SEMIDIRECT PRODUCTS

Ivo Terek

In this write-up, we'll describe the Lie algebra structure induced on $\mathfrak{g}\times\mathfrak{h}$ by a semidirect product Lie group structure on $G\times H$, by essentially elaborating on some details of the computation outlined in https://math.stackexchange.com/a/3378604/118056. We will start by recalling some relevant definitions.

Definition 1

Let G be a Lie group with Lie algebra $\mathfrak{g}$.

(i) The **conjugation by** g is the map $\mathrm{C}^G_g\colon G\to G$ given by $\mathrm{C}^G_g(\hat{g}) = g\hat{g}g^{-1}$.

(ii) The **adjoint representation of** G is the homomorphism $\mathrm{Ad}^G\colon G\to \mathrm{GL}(\mathfrak{g})$ given by $\mathrm{Ad}^G(g) = \mathrm{d}(\mathrm{C}^G_g)_e$.

(iii) The **adjoint representation of** $\mathfrak{g}$ is the homomorphism $\mathrm{ad}^{\mathfrak{g}}\colon \mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ given by $\mathrm{ad}^{\mathfrak{g}} = \mathrm{d}(\mathrm{Ad}^G)_e$.

Theorem 1

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the Lie bracket on $\mathfrak{g}$ satisfies that

$$[X,\hat{X}]_{\mathfrak{g}} = \mathrm{ad}_X\hat{X} \tag{0.1}$$

for all $X,\hat{X}\in\mathfrak{g}$

Definition 2

Let G and H be Lie groups and $\rho\colon G\to\mathrm{Aut}(H)$ be a homomorphism. The **semidirect product** $G\ltimes_\rho H$ of G and H under ρ is the manifold $G\times H$ equipped with the product given by

$$(g,h)(\hat{g},\hat{h}) = (g\hat{g}, h\,\rho(g)\hat{h}), \tag{0.2}$$

for all $(g,h),(\hat{g},\hat{h})\in G\times H$.

Remark. If ρ is understood, we'll write $G\ltimes H$ instead of $G\ltimes_\rho H$. Note that the identity element of $G\ltimes H$ is (e_G,e_H), and that inverses are given by $(g,h)^{-1} = (g^{-1},\rho(g^{-1})h^{-1})$ for every $(g,h)\in G\ltimes H$. Moreover, note that $\{e_G\}\times H$ is a normal subgroup of $G\ltimes H$ (thus justifying why this notation is used, as opposed to $G\rtimes H$). As we will see in the next computations, it is convenient to regard ρ as a bihomomorphism $\rho\colon G\times H\to H$.

Last updated: October 30, 2022. terekcouto.1@osu.edu.

Proposition 1

The conjugations in G, H, and $G \ltimes H$ are related via

$$\mathrm{C}^{G\ltimes H}_{(g,h)}(\hat{g},\hat{h}) = \left(\mathrm{C}^G_g(\hat{g}), \mathrm{C}^H_h(\rho(g)\hat{h})\, h\, \rho(\mathrm{C}^G_g(\hat{g}))h^{-1}\right), \tag{0.3}$$

for every $(g,h), (\hat{g},\hat{h}) \in G \ltimes H$.

Proof: Compute:

$$\begin{aligned} \mathrm{C}^{G\ltimes H}_{(g,h)}(\hat{g},\hat{h}) &= (g,h)(\hat{g},\hat{h})(g,h)^{-1} \\ &= (g\hat{g}, h\,\rho(g)\hat{h})(g^{-1}, \rho(g^{-1})h^{-1}) \\ &= \left(g\hat{g}g^{-1}, h\,\rho(g)\hat{h}\,\rho(g\hat{g})(\rho(g^{-1})h^{-1})\right) \\ &= \left(g\hat{g}g^{-1}, h\,\rho(g)\hat{h}\,h^{-1}h\,\rho(g\hat{g}g^{-1})h^{-1}\right) \\ &= \left(\mathrm{C}^G_g(\hat{g}), \mathrm{C}^H_h(\rho(g)\hat{h})\, h\, \rho(\mathrm{C}^G_g(\hat{g}))h^{-1}\right). \end{aligned} \tag{0.4}$$

□

Proposition 2

The adjoint representations of G, H, and $G \ltimes H$ are related via

$$\mathrm{Ad}^{G\ltimes H}_{(g,h)}(\hat{Y},\hat{Z}) = \left(\mathrm{Ad}^G_g(\hat{Y}), \mathrm{Ad}^H_h(\mathrm{d}(\rho(g,\cdot))_{e_H}\hat{Z}) + h\,\mathrm{d}(\rho(\cdot,h^{-1}))_{e_G}\mathrm{Ad}^G_g(\hat{Y})\right), \tag{0.5}$$

for all $(g,h) \in G \ltimes H$ and $(\hat{Y},\hat{Z}) \in \mathfrak{g} \times \mathfrak{h}$, where by $h\,\mathrm{d}(\rho(\cdot,h^{-1}))_{e_G}\mathrm{Ad}^G_g(\hat{Y})$ we understand the result of applying the derivative at e_H of the left translation by $h \in H$ to the element $\mathrm{d}(\rho(\cdot,h^{-1}))_{e_G}\mathrm{Ad}^G_g(\hat{Y}) \in \mathfrak{h}$.

Proof: We freeze (g,h) in (0.3), and differentiate it with respect to $(\hat{g},\hat{h})$ at (e_G,e_H), in the direction of $(\hat{Y},\hat{Z})$. The first component evidently is $\mathrm{Ad}^G_g(\hat{Y})$ and, for the second one, we use the product rule, noting that setting $\hat{g} = e_G$ in $h\,\rho(\mathrm{C}^G_g(\hat{g}))h^{-1}$ yields e_H, and setting $\hat{h} = e_H$ in $\mathrm{C}^H_h(\rho(g)\hat{h})$ yields e_G. In other words, we just need to differentiate the expressions $\mathrm{C}^H_h(\rho(g)\hat{h})$ and $h\,\rho(\mathrm{C}^G_g(\hat{g}))h^{-1}$ separately. The conclusion follows. □

Proposition 3

The adjoint representations of $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{g} \ltimes \mathfrak{h}$ (the Lie algebra of $G \ltimes H$) are related via

$$\mathrm{ad}^{\mathfrak{g}\ltimes\mathfrak{h}}_{(Y,Z)}(\hat{Y},\hat{Z}) = \left(\mathrm{ad}^{\mathfrak{g}}_Y\hat{Y}, \mathrm{ad}^{\mathfrak{h}}_Z\hat{Z} + \mathrm{d}\rho_{(e_G,e_H)}(Y,\hat{Z}) - \mathrm{d}\rho_{(e_G,e_H)}(\hat{Y},Z)\right), \tag{0.6}$$

for all $(Y,Z), (\hat{Y},\hat{Z}) \in \mathfrak{g} \ltimes \mathfrak{h}$.

Proof: In a similar fashion to what was done in the previous computation, we freeze $(\hat{Y},\hat{Z})$ in (0.5), and differentiate it with respect to (g,h) at (e_G,e_H), in the direction of (Y,Z). Again, the first component trivially equals $\mathrm{ad}^{\mathfrak{g}}_Y\hat{Y}$. There are two terms to be discussed here, both relying on the idea that a full derivative may be written as a sum of partial derivatives and in the product rule.

- $\mathrm{Ad}^H_h(\mathrm{d}(\rho(g,\cdot))_{e_H}\hat{Z})$. Differentiating this yields two terms. To compute the first, we may set $h=e_H$ and differentiate the remaining expression, $\mathrm{d}(\rho(g,\cdot))_{e_H}\hat{Z}$ (as $\mathrm{Ad}^H_{e_H}=\mathrm{Id}_{\mathfrak{h}}$), at $g=e_G$ and in the direction of Y, to obtain $\mathrm{d}\rho_{(e_G,e_H)}(Y,\hat{Z})$. As for the second term, we may set $g=e_G$ and differentiate the remaining expression at $h=e_H$ in the direction of Z. However, $\mathrm{d}(\rho(e_G,\cdot))_{e_H}$ is already the zero map.

- $h\,\mathrm{d}(\rho(\cdot,h^{-1}))_{e_G}\mathrm{Ad}^G_g(\hat{Y})$. Using the product rule together with the same principle as the above point, we see that differentiating this expression yields three terms. To compute the first, setting $g=e_G$ and the second h equal to e_H yields zero, as $\mathrm{d}(\rho(\cdot,e_H))_{e_G}$ is already the zero map. For the second term, we set $g=e_G$ and the first h equal to e_H, so that differentiating $\mathrm{d}(\rho(\cdot,h^{-1}))_{e_G}\hat{Y}$ at $h=e_H$ in the direction of Z yields $-\mathrm{d}\rho_{(e_G,e_H)}(\hat{Y},Z)$, in view of the chain rule together with the fact that the derivative of the inversion map $H\ni h\mapsto h^{-1}\in H$ at $h=e_H$ is $-\mathrm{Id}_{\mathfrak{h}}$. Lastly, setting $h=e_H$ to try and differentiate the remaining expression with respect to g also gives us zero (again because $\mathrm{d}(\rho(\cdot,e_H))_{e_G}$ is the zero map).

With this in place, the conclusion follows. □

Now, Theorem 1 gives us how to compute Lie brackets in $\mathfrak{g}\ltimes\mathfrak{h}$:

Corollary 1

The Lie bracket in $\mathfrak{g}\ltimes\mathfrak{h}$ is given by

$$[(Y,Z),(\hat{Y},\hat{Z})]_{\mathfrak{g}\ltimes\mathfrak{h}}=\left([Y,\hat{Y}]_{\mathfrak{g}},[Z,\hat{Z}]_{\mathfrak{h}}+Y\cdot\hat{Z}-\hat{Y}\cdot Z\right), \tag{0.7}$$

for all $(Y,Z),(\hat{Y},\hat{Z})\in\mathfrak{g}\ltimes\mathfrak{h}$, where the multiplication $\cdot$ is a shorthand for $\mathrm{d}\rho_{(e_G,e_H)}$.

Let's conclude the discussion by noting how (0.7) suggests the definition of a semidirect product of (abstract) Lie algebras. It suffices to remember that the Lie algebra of $\mathrm{Aut}(H)$ equals the algebra $\mathfrak{der}(\mathfrak{h})$ of derivations of $\mathfrak{h}$ when H is connected and simply connected. Replacing $\rho\colon G\to\mathrm{Aut}(H)$ with its derivative $\rho_*\colon\mathfrak{g}\to\mathfrak{der}(\mathfrak{h})$, we have:

Definition 3

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras, and let $\rho_*\colon\mathfrak{g}\to\mathfrak{der}(\mathfrak{h})$ be a homomorphism. The **semidirect product** $\mathfrak{g}\ltimes_{\rho_*}\mathfrak{h}$ of $\mathfrak{g}$ and $\mathfrak{h}$ under ρ_* is the vector space $\mathfrak{g}\times\mathfrak{h}$ equipped with the Lie bracket given by

$$[(Y,Z),(\hat{Y},\hat{Z})]_{\mathfrak{g}\ltimes_{\rho_*}\mathfrak{h}}=\left([Y,\hat{Y}]_{\mathfrak{g}},[Z,\hat{Z}]_{\mathfrak{h}}+\rho_*(Y)(\hat{Z})-\rho_*(\hat{Y})(Z)\right), \tag{0.8}$$

for all $(Y,Z),(\hat{Y},\hat{Z})\in\mathfrak{g}\ltimes_{\rho_*}\mathfrak{h}$.