THE RICCI IDENTITY

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1 Setup

We work on the smooth category, and fix a vector bundle $E \to M$ equipped with a connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, where $\Gamma(E)$ stands for the $C^\infty(M)$-module of smooth sections of $E$. The curvature tensor of $\nabla$ is $R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ defined by

$$R(X,Y)\psi = \nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi - \nabla_{[X,Y]}\psi.$$ (1.1)

Relative to a local trivialization $e_a$ for $E$ and local coordinates $x^j$ for $M$, we write

$$(a) \quad \nabla_{\partial_j} e_a = \Gamma^b_{ja} e_b$$ (b) $\quad R(\partial_j, \partial_k) e_a = R_{jka}^\ b e_b,$$ (1.2)

while repeatedly substituting (1.2-a) into (1.1) to compute (1.2-b) yields

$$R_{jka}^\ b = \partial_j \Gamma^b_{ka} - \partial_k \Gamma^b_{ja} + \Gamma^c_{ja} \Gamma^b_{kc} - \Gamma^c_{ja} \Gamma^b_{kc}.$$ (1.3)

Whenever $\psi \in \Gamma(E)$, we may consider $\nabla \psi$ as a section of $\text{Hom}(TM, E)$. Applying the Leibniz rule for $\nabla$ together with (1.2-a), we have that

$$\nabla_{\partial_j} \psi = (\partial_j \psi^b + \Gamma^b_{ja} \psi^a) e_b,$$ (1.4)

and this motivates defining

$$\nabla_j \psi^b = (\partial_j \psi^b + \Gamma^b_{ja} \psi^a) e_b,$$ (1.5)

Another notation for $\nabla_j \psi^b$ is $\psi^b_j$, but I prefer the former to the latter as it allows us to think of “$\nabla_j = \partial_j + \Gamma^j_j$” as an operator on its own right.

The vector-bundle index $a$, for concrete choices of $E$, may actually stand for a collection of indices coming from $M$. For example, assume that $E = TM\otimes^2$ has a connection induced from some connection in $TM$. If $e_a$ is a local frame for $TM$, then $e_a \otimes e_b$ forms a local trivialization for $E$, in which case we have that

$$\nabla_{\partial_i} (e_a \otimes e_b) = \nabla_{\partial_i} e_a \otimes e_b + e_a \otimes \nabla_{\partial_i} e_b$$
$$= \Gamma^c_{ja} e_c \otimes e_b + e_a \otimes \Gamma^d_{jb} e_d$$
$$= (\Gamma^c_{ja} \delta^d_b + \delta^c_a \Gamma^d_{jb}) e_c \otimes e_d$$ (1.6)

says that $\Gamma^c_{ja} \delta^d_b + \delta^c_a \Gamma^d_{jb}$, where the underlines on $\Gamma^c_{ja} \delta^d_b$ are meant to emphasize that $ab$ is a single $TM\otimes^2$-index, as is $cd$. The point of observing this is that it suffices to develop the theory for vector bundles, and then it may be applied to any tensor bundle over $M$. 
2 Second derivatives

When $TM$ is also equipped with a connection $\nabla^\circ$, the bundle $\text{Hom}(TM, E)$ inherits a connection $\nabla'$ from both $\nabla$ and $\nabla^\circ$. More precisely, if $e_a$ is a local trivialization for $E$ and $x^j$ are local coordinates for $M$, then we may write

$$F = F^a_k \, dx^k \otimes e_a, \quad \text{where } F^a_k \text{ is the } a\text{-th component of } F(\partial_k),$$

(2.1)

for any section $F$ of $\text{Hom}(TM, E)$. The pair $^a_k$ is a single upper $\text{Hom}(TM, E)$-index (while $^k_a$ as in $dx^k \otimes e_a$ would be a lower one). By definition of the induced connection on $\text{Hom}(TM, E)$, we have that

$$(\nabla'_X F)(Y) = \nabla_X (F(Y)) - F(\nabla'^\circ_X Y)$$

(2.2)

Mimicking what was done in (1.4), we set $X = \partial_j$ and $Y = \partial_k$ on (2.2) and compute

$$(\nabla'_{\partial_j} F)(\partial_k) = \nabla_{\partial_j}(F(\partial_k)) - F(\nabla'_{\partial_j} \partial_k) = \nabla_{\partial_j}(F^a_k e_a) - F(\bar{F}^\ell_k \partial_\ell) = (\partial_j F^a_k) e_a + F^a_k \partial_j e_a - \bar{F}^\ell_k F(\partial_\ell)$$

(2.3)

$$= (\partial_j F^a_k) e_a + F^b_k \bar{\Gamma}^a_{jb} e_a - \bar{F}^\ell_k F^b \partial_\ell e_a$$

$$= (\partial_j F^a_k + F^b_k \bar{\Gamma}^a_{jb} - \bar{F}^\ell_k F^b) e_a,$$

giving us that

$$\nabla'_j F^a_k = \partial_j F^a_k + F^b_k \bar{\Gamma}^a_{jb} - \bar{F}^\ell_k F^b.$$  

(2.4)

Usually, the left side of the above is written just as $\nabla'_j F^a_k$, with the dependence of the right side on the auxiliary connection $\nabla^\circ$ being understood. Now, we may take $F = \nabla \psi$ for some section $\psi \in \Gamma(E)$, so that

$$\nabla_j \nabla_k \psi^a = (\partial_j \nabla_k \psi^a) + (\nabla_k \psi^b) \bar{\Gamma}^a_{jb} - \bar{F}^\ell_k (\nabla_\ell \psi^a)$$

(2.5)

Substitute (1.5) into the first two terms on the right side of (2.5) to compute

$$\nabla_j \nabla_k \psi^a = (\partial_j \partial_k \psi^a + \Gamma^a_{kb} \psi^b) + (\partial_j \psi^b + \Gamma^b_{kc} \psi^c) \bar{\Gamma}^a_{jb} - \bar{F}^\ell_k (\nabla_\ell \psi^a)$$

$$= \partial_j \partial_k \psi^a + (\partial_j \Gamma^a_{kb}) \psi^b + \Gamma^a_{kb} \partial_k \psi^b + \Gamma^a_{jb} \partial_k \psi^b + \Gamma^a_{jb} \Gamma^b_{kc} \psi^c - \bar{F}^\ell_k (\nabla_\ell \psi^a)$$

(2.6)

Noting that the first group of three terms in the last line of (2.6) is symmetric in the pair $(j, k)$, we may directly obtain that

$$\nabla_j \nabla_k \psi^a - \nabla_k \nabla_j \psi^a = (\partial_j \Gamma^a_{kb} - \partial_k \Gamma^a_{jb}) \psi^b + (\Gamma^a_{jb} \Gamma^b_{kc} - \Gamma^a_{kb} \Gamma^b_{jc}) \psi^c - \tau^\ell_{jk} \nabla_\ell \psi^a,$$

where $\tau^\ell_{jk}$ are the components of the torsion tensor of $\nabla^\circ$, defined by

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

(2.8)

If $\nabla^\circ$ is torsionfree (i.e., $\tau = 0$), substituting (1.3) into (2.7) and renaming $b \leftrightarrow c$, it follows that

$$\nabla_j \nabla_k \psi^a - \nabla_k \nabla_j \psi^a = R^a_{jkb} \psi^b,$$

(2.9)

as required.
3 Application to tensor bundles

In this section, we let \( \nabla \) be a torsionfree connection on \( M \), and again denote by \( \nabla \) the induced connections on all tensor bundles over \( M \). We also choose the "auxiliary" connection to be \( \nabla^o = \nabla \). Set \( E = TM^{\otimes r} \otimes T^* M^{\otimes s} \). A \( r \)-times contravariant and \( s \)-times covariant tensor field on \( M \) is a section of \( E \), which may be written relative to a local coordinate system \( x^i \) for \( M \) as

\[
T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}. \tag{3.1}
\]

As we are not dealing with metrics and the process of raising/lowering indices, there is no need to stick to the more precise notation \( T_{i_1 \ldots i_r}^{j_1 \ldots j_s} \). To apply (2.9), we need an expression for the curvature tensor of the induced connection on \( E \). Such expression, however, will follow from the Leibniz rule combined with relations

\[
\begin{align*}
(a) \ R(\partial_j, \partial_k) \partial_{i_a} &= R_{jki_a}^p \partial_p \\
(b) \ R(\partial_j, \partial_k) dx^{i_b} &= -R_{jkq}^{i_b} dx^q
\end{align*} \tag{3.2}
\]

from the base cases \((r, s) = (1, 0)\) and \((r, s) = (0, 1)\). Indeed, we have that

\[
R(\partial_j, \partial_k)(\partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}) =
\]

\[
= (R_{jki_1}^p \partial_p) \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}
+ \cdots + \partial_{i_1} \otimes \cdots \otimes (R_{jk{i_r}_p}^r \partial_p) \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}
+ \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes (-R_{jkq}^{i_r} dx^q_1) \otimes \cdots \otimes dx^{j_s}
+ \cdots + \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes \cdots \otimes dx^{j_1} \otimes \cdots \otimes (-R_{jkq}^{i_s} dx^q_s).
\tag{3.3}
\]

With the aid of Kronecker deltas, we may informally rewrite (3.3) as

\[
R(\partial_j, \partial_k)(\partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s})
\]

\[
= \sum_{\ell=1}^r R_{jki_1}^p \delta_{i_1}^{\ell p_1} \cdots \delta_{i_r}^{\ell p_r} \delta_{j_1}^{\ell q_1} \cdots \delta_{j_s}^{\ell q_s} - \sum_{\ell=1}^s R_{jkq}^{i_\ell} \delta_{i_1}^{\ell 1} \cdots \delta_{i_r}^{\ell r} \delta_{j_1}^{\ell q_1} \cdots \delta_{j_s}^{\ell q_s}. \tag{3.4}
\]

The right side of (3.4) is the curvature term present in the right side of (2.9) for our choice of \( E \) (after suitably adjusting dummy indices). Already contracting all the Kronecker deltas possible, we obtain that

\[
\nabla_j \nabla_k T^{i_1 \ldots i_r}_{j_1 \ldots j_s} - \nabla_k \nabla_j T^{i_1 \ldots i_r}_{j_1 \ldots j_s} = R_{jkq}^{i_1} T^{p_2 \ldots i_r}_{j_1 \ldots j_s} + \cdots + R_{jkp}^{i_r} T^{i_1 \ldots i_{r-1}p}_{j_1 \ldots j_s}
- R_{jkq}^{i_1} q T^{i_2 \ldots i_r}_{j_2 \ldots j_s} - \cdots - R_{jkq}^{i_s} T^{i_1 \ldots i_{r-1}q}_{j_1 \ldots j_{s-1}q}. \tag{3.5}
\]

The placement of signs follows (3.2). We list some particular cases of (3.5) below for the reader’s convenience:

(i) \( \nabla_k \nabla_i T_{ij} - \nabla_i \nabla_k T_{ij} = -R_{klq}^{q} T_{ij} - R_{klq}^{q} T_{ij} \).

(ii) \( \nabla_k \nabla_i T_{ij}^p - \nabla_i \nabla_k T_{ij}^p = R_{klp}^{r} T_{ij}^p - R_{klq}^{q} T_{ij}^p. \)

(iii) \( \nabla \nabla_m T_{ij}^k - \nabla_m \nabla \nabla T_{ij}^k = R_{lmp}^{k} T_{ij}^p - R_{lmi}^{q} T_{ij}^q - R_{lmj}^{q} T_{iq}^q. \)
4 Geometric consequences

Here, we let \((M, g)\) be a \(n\)-dimensional pseudo-Riemannian manifold, and \(\nabla\) be its Levi-Civita connection. One consequence of the Ricci identity (3.5) is that

\[
\text{if } (M, g) \text{ has nonzero constant sectional curvature, then } M \text{ does not admit nontrivial parallel vector fields, or parallel 1-forms.} \tag{4.1}
\]

If \(X\) is a parallel vector field on \(M\), then \(R_{ijp}^k X^p = 0\) reads
\[
K^2 \left( g_{jp} \delta_i^k - g_{ip} \delta_j^k \right) X^p = 0,
\]
where \(K \neq 0\) is the sectional curvature of \((M, g)\). Evaluating it and cancelling \(K\) gives us that \(\delta_i^j X_j - \delta_j^i X_i = 0\), and making \(k = i\) yields \((n - 1)X_i = 0\), so that \(X_i = 0\). The argument for 1-forms \(\alpha\) is dual: \(R_{ijk}^q \alpha_q = 0\) reads
\[
K \left( g_{jk} \delta_i^q - g_{ik} \delta_j^q \right) \alpha_q = 0,
\]
and therefore \(g_{jk} \alpha_i - g_{ik} \alpha_j = 0\). Contracting against \(g^{jk}\) leads to \((\dim M - 1) \alpha_i = 0\), and hence \(\alpha_i = 0\).