while the Bianchi identity follows by a direct calculation, with the six terms canceling in pairs. We will call $g \owedge g$ the *fundamental curvature* of (V, g). To milk more examples out of this idea, we regard $g \mapsto g \owedge g$ as a quadratic form in the variable g, and polarize. Recall that if Q is a quadratic form, we can recover the associated bilinear form B by

$$B(x,y) = \frac{Q(x+y) - Q(x) - Q(y)}{2}.$$

Then $Q(x) = B(x,x)$. So, let $T, S \in \mathscr{T}^0_2(V)$. Compute

$$\begin{aligned} &\big((T+S) \owedge (T+S)\big)(x,y,z,w) - (T \owedge T)(x,y,z,w) - (S \owedge S)(x,y,z,w) = \\ &= (T+S)(y,z)(T+S)(x,w) - (T+S)(x,z)(T+S)(y,w) \\ &\quad - T(y,z)T(x,w) + T(x,z)T(y,w) - S(y,z)S(x,w) + S(x,z)S(y,w) \\ &= T(y,z)S(x,w) + S(y,z)T(x,w) - T(x,z)S(y,w) - S(x,z)T(y,w). \end{aligned}$$

This motivates the:

[1] Which we'll assume, throughout the text, that is never negative-definite.

Definition 3. Let $T, S \in \mathcal{T}^0_2(V)$. The *Kulkarni-Nomizu* product of T and S is the tensor $T \owedge S$ defined by

$$2(T \owedge S)(x,y,z,w) = T(y,z)S(x,w) - T(x,z)S(y,w) + S(y,z)T(x,w) - S(x,z)T(y,w).$$

So we have a *symmetric* map $\owedge \colon \mathcal{T}^0_2(V) \times \mathcal{T}^0_2(V) \to \mathcal{T}^0_4(V)$.

Remark.

- Note that the factor 2 in the left side above appeared naturally, as a mere consequence of the polarization process.
- If (e_i) is a basis for V, then we have the coordinate expression

$$2(T \owedge S)_{ijk\ell} = T_{jk}S_{i\ell} - T_{ik}S_{j\ell} + S_{jk}T_{i\ell} - S_{ik}T_{j\ell}.$$

It is also easy to see that the product $T \owedge S$ not only always satisfies symmetry (i) from Definition 1, but also satisfies symmetry (iii). So, to say that $T \owedge S$ is a curvature-like tensor, it remains to check symmetry (ii), the Bianchi identity. However, it is not guaranteed to hold. Brute force says that (twice) the left side of the Bianchi identity for $(T \owedge S)(x,y,z,w)$ actually equals the sum of six terms

$$\begin{aligned}(T(y,z) - T(z,y))S(x,w) &+ (T(z,x) - T(x,z))S(y,w) \\ &+ (T(x,y) - T(y,x))S(z,w) + (\text{three more terms, but switching } T \leftrightarrow S).\end{aligned}$$

This definitely vanishes if both T and S are symmetric tensors. Thus we have proved the:

Proposition 4. *Let $T, S \in \mathcal{T}^0_2(V)$. If both T and S are symmetric, then $T \owedge S \in \mathcal{R}(V)$ is curvaturelike.*

Remark.

- We'll let $\mathcal{S}(V)$ denote the space of all type $(0,2)$ symmetric tensors.
- Focusing on such symmetries allows us to give another interpretation of the Kulkarni-Nomizu product. If $R \in \mathcal{R}(V)$, symmetry (i) says that R defines a map $R' \colon V^{\wedge 2} \times V^{\wedge 2} \to \mathbb{R}$, characterized by its action on pairs of 2-blades as $R'(x \wedge y, z \wedge w) \doteq R(x,y,z,w)$. Now, symmetry (iii) induces yet another linear map $R'' \colon V^{\wedge 2} \odot V^{\wedge 2} \to \mathbb{R}$ (here $\odot$ denotes symmetric product), characterized by $R''((x \wedge y) \odot (z \wedge w)) = R(x,y,z,w)$. On the other hand, if V is equipped with a pseudo-Euclidean scalar product g, we have an induced scalar product in $V^{\wedge 2}$ (also denotes by g), acting on 2-blades by

$$g(x \wedge y, z \wedge y) = \det \begin{pmatrix} g(x,z) & g(x,w) \\ g(y,z) & g(y,w) \end{pmatrix} = g(x,z)g(y,w) - g(x,w)g(y,z).$$

  This induced scalar product is precisely $-(g \owedge g)''$. So in some sense, $\owedge$ is the identity (up to a sign depending on the conventions adopted).

Now that we have this small factory of examples, given by the operation $\owedge$, let's move on and establish a few algebraic features of curvaturelike tensors.

Proposition 5. *If* $\dim V = n$, *then* $\dim \mathscr{R}(V) = n^2(n^2-1)/12$.

Proof: As mentioned in the previous remark, we can see any curvaturelike tensor $R \in \mathscr{R}(V)$ as a symmetric bilinear map $V^{\wedge 2} \odot V^{\wedge 2} \to \mathbb{R}$ or, equivalently, as a self-adjoint endomorphism of $V^{\wedge 2}$. Moreover, $\mathscr{R}(V)$ is precisely the space of such endomorphisms which lie in the kernel of the "Bianchi" map, which is a surjection onto $V^{\wedge 4}$, as it is a multiple of the skew-symmetrization operation. Thus

$$\dim(V^{\wedge 2} \odot V^{\wedge 2}) = \dim \mathscr{R}(V) + \dim(V^{\wedge 4}).$$

Since $\dim(V^{\wedge 2}) = n(n-1)/2$, we have that

$$\dim \mathscr{R}(V) = \frac{1}{2}\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2}+1\right) - \binom{n}{4} = \frac{n^2(n^2-1)}{12}.$$

$\square$

In particular, when $\dim V = 2$, we have that $\dim \mathscr{R}(V) = 1$. In this case, if V is equipped with a pseudo-Euclidean scalar product g, it follows that $\mathscr{R}(V) = \mathbb{R}(g \owedge g)$, which is to say, any $R \in \mathscr{R}(V)$ is of the form $R = Kg \owedge g$, for some constant $K \in \mathbb{R}$ (to be thought as the "g-Gaussian curvature" of R). In particular, taking an orthonormal basis (e_1, e_2) of V with e_1 spacelike, we evaluate both sides on the 4-uple (e_1, e_2, e_2, e_1) to conclude that $K = \varepsilon R(e_1, e_2, e_2, e_1)$, where $\varepsilon = g(e_2, e_2)$ is 1 if g is Euclidean, and -1 if g is Lorentzian.

When $\dim V > 2$, our goal is to decompose the space of curvaturelike tensors as something of the form $\mathscr{R}(V) = \mathbb{R}(g \owedge g) \oplus$ (some extra factors). To describe said extra factors, we'll recall briefly the notion of *contraction*, and focus on the particular contraction known as the *Ricci contraction*.

If $T \in \mathscr{T}^r_s(V)$ and $1 \leq a \leq r$, $1 \leq b \leq s$, we define $\operatorname{tr}^a_b(T) \in \mathscr{T}^{r-1}{}_{s-1}(V)$ in terms of components relative to a basis by

$$(\operatorname{tr}^a_b(T))^{i_1 \ldots i_{r-1}}{}_{j_1 \ldots j_{s-1}} = T^{i_1 \ldots i_{a-1} k i_a \ldots i_{r-1}}{}_{j_1 \ldots j_{b-1} k j_b \ldots j_{s-1}},$$

where k appear in the a-th upper slot and the b-th lower slot, and this does not depend on the choice of basis, due to the coordinate transformation law for tensors. When V has a pseudo-Euclidean inner product (which we'll assume is always the case, from here onwards) and r or s is at least 2, we also define *metric traces* $\operatorname{tr}_{a,b}(T) \in \mathscr{T}^r_{s-2}(V)$ and $\operatorname{tr}^{a,b}(T) \in \mathscr{T}^{r-2}{}_s(V)$ by raising or lowering the an index, and applying tr^a_b. That is, we put

$$\operatorname{tr}_{a,b}(T)(f^1, \ldots, f^r, x_1, \ldots, x_{s-2}) \doteq \operatorname{tr}_{1,2}\big(T(f^1, \ldots, f^r, x_1, \ldots, \bullet, \ldots, \bullet, \ldots, x_{s-2})\big),$$

and

$$\operatorname{tr}^{a,b}(T)(f^1, \ldots, f^{r-2}, x_1, \ldots, x_s) \doteq \operatorname{tr}^{1,2}\big(T(f^1, \ldots, \bullet, \ldots, \bullet, \ldots, f^{r-2}, x_1, \ldots, x_s)\big),$$

where the •s are in the correct slots, and those are given in terms of a basis by

$$(\mathrm{tr}_{a,b}(T))^{i_1\ldots i_r}{}_{j_1\ldots j_{s-2}} = g^{k\ell}T^{i_1\ldots i_r}{}_{j_1\ldots k\ldots \ell\ldots j_{s-2}} \quad \text{and} \quad (\mathrm{tr}^{a,b}(T))^{i_1\ldots i_{r-2}}{}_{j_1\ldots j_s} = g_{k\ell}T^{i_1\ldots k\ldots \ell\ldots i_r}{}_{j_1\ldots j_s},$$

where $(g_{ij})_{i,j=1}^n$ are the components of g relative to the chosen basis, and $(g^{ij})_{i,j=1}^n$ is the inverse matrix. For a further discussion and several examples, see [3].

Example 6. Some intuition for that operation is given by the following situation: given $\alpha \in \mathcal{T}^0_2(V)$, the non-degeneracy of g ensures the existence of $A \in \mathrm{Lin}(V)$ such that $\alpha(x,y) = g(Ax,y)$, for all $x,y \in V$. If (e_i) denotes the chosen basis, making $x = e_j$ and $y = e_k$ yields $\alpha_{jk} = g(Ae_j, e_k) = g(A^i{}_j e_i, e_k) = A^i{}_j g_{ik}$, and so $A^i{}_j = g^{ik}\alpha_{jk}$. Meaning that $\mathrm{tr}_{1,2}(\alpha) = \mathrm{tr}(A)$.

So, let's look at all possible contractions of a curvaturelike tensor $R \in \mathscr{R}(V)$. Since R is skew in the first pair and skew in the last pair, $\mathrm{tr}_{1,2}(R) = \mathrm{tr}_{3,4}(R) = 0$. Also, applying symmetry (i) from Definition 1 twice says that $\mathrm{tr}_{2,4}(R) = \mathrm{tr}_{1,3}(R)$. Also, we have that $\mathrm{tr}_{2,3}(R) = -\mathrm{tr}_{1,3}(R) = \mathrm{tr}_{1,4}(R)$. That is to say, up to sign, $\mathrm{tr}_{1,4}(R)$ is the only non-trivial contraction of R.

Definition 7. Let $R \in \mathscr{R}(V)$. The *Ricci contraction* of R is $\mathrm{Ric}(R) \in \mathcal{T}^0_2(V)$ given by $\mathrm{Ric}(R) \doteq \mathrm{tr}_{1,4}(R)$. The *scalar contraction* of R is the number $\mathrm{s}(R) = \mathrm{tr}_{1,2}(\mathrm{Ric}(R))$.

Remark.

- Note that since $R(z,x,y,w) = R(x,z,w,y) = R(w,y,x,z)$, the symmetry of g allows us to trace in the pair (z,w) to get $\mathrm{Ric}(R)(x,y) = \mathrm{Ric}(R)(y,x)$, meaning that $\mathrm{Ric}(R) \in \mathcal{S}(V)$. In terms of coordinates, $R_{ij} = g^{\ell m}R_{\ell ijm}$ and $\mathrm{s} = g^{ij}R_{ij}$.
- When V has a pseudo-Euclidean scalar product, every $R \in \mathscr{R}(V)$ defines a map $V^3 \to V$ (which we'll also denote by R) such that $R(x,y,z,w) = g(R(x,y)z,w)$, for all $x,y,z,w \in V$. Writing $R(x,y)z$ instead of $R(x,y,z)$ is customary in Geometry, as one usually sees $R(x,y)\colon V \to V$ as an endomorphism of V. Note that $\mathrm{tr}\,R(x,y) = 0$ for all $x,y \in V$, in view of the symmetries or the original R. With this terminology, one may also write $\mathrm{Ric}(R)(y,z) = \mathrm{tr}(x \mapsto R(x,y)z)$.

Example 8. Consider the fundamental curvature $g \owedge g$, given in terms of a basis by the expression $(g \owedge g)_{ijk\ell} = g_{jk}g_{i\ell} - g_{ik}g_{j\ell}$. So we compute the Ricci contraction as

$$\mathrm{Ric}(g \owedge g)_{jk} = g^{i\ell}g_{jk}g_{i\ell} - g^{i\ell}g_{ik}g_{j\ell} = ng_{jk} - \delta^\ell_k g_{j\ell} = (n-1)g_{jk},$$

which means that $\mathrm{Ric}(g \owedge g)(x,y) = (n-1)g(x,y)$. Procceeding, we get

$$\mathrm{s}(g \owedge g) = g^{ij}(n-1)g_{ij} = n(n-1).$$

We can do a similar computation for a general product:

Proposition 9. *Let $T, S \in \mathcal{S}(V)$, and consider $T \owedge S \in \mathscr{R}(V)$. Then:*

$$2\mathrm{Ric}(T \owedge S)(x,y) = \mathrm{tr}_{1,2}(S)T(x,y) - \mathrm{tr}_{1,4}(T \otimes S)(y,x) + \mathrm{tr}_{1,2}(T)S(x,y) - \mathrm{tr}_{1,4}(S \otimes T)(y,x).$$

Proof: Directly compute

$$\begin{aligned} 2\,\mathrm{Ric}(T \owedge S)_{jk} &= g^{i\ell}T_{jk}S_{i\ell} - g^{i\ell}T_{ik}S_{j\ell} + g^{i\ell}S_{jk}T_{i\ell} - g^{i\ell}S_{ik}T_{j\ell} \\ &= (g^{i\ell}S_{i\ell})T_{jk} - g^{i\ell}(T\otimes S)_{ikj\ell} + (g^{i\ell}T_{i\ell})S_{jk} - g^{i\ell}(S\otimes T)_{ikj\ell} \\ &= \mathrm{tr}_{1,2}(S)T_{jk} - \mathrm{tr}_{1,4}(T\otimes S)_{kj} + \mathrm{tr}_{1,2}(T)S_{jk} - \mathrm{tr}_{1,4}(S\otimes T)_{kj}. \end{aligned}$$

□

As an important particular case, we get the:

Corollary 10. *Let $T \in \mathcal{S}(V)$. Then* $2\mathrm{Ric}(T \owedge g) = (n-2)T + \mathrm{tr}_{1,2}(T)g$. *It also follows that* $\mathrm{s}(T \owedge g) = (n-1)\mathrm{tr}_{1,2}(T)$.

Proof: It is instructive to repeat the previous proof:

$$\begin{aligned} 2\,\mathrm{Ric}(T \owedge g)_{jk} &= g^{i\ell}T_{jk}g_{i\ell} - g^{i\ell}T_{ik}g_{j\ell} + g^{i\ell}g_{jk}T_{i\ell} - g^{i\ell}g_{ik}T_{j\ell} \\ &= nT_{jk} - \delta^i_j T_{ik} + \mathrm{tr}_{1,2}(T)g_{jk} - \delta^\ell_k T_{j\ell} \\ &= (n-2)T_{jk} + \mathrm{tr}_{1,2}(T)g_{jk}. \end{aligned}$$

For the last statement, we compute

$$2\mathrm{s}(T \owedge g) = (n-2)\mathrm{tr}_{1,2}(T) + \mathrm{tr}_{1,2}(T)n = (2n-2)\mathrm{tr}_{1,2}(T)$$

and simplify. □

Remark. In the same way that the space $\mathbb{R}(g \owedge g)$ appeared before, let's also set the notation $g \owedge \mathcal{S}(V) \doteq \{g \owedge T \in \mathscr{R}(V) \mid T \in \mathcal{S}(V)\}$, and

$$g \owedge \mathcal{S}(V)_0 \doteq \{g \owedge T \in \mathscr{R}(V) \mid T \in \mathcal{S}(V) \text{ has } \mathrm{tr}_{1,2}(T) = 0\}.$$

Corollary 11. *If $n \geq 3$, the maps $g \owedge \_ \colon \mathcal{S}(V) \to g \owedge \mathcal{S}(V)$ and $\mathrm{Ric}\colon g \owedge \mathcal{S}(V) \to \mathcal{S}(V)$ are isomorphisms.*

Proof: We check that the composition $\mathcal{S}(V) \to \mathcal{S}(V)$ is an isomorphism. For that, it suffices to show that it is injective. So, assume that $T \in \mathcal{S}(V)$ is a tensor such that $\mathrm{Ric}(g \owedge T) = 0$. Then $(n-2)T + \mathrm{tr}_{1,2}(T)g = 0$. Trace that to get $(2n-2)\mathrm{tr}_{1,2}(T) = 0$. So $\mathrm{tr}_{1,2}(T) = 0$, hence $(n-2)T = 0$ and thus $T = 0$, as wanted. So both $g \owedge \_$ and Ric are injective. But $g \owedge \_$ is surjective by definition, hence an isomorphism. Since $g \owedge \_$ and the full composition are isomorphisms, Ric is an isomorphism as well. □

To move on, recall that given any endomorphism $A \in \mathrm{Lin}(V)$, the *traceless part* of A is $A_0 \in \mathrm{Lin}(V)$ given by $A_0 = A - (\mathrm{tr}(A)/n)\,\mathrm{Id}_V$. Keeping the same notation of Example 6, we may convert this to a relation between bilinear maps by the relation $\alpha_0 = \alpha - (\mathrm{tr}_{1,2}(\alpha)/n)g$. So $\mathrm{tr}(A_0) = 0$ and $\mathrm{tr}_{1,2}(\alpha_0) = 0$. This idea will give us simpler objects related to a given curvaturelike tensor $R \in \mathscr{R}(V)$. For example, we have the:

Definition 12. Let $R \in \mathscr{R}(V)$. The *Einstein tensor* of R is $G(R) \in \mathcal{S}(V)$ defined by

$$G(R) = \mathrm{Ric}(R) - \frac{\mathrm{s}(R)}{n}g.$$

Example 13. For the fundamental curvature of (V, g), we have

$$G(g \owedge g) = \mathrm{Ric}(g \owedge g) - \frac{\mathrm{s}(g \owedge g)}{n} g = (n-1)g - \frac{n(n-1)}{n} g = 0.$$

In the same way we could form the traceless part of $\mathrm{Ric}(R)$, we'd like to form the Ricci-traceless part of R. So, to motivate the next definition, we'll look for a tensor of the particular form

$$W(R) = \mathrm{Ric}(R) - \lambda(g \owedge h(R))$$

with $\mathrm{Ric}(W(R)) = 0$, for some $h(R) \in \mathcal{S}(V)$ and $\lambda \in \mathbb{R}$. It will also be convenient to look for $h(R)$ of the form $h(R) = \mathrm{Ric}(R) - \mu g$, for some $\mu \in \mathbb{R}$. So our goal is to find λ and μ making all of this work. Tracing the expression for $W(R)$, we get $0 = \mathrm{Ric}(W(R)) = \mathrm{Ric}(R) - \lambda \mathrm{Ric}(g \owedge h(R))$, and so

$$\begin{aligned}
\mathrm{Ric}(R) = \lambda \mathrm{Ric}(g \owedge h(R)) &= \frac{\lambda}{2}((n-2)h(R) + \mathrm{tr}_{1,2}(h(R))g) \\
&= \frac{\lambda}{2}((n-2)(\mathrm{Ric}(R) - \mu g) + (\mathrm{s}(R) - n\mu)g) \\
&= \frac{\lambda}{2}((n-2)\mathrm{Ric}(R) + (\mathrm{s}(R) - 2(n-1)\mu)g).
\end{aligned}$$

This way, we see that $\mu = \mathrm{s}(R)/(2n-2)$ and $\lambda = 2/(n-2)$ fit the bill.

Definition 14. Let $R \in \mathscr{R}(V)$.

(i) The *Schouten tensor* of R is $h(R) \in \mathcal{S}(V)$ given by

$$h(R) = \mathrm{Ric}(R) - \frac{\mathrm{s}(R)}{2(n-1)} g.$$

(ii) The *Weyl tensor* of R is $W(R) \in \mathscr{R}(V)$ given (when $n > 2$) by

$$W(R) = R - \frac{2}{n-2}(g \owedge h(R)).$$

We'll also put $\mathscr{W}(V) = \{W \in \mathscr{R}(V) \mid \mathrm{Ric}(W) = 0\}$.

Remark.

- So, from here on, when R is understood, we will denote the associated tensors to R just by $\mathrm{Ric}, \mathrm{s}, h$ and W. Some places denote the Weyl tensor by the letter C instead of W, as it actually controls *conformal* flatness.

- As expected, when $n \geq 3$, we have that $W = R$ if and only if $\mathrm{Ric} = 0$. To wit, if $\mathrm{Ric} = 0$, then $\mathrm{s} = 0$, so that $h = 0$ and thus $W = R$. Conversely, if $W = R$, then $g \owedge h = 0$, and Corollary 11 gives $h = 0$, so that $\mathrm{tr}_{1,2}(h) = 0$ readily implies $\mathrm{s} = 0$, and we return to the definition of h to get that $\mathrm{Ric} = 0$.

- When $n = 3$, we always have $W = 0$. The reason is that in this particular case, $\mathrm{Ric}(R)$ completely determines R.

Example 15. Take $K \in \mathbb{R}$ and consider $R = Kg \owedge g$. We have seen in Example 8 that $\mathrm{Ric} = (n-1)Kg$, and so $\mathrm{s} = n(n-1)K$. This way we get

$$h = (n-1)Kg - \frac{n(n-1)K}{2(n-1)}g = \frac{n-2}{2}Kg,$$

and then

$$W = Kg \owedge g - \frac{2}{n-2}g \owedge \left(\frac{n-2}{2}Kg\right) = Kg \owedge g - Kg \owedge g = 0.$$

Example 16. Let $T \in \mathcal{S}(V)$, and consider $R = g \owedge T$. We have seen in Corollary 10 that $\mathrm{Ric} = (1/2)((n-2)T + \mathrm{tr}_{1,2}(T)g)$ and $\mathrm{s} = (n-1)\mathrm{tr}_{1,2}(T)$. With this, we compute

$$h = \frac{1}{2}((n-2)T + \mathrm{tr}_{1,2}(T)g) - \frac{(n-1)\mathrm{tr}_{1,2}(T)}{2(n-1)}g = \frac{n-2}{2}T,$$

so that

$$W = g \owedge T - \frac{2}{n-2}g \owedge \left(\frac{n-2}{2}T\right) = g \owedge T - g \owedge T = 0.$$

We also have that

$$G = \left(\frac{n-2}{2}\right)\left(T - \frac{g}{n}\right).$$

The two previous examples illustrate particular cases of the:

Theorem 17. *We have the decompositions:*

(i) $\mathcal{R}(V) = \mathbb{R}(g \owedge g) \oplus (g \owedge \mathcal{S}(V)_0) \oplus \mathcal{W}(V)$.

(ii) $g \owedge \mathcal{S}(V) = \mathbb{R}(g \owedge g) \oplus (g \owedge \mathcal{S}(V)_0)$.

Furthermore, the explicit decomposition of an element $R \in \mathcal{R}(V)$ is

$$R = \frac{\mathrm{s}}{n(n-1)}g \owedge g + \frac{2}{n-2}g \owedge G + W.$$

Proof: Let's first establish the expression for R (and hence the sums – not yet direct – in items (i) and (ii)). Of course one can just start from the right-hand side and simplify everything until only R is obtained, but it is more enlightening to see how such expression is discovered:

$$\begin{aligned} R &= \frac{2}{n-2}g \owedge h + W \\ &= \frac{2}{n-2}g \owedge \left(\mathrm{Ric} - \frac{\mathrm{s}}{2(n-1)}g\right) + W \\ &= \frac{2}{n-2}g \owedge \left(G + \frac{\mathrm{s}}{n}g - \frac{\mathrm{s}}{2(n-1)}g\right) + W \\ &= \frac{2}{n-2}g \owedge \left(G + \frac{(n-2)\mathrm{s}}{2n(n-1)}g\right) + W \\ &= \frac{\mathrm{s}}{n(n-1)}g \owedge g + \frac{2}{n-2}g \owedge G + W. \end{aligned}$$

The sums are direct by Corollary 11 (which gives $\mathbb{R}(g \owedge g) \cap (g \owedge \mathcal{S}(V)_0) = \{0\}$ and $(g \owedge \mathcal{S}(V)) \cap \mathcal{W}(V) = \{0\}$). $\square$

So, from the above decomposition, we see that W may be regarded as the "remainder of the $\owedge$-division" of R by g. This gives us a last corollary:

Corollary 18. *Two curvaturelike tensors $R_1, R_2 \in \mathscr{R}(V)$ have the same Weyl-component if and only if the difference $R_1 - R_2$ is $\owedge$-divisible by g.*

References

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