COMPUTATIONS ON PP-WAVE SPACETIMES

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Definition 1. A Lorentzian manifold (M, g), $\dim M = n+2$, is called a *pp-wave spacetime* if it admits a parallel null field $\boldsymbol{L} \in \mathfrak{X}(M)$ such that the connection induced in the quotient screen bundle $\boldsymbol{L}^\perp/\mathbb{R}\boldsymbol{L} \to M$ is flat, or, in other words, we may write $R(\boldsymbol{X},\boldsymbol{Y})\colon \boldsymbol{L}^\perp \to \mathbb{R}\boldsymbol{L}$ for all $\boldsymbol{X},\boldsymbol{Y} \in \mathfrak{X}(M)$.

Theorem 2 (Brinkmann). *Each point in a pp-wave spacetime (M,g) admits a coordinate neighborhood with coordinates $(x^+, x^-, x^1, \ldots, x^n)$ for which the metric is expressed as*

$$\mathrm{g} = 2H\,(\mathrm{d}x^+)^2 + 2\,\mathrm{d}x^+\,\mathrm{d}x^- + \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j,$$

where H is a smooth function not depending on x^- and i,j range from 1 to n. On such coordinates, we have $\boldsymbol{L} = \partial_-$. Given $p \in M$, such coordinates may be chosen centered at p and such that $H(x^+,x^-,\mathbf{0}) = (\partial_i H)(x^+,x^-,\mathbf{0}) = 0$ for all i and x^+ ranging over an interval centered at 0 (those are called normal Brinkmann coordinates).

Remark. (M,g) is called a *plane wave* if the data function H is quadratic on the variables (x^i) with coefficients depending on x^+, that is, if it has the form

$$H(x^+,x^-,x^1,\ldots,x^n) = H_{ij}(x^+)x^ix^j + H_k(x^+)x^k + H_0(x^+).$$

Moreover, note that $\mathrm{g}(\partial_-,\cdot) = \mathrm{d}x^+$, so it will also follow that $\mathrm{d}x^+$ is a parallel null 1-form.

Proof: Since $\boldsymbol{L}$ is parallel, the 1-form $\mathrm{g}(\boldsymbol{L},\cdot)$ is closed, so Poincaré's Lemma gives a local smooth function x^+ such that $\mathrm{g}(\boldsymbol{L},\cdot) = \mathrm{d}x^+$. Next, since $\boldsymbol{L}$ is a parallel distribution, so is $\boldsymbol{L}^\perp = \ker \mathrm{d}x^+$ (for if $\boldsymbol{X} \in \mathfrak{X}(M)$ and $\boldsymbol{Y} \in \Gamma(\boldsymbol{L}^\perp)$ are arbitrary, then we have that $\mathrm{g}(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{L}) = -\mathrm{g}(\boldsymbol{Y},\nabla_{\boldsymbol{X}}\boldsymbol{L}) = 0$, so $\nabla_{\boldsymbol{X}}\boldsymbol{Y} \in \Gamma(\boldsymbol{L}^\perp)$). Thus, the Levi-Civita connection induces connections on all the fibers of x^+, which are the integral hypersurfaces of $\boldsymbol{L}^\perp$ (even though g is degenerate on them). We now claim that those induced connections are all flat. To wit, given $\boldsymbol{X},\boldsymbol{Y} \in \Gamma(\boldsymbol{L}^\perp)$ and $\boldsymbol{Z},\boldsymbol{W} \in \mathfrak{X}(M)$, pair-symmetry of the Riemann tensor gives that $R(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) = R(\boldsymbol{Z},\boldsymbol{W},\boldsymbol{X},\boldsymbol{Y}) = 0$, since $R(\boldsymbol{Z},\boldsymbol{W})\boldsymbol{X}$ is a multiple of $\boldsymbol{L}$ (by definition of a pp-wave) and $\boldsymbol{Y} \in \Gamma(\boldsymbol{L}^\perp)$. With this in place, choose a point $p \in M$ and consider the integral hypersurface of $\boldsymbol{L}^\perp$ passing through p. Take orthonormal parallel fields $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$ *along such hypersurface*, and take a geodesic $\gamma\colon I \to M$ starting at p, orthogonal to $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$, and transverse to hypersurface. The transversality condition allows us to assume[1] that $\mathrm{d}x^+|_{\gamma(t)}(\gamma'(t)) = 1$

[1] This is a general phenomenon: if M is a smooth manifold, $f\colon M \to \mathbb{R}$ is smooth, and $\alpha\colon I \to M$ is a curve transverse to the fibers of f, then $(f\circ\alpha)' \neq 0$. So let h be the inverse of $f\circ\alpha$ defined on its range, and let $\gamma = \alpha\circ h$. Then $(f\circ\gamma)'(t) = (f\circ\alpha\circ h)'(t) = (f\circ\alpha)'(h(t))h'(t) = 1$, and this implies that $f(\gamma(t)) = t + t_0$ for some $t_0 \in \mathbb{R}$ — one can even arrange for $t_0 = 0$ by reparametrizing γ further.

for all $t \in I$ or, in other words, $\mathrm{g}(\boldsymbol{L},\gamma') = 1$. That is, we may assume that x^+ is the geodesic parameter. The fields $\boldsymbol{L}, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$, being parallel (along the hypersurface), give coordinates $(x^-, x^1, \ldots, x^n)$ on the integral hypersurface passing through p. For each x^+, we consider the parallel transport of all of those fields from $p = \gamma(0)$ to $\gamma(x^+)$, and then we extend those to parallel fields along the integral hypersurface of $\boldsymbol{L}^\perp$ passing through $\gamma(x^+)$ — such extensions (denoted with the same letters) give coordinates $(x^-, x^1, \ldots, x^n)$ on each hypersurface. Thus, given $q \in M$ near enough p, the coordinate $x^+(q)$ will be the value corresponding to the hypersurface passing through q, and the remaining values $x^-(q), x^1(q), \ldots, x^n(q)$ are determined by the hypersurface coordinates just defined. Since parallel translations are isometries and $\mathrm{g}(\partial_i, \partial_j) = \mathrm{g}(\boldsymbol{X}_i, \boldsymbol{X}_j) = \delta_{ij}$ on each hypersurface, these relations hold on the entire coordinate domain. By construction, ∂_+ (which in particular satisfies $\partial_+|_{\gamma(x^+)} = \gamma'(x^+)$) is orthogonal to all the ∂_i, and we have $\mathrm{g}(\partial_+, \partial_-) = 1$. The function $H = \mathrm{g}(\partial_+, \partial_+)/2$ satisfies $\partial_- H = 0$ because ∂_- is parallel. For the adjustment needed to get *normal* Brinkmann coordinates, see Lemma 3.1 in [1]. □

Let's compute the Christoffel symbols, assuming that $\lambda, \mu, \nu, \delta \in \{+, -, 1, \ldots, n\}$ and using

$$\Gamma^\lambda_{\mu\nu} = \frac{g^{\lambda\delta}}{2}(\partial_\mu g_{\delta\nu} + \partial_\nu g_{\delta\mu} - \partial_\delta g_{\mu\nu}).$$

Note that

$$(g_{\mu\nu}) = \begin{pmatrix} 2H & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{Id}_n \end{pmatrix} \quad \text{and} \quad (g^{\mu\nu}) = \begin{pmatrix} 0 & 1 & \mathbf{0} \\ 1 & -2H & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{Id}_n \end{pmatrix}$$

In particular, note that unless $\mu = \lambda = +$, we automatically have $\partial_\mu g_{\nu\lambda} = 0$.

- If $\lambda = +$: since $g^{\delta +} = \delta^{\delta -}$, we'll have terms with ∂_- derivatives and only $g_{\bullet\bullet}$ terms where $-$ enters, so $\Gamma^+_{\mu\nu} = 0$ for all μ and ν.

- If $\lambda = -$: this time we have to deal with $g^{\delta -}$, so

$$\Gamma^-_{\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu +} + \partial_\nu g_{\mu +} - \partial_+ g_{\mu\nu}) - H(\partial_\mu g_{\nu -} + \partial_\nu g_{\mu -} - \partial_- g_{\mu\nu}).$$

  By previous comments, all terms multiplying H vanish. Systematically, we see that $\Gamma^-_{++} = \partial_+ H$ and $\Gamma^-_{+j} = \partial_j H$, while the remaining symbols vanish.

- If $\lambda = k$: the only non-zero $g^{\delta k}$ are when $\delta = \ell$ is latin, so we have

$$\Gamma^k_{\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu k} + \partial_\nu g_{\mu k} - \partial_k g_{\mu\nu}) = -\frac{\partial_k g_{\mu\nu}}{2}.$$

  Hence $\Gamma^k_{++} = -\partial_k H = -\partial^k H$ and all the remaining symbols vanish.

So, the non-zero symbols are $\Gamma^-_{++} = \partial_+ H$, $\Gamma^-_{+j} = \partial_j H$ and $\Gamma^k_{++} = -\partial^k H$. Let's summarize our computations so far.

Proposition 3. *If (M, g) is a pp-wave spacetime, then relative to Brinkmann coordinates we have that the Levi-Civita connection ∇ is described by*

$$(\nabla_{\partial_\mu}\partial_\nu) = \begin{pmatrix} -(\partial^k H)\partial_k + (\partial_+ H)\partial_- & \mathbf{0} & (\partial_j H)\partial_- \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (\partial_j H)\partial_- & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Alternatively, we may write all covariant derivatives as

(i) $\nabla\partial_+ = -\mathrm{d}x^+ \otimes \mathrm{grad}_{\boldsymbol{x}} H + \mathrm{d}H \otimes \partial_-$;

(ii) $\nabla\partial_- = 0$;

(iii) $\nabla\partial_i = \partial_i H \,\mathrm{d}x^+ \otimes \partial_-$,

where $\mathrm{grad}_{\boldsymbol{x}} H = (\partial^k H)\partial_k$ *is the "flat gradient" of H.*

With this in place, we systematically move on to curvature operators, simply using the definition

$$R(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z} = \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z}.$$

- $R(\partial_+, \partial_-) = 0$ since ∂_- is parallel and $\partial_- H = 0$.
- $R(\partial_+, \partial_j)\partial_+ = (\partial_j\partial^k H)\partial_k$ and $R(\partial_+, \partial_j)\partial_k = -(\partial_j\partial_k H)\partial_-$.
- $R(\partial_-, \partial_j) = 0$ again since ∂_- is parallel and $\partial_- H = 0$.
- $R(\partial_i, \partial_j) = 0$ since $\partial_i\partial_j H = \partial_j\partial_i H$ and $\nabla_{\partial_i}\partial_k = \mathbf{0}$.

In essence, the only non-zero curvature operators are

$$R(\partial_+, \partial_j) = (\partial_j\partial^k H)\,\mathrm{d}x^+ \otimes \partial_k - (\partial_j\partial_k H)\,\mathrm{d}x^k \otimes \partial_-.$$

Equivalently, the only non-zero curvature components (up to symmetry) are

$$R_{+j+}{}^{k} = \partial_j\partial^k H \quad \text{and} \quad R_{+jk}{}^{-} = -\partial_j\partial_k H.$$

Proposition 4. *A pp-wave spacetime if flat if and only if, relative to any system of Brinkmann coordinates, the "flat Hessian"* $\mathrm{Hess}_{\boldsymbol{x}} H = \partial_i\partial_j H\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ *vanishes.*

With this in place, we may compute the Ricci tensor of (M, g).

Proposition 5. *The Ricci tensor of a pp-wave spacetime is given by*

$$\mathrm{Ric} = -\triangle H\,\mathrm{d}x^+ \otimes \mathrm{d}x^+,$$

where $\triangle H = \partial_k\partial^k H$ stands for the "flat Laplacian" induced by the Brinkmann coordinates. Hence every pp-wave spacetime is scalar-flat and Ricci-recurrent.

Next, let's find out when is a pp-wave locally symmetric. Namely, let's compute the covariant differential ∇R.

- $\nabla_{\partial_+} R$: the only non-zero operator is $(\nabla_{\partial_+} R)(\partial_+, \partial_j) = \nabla_{\partial_+}(R(\partial_+, \partial_j))$, whose only non-zero values are found to be

$$[(\nabla_{\partial_+} R)(\partial_+, \partial_j)]\partial_+ = (\partial_+\partial_j\partial^k H)\partial_k \quad \text{and} \quad [(\nabla_{\partial_+} R)(\partial_+, \partial_j)]\partial_k = -(\partial_+\partial_j\partial_k H)\partial_-.$$

- $\nabla_{\partial_-} R = 0$ because ∂_- is parallel and $\partial_- g_{\mu\nu} = 0$ for all μ, ν.

- $\nabla_{\partial_i} R$: the only non-zero operator is $(\nabla_{\partial_i} R)(\partial_+, \partial_j)$, whose only non-zero values are found to be

$$[(\nabla_{\partial_i} R)(\partial_+, \partial_j)]\partial_+ = (\partial_i\partial_j\partial^k H)\partial_k \quad \text{and} \quad [(\nabla_{\partial_i} R)(\partial_+, \partial_j)]\partial_k = -(\partial_i\partial_j\partial_k H)\partial_-.$$

Theorem 6. *A pp-wave spacetime (M, g) with parallel null field $\boldsymbol{L}$ is a plane wave spacetime if and only if $\nabla_{\boldsymbol{Z}} R = 0$ for all $\boldsymbol{Z} \in \Gamma(\boldsymbol{L}^\perp)$.*

Theorem 7. *A pp-wave spacetime (M, g) is locally symmetric if and only if, relative to Brinkmann coordinates, the "flat Hessian" matrix of the data function H is a constant. In particular, this is equivalent to (M, g) being a plane wave for which H is a quadratic polynomial in the variables (x^i) with* constant coefficients.

As for the next step, we look at the Weyl tensor. Since $\dim M = n + 2$ and $\mathrm{s} = 0$, we may generally write

$$\mathrm{W}_{\mu\nu\lambda}{}^{\delta} = R_{\mu\nu\lambda}{}^{\delta} - \frac{1}{n}(R_{\nu\lambda}\delta^\delta_\mu - R_{\mu\lambda}\delta^\delta_\nu + g_{\nu\lambda}g^{\delta\varepsilon}R_{\mu\varepsilon} - g_{\mu\lambda}g^{\delta\varepsilon}R_{\nu\varepsilon}).$$

For a pp-wave spacetime, we may use again that $g^{\delta+} = \delta^{\delta-}$ as well as $R_{\mu\nu} = -\delta_{\mu+}\delta_{\nu+}\triangle H$ to simplify things to

$$\mathrm{W}_{\mu\nu\lambda}{}^{\delta} = R_{\mu\nu\lambda}{}^{\delta} + \frac{\triangle H}{n}(\delta_{\nu+}\delta_{\lambda+}\delta^\delta_\mu - \delta_{\mu+}\delta_{\lambda+}\delta^\delta_\nu + g_{\nu\lambda}\delta^{\delta-}\delta_{\mu+} - g_{\mu\lambda}\delta^{\delta-}\delta_{\nu+}).$$

We'll follow the same order we used to compute curvature operators.

- $\mathrm{W}(\partial_+, \partial_-) = 0$, since $g_{\lambda-} = \delta_{\lambda+}$.

- $\mathrm{W}(\partial_+, \partial_j)$: the formula from above simplifies to

$$\mathrm{W}_{+j\lambda}{}^{\delta} = R_{+j\lambda}{}^{\delta} + \frac{\triangle H}{n}(-\delta_{\lambda+}\delta^\delta_j + g_{j\lambda}\delta^{\delta-}).$$

Choosing $\lambda = +$ gives that

$$\mathrm{W}(\partial_+, \partial_j)\partial_+ = R(\partial_+, \partial_j)\partial_+ - \frac{\triangle H}{n}\partial_j \implies \mathrm{W}(\partial_+, \partial_j)\partial_+ = (\partial_j\partial^k H)\partial_k - \frac{\triangle H}{n}\partial_j$$

and choosing $\lambda = k$ gives

$$\mathrm{W}(\partial_+, \partial_j)\partial_k = R(\partial_+, \partial_j)\partial_k + \frac{\triangle H}{n}\delta_{jk}\partial_- \implies \mathrm{W}(\partial_+, \partial_j)\partial_k = \left(-\partial_j\partial_k H + \frac{\triangle H}{n}\delta_{jk}\right)\partial_-$$

- $\mathrm{W}(\partial_-, \partial_j) = 0$ as all the Kronecker deltas vanish.
- $\mathrm{W}(\partial_i, \partial_j) = 0$ as all the Kronecker deltas vanish again.

Alternatively, one can avoid coordinate computations and directly appeal to the orthogonal decomposition of the fully covariant R, which yields:

Proposition 8. *The Weyl tensor of a pp-wave spacetime (M, g) is given in Brinkmann coordinates by*

$$\mathrm{W} = R + 2\frac{\triangle H}{n}\mathrm{g} \owedge (\mathrm{d}x^+)^2.$$

In particular, a pp-wave spacetime is conformally flat if and only if the "flat Hessian" of the data function H is "scalar", that is, if

$$\mathrm{Hess}_x H = \frac{\triangle H}{n}\,\mathrm{Id}_n.$$

Our last goal here will be to compute $\nabla \mathrm{W}$ and give conditions for a pp-wave spacetime to be a ECS manifold (*essentially conformally symmetric manifold*). A straightforward computation (using that ∂_- is parallel and that none of the $\nabla_{\partial_\mu}\partial_\nu$ has a ∂_+-component) shows that

$$(\nabla_{\partial_\mu}\mathrm{W})(\partial_+, \partial_-) = (\nabla_{\partial_\mu}\mathrm{W})(\partial_-, \partial_j) = (\nabla_{\partial_\mu}\mathrm{W})(\partial_i, \partial_j) = [(\nabla_{\partial_\mu}\mathrm{W})(\partial_+, \partial_j)]\partial_- = 0$$

for all choices of indices, while all the other non-zero values are given by

$$\begin{aligned}
[(\nabla_{\partial_\mu}\mathrm{W})(\partial_+, \partial_j)]\partial_+ &= (\partial_\mu\partial_j\partial^k H)\partial_k - \frac{\partial_\mu(\triangle H)}{n}\partial_j \\
[(\nabla_{\partial_\mu}\mathrm{W})(\partial_+, \partial_j)]\partial_k &= \left(-\partial_\mu\partial_j\partial_k H + \frac{\partial_\mu(\triangle H)}{n}\delta_{jk}\right)\partial_-
\end{aligned}$$

Note that for $\mu = -$, the right side of the above two relations is just zero as $\partial_- H = 0$. (i.e., a tautology). The computations done to verify the first one for $\mu = +$ and $\mu = i$ are slightly different (but equally easy). Verifying the second equation, in turn, is again straightforward using that $\mathrm{W}(\cdot, \cdot)\partial_- = 0$.

Proposition 9. *A pp-wave spacetime (M, g) has parallel Weyl tensor if and only if, relative to Brinkmann coordinates, we have*

$$\mathrm{Hess}_x H = \frac{\triangle H}{n}\,\mathrm{Id}_n + A$$

for some constant $A \in \mathfrak{sl}_n(\mathbb{R})$.

Corollary 10. *Every pp-wave spacetime with parallel Weyl tensor is a plane wave.*

Proof: Fixed x^+, we may as well assume the ambient space is the standard Euclidean space. Let $\psi = \triangle H/n$. The goal is to show that $\mathrm{d}\psi = 0$, as the Hessian being constant will imply that (M, g) is a plane wave. Recall the formula $\mathrm{d}^\nabla(\mathrm{Hess}\, H) = R(\cdot, \cdot, \nabla H, \cdot)$. Applying d^∇ on both sides of $\mathrm{Hess}\, H = \psi \mathrm{g} + A$ and using the flatness assumption gives $0 = \mathrm{d}\psi \wedge \mathrm{g}$. So $\mathrm{d}\psi = 0$ as g has full-rank. □

Proposition 11. *A pp-wave spacetime (M, g) is ECS if and only if, relative to Brinkmann coordinates, we have*

$$\mathrm{Hess}_x H = \frac{\triangle H}{n}\,\mathrm{Id}_n + A$$

for some non-zero $A \in \mathfrak{sl}_n(\mathbb{R})$ *and* $\partial_+ \triangle H \neq 0$.

Theorem 12. *Every point in a ECS* plane wave (M, g) *admits a coordinate neighborhood with coordinates $(t, s, (x^i))$ for which the metric is expressed as*

$$\mathrm{g} = \kappa\, \mathrm{d}t^2 + \mathrm{d}t\, \mathrm{d}s + \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j,$$

where κ is the (smooth) map given by $\kappa(t, s, \boldsymbol{v}) = f(t)\langle \boldsymbol{v}, \boldsymbol{v}\rangle + \langle A\boldsymbol{v}, \boldsymbol{v}\rangle$, with f non-constant and $A \in \mathfrak{sl}_n(\mathbb{R})$ non-zero.

Proof: By Theorem 2, we may already start with a system of *normal* Brinkmann coordinates $(x^+, x^-, (x^i))$ centered at the given point. Now let $t = x^+$ and $s = 2x^-$. We may write $H = H_{ij}(x^+)x^i x^j$, so that Proposition 11 gives us $A \in \mathfrak{sl}_n(\mathbb{R})$, $A \neq 0$, such that

$$2H_{ij}(x^+) = (\partial_i \partial_j H)(x^+) = \frac{(\triangle H)(x^+)}{n}\delta_{ij} + a_{ij}.$$

Let $f(t) = (\triangle H)(t)/n$. With this we clearly have

$$\mathrm{g} = 2H\,(\mathrm{d}x^+)^2 + 2\,\mathrm{d}x^+\,\mathrm{d}x^- + \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \kappa\,\mathrm{d}t^2 + \mathrm{d}t\,\mathrm{d}s + \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j,$$

as required. □

References

[1] Globke, W.; Leistner, T.; *Locally homogeneous pp-waves*, Journal of Geometry and Physics 108, pp. 83–101, 2016.