THE RICCI IDENTITY

Ivo Terek

1 Setup

We work on the smooth category, and fix a vector bundle $E \to M$ equipped with a connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, where $\Gamma(E)$ stands for the $C^\infty(M)$-module of smooth sections of E. The curvature tensor of ∇ is $R\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ defined by

$$R^\nabla(X,Y)\psi = \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi. \tag{1.1}$$

Relative to a local trivialization e_a for E and local coordinates x^j for M, we write

$$\text{(a) } \nabla_{\partial_j} e_a = \mathit{\Gamma}_{ja}^b e_b \qquad \text{(b) } R(\partial_j,\partial_k)e_a = R_{jka}{}^b e_b, \tag{1.2}$$

while repeatedly substituting (1.2-a) into (1.1) to compute (1.2-b) yields

$$R_{jka}{}^b = \partial_j \mathit{\Gamma}_{ka}^b - \partial_k \mathit{\Gamma}_{ja}^b + \mathit{\Gamma}_{ka}^c \mathit{\Gamma}_{jc}^b - \mathit{\Gamma}_{ja}^c \mathit{\Gamma}_{kc}^b. \tag{1.3}$$

Whenever $\psi \in \Gamma(E)$, we may consider $\nabla\psi$ as a section of $\mathrm{Hom}(TM, E)$. Applying the Leibniz rule for ∇ together with (1.2-a), we have that

$$\nabla_{\partial_j}\psi = (\partial_j\psi^b + \mathit{\Gamma}_{ja}^b\psi^a)e_b, \tag{1.4}$$

and this motivates defining

$$\nabla_j\psi^b = \partial_j\psi^b + \mathit{\Gamma}_{ja}^b\psi^a, \text{ so that } \nabla\psi = \nabla_j\psi^b \mathrm{d}x^j \otimes e_b. \tag{1.5}$$

Another notation for $\nabla_j\psi^b$ is $\psi^b_{;j}$, but I prefer the former to the latter as it allows us to think of "$\nabla_j = \partial_j + \mathit{\Gamma}_j$" as an operator on its own right.

The vector-bundle index a, for concrete choices of E, may actually stand for a collection of indices coming from M. For example, assume that $E = TM^{\otimes 2}$ has a connection induced from some connection in TM. If e_a is a local frame for TM, then $e_a \otimes e_b$ forms a local trivialization for E, in which case we have that

$$\begin{aligned} \nabla_{\partial_j}(e_a \otimes e_b) &= \nabla_{\partial_j} e_a \otimes e_b + e_a \otimes \nabla_{\partial_j} e_b \\ &= \mathit{\Gamma}_{ja}^c e_c \otimes e_b + e_a \otimes \mathit{\Gamma}_{jb}^d e_d \\ &= (\mathit{\Gamma}_{ja}^c \delta_b^d + \delta_a^c \mathit{\Gamma}_{jb}^d) e_c \otimes e_d \end{aligned} \tag{1.6}$$

says that $\mathit{\Gamma}_{j,\underline{ab}}^{\underline{cd}} = \mathit{\Gamma}_{ja}^c\delta_b^d + \delta_a^c\mathit{\Gamma}_{jb}^d$, where the underlines on $\mathit{\Gamma}_{j,\underline{ab}}^{\underline{cd}}$ are meant to emphasize that ab is a single $TM^{\otimes 2}$-index, as is cd. The point of observing this is that it suffices to develop the theory for vector bundles, and then it may be applied to any tensor bundle over M.

2 Second derivatives

When TM is also equipped with a connection ∇°, the bundle $\mathrm{Hom}(TM,E)$ inherits a connection ∇' from both ∇ and ∇°. More precisely, if e_a is a local trivialization for E and x^j are local coordinates for M, then we may write

$$F = F^a_k\,\mathrm{d}x^k \otimes e_a, \text{ where } F^a_k \text{ is the } a\text{-th component of } F(\partial_k), \tag{2.1}$$

for any section F of $\mathrm{Hom}(TM,E)$. The pair ${}^{a}_{k}$ is a single *upper* $\mathrm{Hom}(TM,E)$-index (while ${}^{k}_{a}$ as in $\mathrm{d}x^k \otimes e_a$ would be a *lower* one). By definition of the induced connection on $\mathrm{Hom}(TM,E)$, we have that

$$(\nabla'_X F)(Y) = \nabla_X(F(Y)) - F(\nabla^\circ_X Y) \tag{2.2}$$

Mimicking what was done in (1.4), we set $X = \partial_j$ and $Y = \partial_k$ on (2.2) and compute

$$\begin{aligned}(\nabla'_{\partial_j}F)(\partial_k) &= \nabla_{\partial_j}(F(\partial_k)) - F(\nabla^\circ_{\partial_j}\partial_k)\\ &= \nabla_{\partial_j}(F^a_k e_a) - F(\mathring{\Gamma}^\ell_{jk}\partial_\ell)\\ &= (\partial_j F^a_k)e_a + F^a_k\partial_j e_a - \mathring{\Gamma}^\ell_{jk}F(\partial_\ell)\\ &= (\partial_j F^a_k)e_a + F^b_k\Gamma^a_{jb}e_a - \mathring{\Gamma}^\ell_{jk}F^a_\ell e_a\\ &= (\partial_j F^a_k + F^b_k\Gamma^a_{jb} - \mathring{\Gamma}^\ell_{jk}F^a_\ell)e_a,\end{aligned} \tag{2.3}$$

giving us that

$$\nabla'_j F^a_k = \partial_j F^a_k + F^b_k\Gamma^a_{jb} - \mathring{\Gamma}^\ell_{jk}F^a_\ell. \tag{2.4}$$

Usually, the left side of the above is written just as $\nabla_j F^a_k$, with the dependence of the right side on the auxiliary connection ∇° being understood. Now, we may take $F = \nabla\psi$ for some section $\psi \in \Gamma(E)$, so that

$$\nabla_j\nabla_k\psi^a = \partial_j(\nabla_k\psi^a) + (\nabla_k\psi^b)\Gamma^a_{jb} - \mathring{\Gamma}^\ell_{jk}(\nabla_\ell\psi^a) \tag{2.5}$$

Substitute (1.5) into the first two terms on the right side of (2.5) to compute

$$\begin{aligned}\nabla_j\nabla_k\psi^a &= \partial_j(\partial_k\psi^a + \Gamma^a_{kb}\psi^b) + (\partial_k\psi^b + \Gamma^b_{kc}\psi^c)\Gamma^a_{jb} - \mathring{\Gamma}^\ell_{jk}(\nabla_\ell\psi^a)\\ &= \partial_j\partial_k\psi^a + (\partial_j\Gamma^a_{kb})\psi^b + \Gamma^a_{kb}\partial_j\psi^b + \Gamma^a_{jb}\partial_k\psi^b + \Gamma^a_{jb}\Gamma^b_{kc}\psi^c - \mathring{\Gamma}^\ell_{jk}(\nabla_\ell\psi^a)\\ &= (\partial_j\partial_k\psi^a + \Gamma^a_{kb}\partial_j\psi^b + \Gamma^a_{jb}\partial_k\psi^b) + (\partial_j\Gamma^a_{kb})\psi^b + \Gamma^a_{jb}\Gamma^b_{kc}\psi^c - \mathring{\Gamma}^\ell_{jk}(\nabla_\ell\psi^a).\end{aligned} \tag{2.6}$$

Noting that the first group of three terms in the last line of (2.6) is symmetric in the pair (j,k), we may directly obtain that

$$\nabla_j\nabla_k\psi^a - \nabla_k\nabla_j\psi^a = (\partial_j\Gamma^a_{kb} - \partial_k\Gamma^a_{jb})\psi^b + (\Gamma^a_{jb}\Gamma^b_{kc} - \Gamma^a_{kb}\Gamma^b_{jc})\psi^c - \tau^\ell_{jk}\nabla_\ell\psi^a, \tag{2.7}$$

where τ^ℓ_{jk} are the components of the torsion tensor of ∇°, defined by

$$\tau(X,Y) = \nabla^\circ_X Y - \nabla^\circ_Y X - [X,Y]. \tag{2.8}$$

If ∇° is torsionfree (i.e., $\tau = 0$), substituting (1.3) into (2.7) and renaming $b \leftrightarrow c$, it follows that

$$\nabla_j\nabla_k\psi^a - \nabla_k\nabla_j\psi^a = R_{jkb}{}^a\psi^b, \tag{2.9}$$

as required.

3 Application to tensor bundles

In this section, we let ∇ be a *torsionfree* connection on M, and again denote by ∇ the induced connections on all tensor bundles over M. We also choose the "auxiliary" connection to be $\nabla^\circ = \nabla$. Set $E = TM^{\otimes r} \otimes T^*M^{\otimes s}$. A r-times contravariant and s-times covariant tensor field on M is a section of E, which may be written relative to a local coordinate system x^j for M as

$$T = T^{i_1\ldots i_r}_{j_1\ldots j_s}\partial_{i_1}\otimes\cdots\otimes\partial_{i_r}\otimes \mathrm{d}x^{j_1}\otimes\cdots\otimes\mathrm{d}x^{j_s}. \tag{3.1}$$

As we are not dealing with metrics and the process of raising/lowering indices, there is no need to stick to the more precise notation $T^{i_1\ldots i_r}{}_{j_1\ldots j_s}$. To apply (2.9), we need an expression for the curvature tensor of the induced connection on E. Such expression, however, will follow from the Leibniz rule combined with relations

$$\text{(a) } R(\partial_j,\partial_k)\partial_{i_a} = R_{jki_a}{}^p\partial_p \qquad\qquad \text{(b) } R(\partial_j,\partial_k)\mathrm{d}x^{j_b} = -R_{jkq}{}^{j_b}\mathrm{d}x^q \tag{3.2}$$

from the base cases $(r,s) = (1,0)$ and $(r,s) = (0,1)$. Indeed, we have that

$$\begin{aligned} R(\partial_j,\partial_k)\big(\partial_{i_1}\otimes\cdots\otimes\partial_{i_r}\otimes\mathrm{d}x^{j_1}\otimes\cdots\otimes\mathrm{d}x^{j_s}\big) &= \\ = (R_{jki_1}{}^{p_1}\partial_{p_1})\otimes\cdots\otimes\partial_{i_r}\otimes\mathrm{d}x^{j_1}\otimes\cdots\otimes\mathrm{d}x^{j_s} \\ +\cdots+\partial_{i_1}\otimes\cdots\otimes(R_{jki_r}{}^{p_r}\partial_{p_r})\otimes\mathrm{d}x^{j_1}\otimes\cdots\otimes\mathrm{d}x^{j_s} \\ +\partial_{i_1}\otimes\cdots\otimes\partial_{i_r}\otimes(-R_{jkq_1}{}^{j_1}\mathrm{d}x^{q_1})\otimes\cdots\otimes\mathrm{d}x^{j_s} \\ +\cdots+\partial_{i_1}\otimes\cdots\otimes\partial_{i_r}\otimes\cdots\otimes\mathrm{d}x^{j_1}\otimes\cdots\otimes(-R_{jkq_s}{}^{j_s}\mathrm{d}x^{q_s}). \end{aligned} \tag{3.3}$$

With the aid of Kronecker deltas, we may informally rewrite (3.3) as

$$\begin{aligned} &\frac{R(\partial_j,\partial_k)\big(\partial_{i_1}\otimes\cdots\otimes\partial_{i_r}\otimes\mathrm{d}x^{j_1}\otimes\cdots\otimes\mathrm{d}x^{j_s}\big)}{\partial_{p_1}\otimes\cdots\otimes\partial_{p_r}\otimes\mathrm{d}x^{q_1}\otimes\cdots\otimes\mathrm{d}x^{q_s}} = \\ &= \sum_{\ell=1}^r R_{jki_\ell}{}^{p_\ell}\delta^{p_1}_{i_1}\ldots\widehat{\delta^{p_\ell}_{i_\ell}}\ldots\delta^{p_r}_{i_r}\delta^{j_1}_{q_1}\ldots\delta^{j_s}_{q_s} - \sum_{\ell=1}^s R_{jkq_\ell}{}^{j_\ell}\delta^{p_1}_{i_1}\ldots\delta^{p_r}_{i_1}\delta^{j_1}_{q_1}\ldots\widehat{\delta^{q_\ell}_{q_\ell}}\ldots\delta^{j_s}_{q_s}. \end{aligned} \tag{3.4}$$

The right side of (3.4) is the curvature term present in the right side of (2.9) for our choice of E (after suitably adjusting dummy indices). Already contracting all the Kronecker deltas possible, we obtain that

$$\begin{aligned} \nabla_j\nabla_k T^{i_1\ldots i_r}_{j_1\ldots j_s} - \nabla_k\nabla_j T^{i_1\ldots i_r}_{j_1\ldots j_s} = R_{jkp}{}^{i_1}T^{pi_2\ldots i_r}_{j_1\ldots j_s} + \cdots + R_{jkp}{}^{i_r}T^{i_1\ldots i_{r-1}p}_{j_1\ldots j_s} \\ - R_{jkj_1}{}^{q}T^{i_1\ldots i_r}_{qj_2\ldots j_s} - \cdots - R_{jkj_s}{}^{q}T^{i_1\ldots i_r}_{j_1\ldots j_{s-1}q}. \end{aligned} \tag{3.5}$$

The placement of signs follows (3.2). We list some particular cases of (3.5) below for the reader's convenience:

(i) $\nabla_k\nabla_\ell T_{ij} - \nabla_\ell\nabla_k T_{ij} = -R_{k\ell i}{}^q T_{qj} - R_{k\ell j}{}^q T_{iq}$.

(ii) $\nabla_k\nabla_\ell T^r_i - \nabla_\ell\nabla_k T^r_i = R_{k\ell p}{}^r T^p_i - R_{k\ell i}{}^q T^r_q$.

(iii) $\nabla_\ell\nabla_m T^k_{ij} - \nabla_m\nabla_\ell T^k_{ij} = R_{\ell mp}{}^k T^p_{ij} - R_{\ell mi}{}^q T^k_{qj} - R_{\ell mj}{}^q T^k_{iq}$.

4 Geometric consequences

Here, we let (M, g) be a n-dimensional pseudo-Riemannian manifold, and ∇ be its Levi-Civita connection. One consequence of the Ricci identity (3.5) is that

$$\text{if } (M,\mathrm{g}) \text{ has nonzero constant sectional curvature, then } M \text{ does not admit nontrivial parallel vector fields, or parallel 1-forms.} \tag{4.1}$$

If X is a parallel vector field on M, then ${R_{ijp}}^k X^p = 0$ reads $K(g_{jp}\delta_i^k - g_{ip}\delta_j^k)X^p = 0$, where $K \neq 0$ is the sectional curvature of (M, g). Evaluating it and cancelling K gives us that $\delta_i^k X_j - \delta_j^k X_i = 0$, and making $k = i$ yields $(n-1)X_i = 0$, so that $X_i = 0$. The argument for 1-forms α is dual: ${R_{ijk}}^q \alpha_q = 0$ reads $K(g_{jk}\delta_i^q - g_{ik}\delta_j^q)\alpha_q = 0$, and therefore $g_{jk}\alpha_i - g_{ik}\alpha_j = 0$. Contracting against g^{jk} leads to $(\dim M - 1)\alpha_i = 0$, and hence $\alpha_i = 0$.