ON TOTAL SPACES OF TAUTOLOGICAL LINE BUNDLES

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1 The tautological line bundle $E_1 \to \mathbb{P}V$

1.1 Setup, trivializations, and transition mappings

Given any $(n+1)$-dimensional vector space V over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, one may form the *tautological line bundle* over the projective space $\mathbb{P}V$, which assigns to each point $L \in \mathbb{P}V$ the one-dimensional vector space L itself. In other words, the fiber over L is L. Writing E_1 for the total space of such a bundle, we have that

$$E_1 = \bigsqcup_{L \in \mathbb{P}V} L = \bigcup_{L \in \mathbb{P}V} (\{L\} \times L) = \{(L, v) \in \mathbb{P}V \times V \mid v \in L\}. \tag{1.1}$$

Write $\pi\colon E_1 \to \mathbb{P}V$ for the projection given by $\pi(L, v) = L$. For every linear functional $f \in V^* \smallsetminus \{0\}$, we may construct a local trivialization χ_f for E_1 by noting that

$$\text{the set } U_f = \{L \in \mathbb{P}V \mid f[L] = \mathbb{R}\} \text{ is open in } \mathbb{P}V, \tag{1.2}$$

by definition of quotient topology, and defining

$$\chi_f\colon \pi^{-1}[U_f] \to U_f \times \mathbb{R} \text{ by } \chi_f(L, v) = (L, f(v)), \tag{1.3}$$

whose inverse is the mapping

$$\begin{gathered}\chi_f^{-1}\colon U_f \times \mathbb{R} \to \pi^{-1}[U_f] \text{ defined by } \chi_f^{-1}(L, \lambda) = (L, \lambda x / f(x)), \\ \text{where a nonzero element } x \in L \smallsetminus \{0\} \text{ (so } L = \mathbb{K}x) \text{ is chosen at will.}\end{gathered} \tag{1.4}$$

To see that the choice of x in (1.4) does not matter, observe that replacing x with any multiple μx, $\mu \in \mathbb{K} \smallsetminus \{0\}$, it follows that $\lambda(\mu x)/f(\mu x) = \lambda \mu x / (\mu f(x)) = \lambda x / f(x)$, by linearity of f.

Each restriction $\{L\} \times L \ni (L, v) \mapsto f(v) \in \mathbb{R}$ of χ_f in (1.3) is a linear isomorphism due to $L \in U_f$, and so $\{(U_f, \chi_f)\}_{f \in V^* \smallsetminus \{0\}}$ is an atlas of trivializations for E_1. We proceed to describe its transition functions. To do so, we consider a second linear functional $h \in V^* \smallsetminus \{0\}$ such that $U_f \cap U_h \neq \varnothing$, as well as the composition

$$\chi_f \circ \chi_h^{-1}\colon (U_f \cap U_h) \times \mathbb{R} \to (U_f \cap U_h) \times \mathbb{R}, \tag{1.5}$$

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easily computed – by (1.4) – as

$$(\chi_f \circ \chi_h^{-1})(L,\lambda) = \chi_f\left(L, \frac{\lambda x}{h(x)}\right) = \left(L, f\left(\frac{\lambda x}{h(x)}\right)\right) = \left(L, \frac{f(x)}{h(x)}\lambda\right). \tag{1.6}$$

The ratio $f(x)/h(x)$, however, does not depend on the choice of $x \in L \setminus \{0\}$, but instead only on the line L itself. Therefore the transition functions

$g_{fh}\colon U_f \cap U_h \to \mathrm{GL}_1(\mathbb{K}) = \mathbb{K}^\times$ are given by $g_{fh}(L) = f(x)/h(x)$, where a nonzero element $x \in L \setminus \{0\}$ (so $L = \mathbb{K}x$) is chosen at will. (1.7)

1.2 Manifold-charts for E_1

Now, we recall that for any smooth vector bundle $E \to M$, charts for E can be built from charts for M together with trivializations for E. The situation for the tautological line bundle $E_1 \to PV$ considered here is particularly nice, as PV admits an atlas $\{(U_f, \varphi_f)\}_{f \in V^* \setminus \{0\}}$ whose domains are the same sets U_f defined in (1.2). Namely, we have that

each $\varphi_f\colon U_f \to f^{-1}(1)$ is given by $\varphi_f(L) = x/f(x)$, where a nonzero element $x \in L \setminus \{0\}$ (so $L = \mathbb{K}x$) is chosen at will. (1.8)

See Figure 1 for a geometric interpretation.

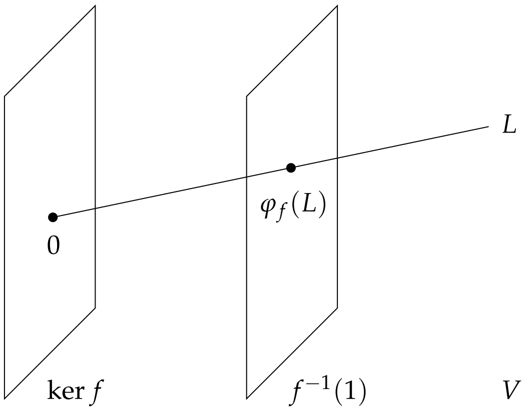
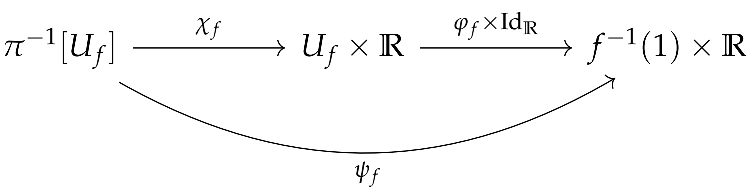


Figure 1: Hyperplane-valued coordinate charts for PV.

The corresponding charts for E_1 will be given by the compositions



(1.9)

More precisely, we have that

$\psi_f\colon \pi^{-1}[U_f] \to f^{-1}(1) \times \mathbb{R}$ is given by $\psi_f(L,v) = (x/f(x), f(v))$, where a nonzero element $x \in L \setminus \{0\}$ (so $L = \mathbb{K}x$) is chosen at will. (1.10)

We may again consider a second linear functional $h \in V^* \smallsetminus \{0\}$ such that $U_f \cap U_h \neq \varnothing$, and directly compute the chart transitions

$$\psi_f \circ \psi_h^{-1} \colon \psi_h[\pi^{-1}[U_f] \cap \pi^{-1}[U_h]] \to \psi_f[\pi^{-1}[U_f] \cap \pi^{-1}[U_h]]. \tag{1.11}$$

Before doing so, observe that $\psi_f[\pi^{-1}[U_f] \cap \pi^{-1}[U_h]] = (f^{-1}(1) \smallsetminus \ker h) \times \mathbb{R}$ – and similarly for $\psi_h[\pi^{-1}[U_f] \cap \pi^{-1}[U_h]] = (h^{-1}(1) \smallsetminus \ker f) \times \mathbb{R}$ – are disconnected. For instance,

$$\text{the connected components of } f^{-1}(1) \smallsetminus \ker h \text{ are the two intersections } f^{-1}(1) \cap h^{-1}(0,\infty) \text{ and } f^{-1}(1) \cap h^{-1}(-\infty,0). \tag{1.12}$$

With this in place, we compute $\psi_f \circ \psi_h^{-1} \colon (h^{-1}(1) \smallsetminus \ker f) \times \mathbb{R} \to (f^{-1}(1) \smallsetminus \ker h) \times \mathbb{R}$ as

$$\begin{aligned}
\psi_f \circ \psi_h^{-1}(u,\lambda) &= ((\varphi_f \times \mathrm{Id}_{\mathbb{R}}) \circ \chi_f) \circ ((\varphi_h \times \mathrm{Id}_{\mathbb{R}}) \circ \chi_h)^{-1}(u,\lambda) \\
&= (\varphi_f \times \mathrm{Id}_{\mathbb{R}}) \circ \chi_f \circ \chi_h^{-1} \circ (\varphi_h \times \mathrm{Id}_{\mathbb{R}})^{-1}(u,\lambda) \\
&= (\varphi_f \times \mathrm{Id}_{\mathbb{R}}) \circ (\chi_f \circ \chi_h^{-1}) \circ (\varphi_h^{-1} \times \mathrm{Id}_{\mathbb{R}})(u,\lambda) \\
&= (\varphi_f \times \mathrm{Id}_{\mathbb{R}}) \circ (\chi_f \circ \chi_h^{-1})(\mathbb{K}u,\lambda) \\
&\overset{(*)}{=} (\varphi_f \times \mathrm{Id}_{\mathbb{R}})\,(\mathbb{K}u, \lambda f(u)) \\
&= \left(\frac{u}{f(u)}, \lambda f(u)\right)
\end{aligned} \tag{1.13}$$

where on $(*)$ we use (1.6) with $x = u$ together with $h(u) = 1$. Writing $F = \psi_f \circ \psi_h^{-1}$ for simplicity, we have that $\mathrm{d}F_{(u,\lambda)} \colon \ker h \times \mathbb{R} \to \ker f \times \mathbb{R}$ is given by

$$\mathrm{d}F_{(u,\lambda)}(w,\xi) = \left(\frac{f(u)w - f(w)u}{f(u)^2}, \xi f(u) + \lambda f(w)\right). \tag{1.14}$$

At this point, it makes no sense to ask ourselves whether F is orientation-preserving or orientation-reversing, as our charts for $\mathrm{P}V$ are not valued in $\mathbb{R}^n$.

1.3 Intrinsic orientability?

Provided V itself is real and oriented, there is a way to assign orientations for $\ker f$ and $\ker h$, and thus proceed with the discussion. To do it, fix a volume form $\Omega \in [V^*]^{\wedge(n+1)} \smallsetminus \{0\}$, and consider a basis $\mathcal{B} = (v_1, \ldots, v_n)$ for $\ker f$. The linear functional $\Omega(v_1, \ldots, v_n, \cdot) \colon V \to \mathbb{R}$ vanishes on $\ker f$, and therefore induces a nonzero – due to linear independence of $\mathcal{B}$ – functional $\Omega_{\mathcal{B}} \colon V/\ker f \to \mathbb{R}$, as does f itself, say $\widetilde{f} \colon V/\ker f \to \mathbb{R}$. As $V/\ker f$ is one-dimensional, we have that $\Omega_{\mathcal{B}} = \alpha \widetilde{f}$ for some scalar $\alpha \in \mathbb{R} \smallsetminus \{0\}$. We will say that $\mathcal{B}$ is *positive* or *negative* according to whether α is positive or negative, respectively. Observe that while $\ker f = \ker(\lambda f)$ for every $\lambda \in \mathbb{R} \smallsetminus \{0\}$, the orientation will change if $\lambda < 0$, so that the choice of "gauge" functional realizing a hyperplane as its kernel does matter.

One strategy would be to assume from here on that $\mathbb{K} = \mathbb{R}$ and that a volume form Ω for V is fixed, and verify whether $\mathrm{d}F_{(u,\lambda)}$ takes positive bases for $\ker h \times \mathbb{R}$ onto positive bases for $\ker f \times \mathbb{R}$, but this sounds very unpleasant to do.

1.4 Coordinate computations

We assume that $V = \mathbb{R}^{n+1}$ and write $\mathbb{RP}^n = \mathrm{P}(\mathbb{R}^{n+1})$. Instead of considering the full atlases $\{(U_f, \chi_f)\}_{f\in V^*\smallsetminus\{0\}}$ and $\{(U_\varphi, \varphi_f)\}_{f\in V^*\smallsetminus\{0\}}$ of trivializations for E_1 and charts for $\mathbb{RP}^n$, respectively, we let f range over the set $\{\pi_0, \ldots, \pi_n\}$ of coordinate projections $\pi_j\colon \mathbb{R}^{n+1} \to \mathbb{R}$, and write simply $U_j = U_{\pi_j}$ and $\chi_j = \chi_{\pi_j}$, for $0 \le j \le n$. In particular, deleting the j-th coordinate describes an (affine) isomorphism between each hyperplane $\pi_j^{-1}(1)$ and $\mathbb{R}^n$.

Whenever $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \smallsetminus \{0\}$, we will write $[x_0 : \cdots : x_n] = \mathbb{R}x \in \mathbb{RP}^n$ for the so-called *homogeneous coordinates* of $\mathbb{R}x$. With this notation in place, the domains U_j – see (1.2) – become

$$U_i = \{[x_0 : \cdots : x_n] \in \mathbb{RP}^n \mid x_i \neq 0\}, \quad 0 \le i \le n, \tag{1.15}$$

while the charts (1.8) now read

$$\varphi_i\colon U_i \to \mathbb{R}^n, \qquad \varphi_i([x_0 : \cdots : x_n]) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right). \tag{1.16}$$

The trivializations χ_i for E_1 – cf. (1.3) – are given by

$$\chi_i\colon \pi^{-1}[U_i] \to U_i \times \mathbb{R}, \qquad \chi_i([x_0 : \cdots : x_n], (v_0, \ldots, v_n)) = ([x_0 : \cdots : x_n], v_i), \tag{1.17}$$

with transition maps $g_{ij}\colon U_i \cap U_j \to \mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$ given by

$$g_{ij}([x_0 : \cdots : x_n]) = \frac{x_i}{x_j}, \tag{1.18}$$

according to (1.7). The manifold-charts for E_1, defined in (1.10), simply reduce to the mappings $\psi_i\colon \pi^{-1}[U_i] \to \mathbb{R}^{n+1}$, given by

$$\psi_i([x_0 : \cdots : x_n], (v_0, \ldots, v_n)) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}, v_i\right). \tag{1.19}$$

Finally, to describe the transition maps computed in (1.13), for $i < j$, start noting that

$$\varphi_j[U_i \cap U_j] = \mathbb{R}^n_{t_i \neq 0} \quad \text{and} \quad \varphi_i[U_i \cap U_j] = \mathbb{R}^n_{t_{j-1} \neq 0} \tag{1.20}$$

are disconnected (compare it with (1.12)) so that $\psi_i \circ \psi_j^{-1}\colon \mathbb{R}^n_{t_i\neq 0} \times \mathbb{R} \to \mathbb{R}^n_{t_{j-1}\neq 0} \times \mathbb{R}$ is given by

$$\begin{aligned}(\psi_i \circ \psi_j^{-1})(t_1, \ldots, t_n, s) &= \\ &= \psi_i([t_1 : \cdots : t_{j-1} : 1 : t_j : \cdots : t_n], (t_1 s, \ldots, t_{j-1}s, s, t_j s, \ldots, t_n s)) \\ &= \left(\frac{t_1}{t_i}, \ldots, \frac{t_{i-1}}{t_i}, \frac{t_{i+1}}{t_i}, \ldots, \frac{t_{j-1}}{t_i}, \frac{1}{t_i}, \frac{t_j}{t_i}, \ldots, \frac{t_n}{t_i}, t_i s\right).\end{aligned} \tag{1.21}$$

The Jacobian matrix of $\psi_i \circ \psi_j^{-1}$ is best described in particular cases.

In $\mathbb{R}P^1$, $\psi_0 \circ \psi_1^{-1}(t,s) = (1/t, ts)$ has

$$\mathrm{d}(\psi_0 \circ \psi_1^{-1})(t,s) = \begin{bmatrix} -1/t^2 & 0 \\ s & t \end{bmatrix}, \quad \det \mathrm{d}(\psi_0 \circ \psi_1^{-1})(t,s) = -\frac{1}{t}. \tag{1.22}$$

In $\mathbb{R}P^2$, we have

$$\begin{aligned} \psi_0 \circ \psi_1^{-1}(t_1,t_2,s) &= \left(\frac{1}{t_1}, \frac{t_2}{t_1}, t_1 s\right) \\ \psi_0 \circ \psi_2^{-1}(t_1,t_2,s) &= \left(\frac{t_2}{t_1}, \frac{1}{t_1}, t_1 s\right) \\ \psi_1 \circ \psi_2^{-1}(t_1,t_2,s) &= \left(\frac{t_1}{t_2}, \frac{1}{t_2}, t_2 s\right), \end{aligned} \tag{1.23}$$

with Jacobians

$$\begin{bmatrix} -1/t_1^2 & 0 & 0 \\ -t_2/t_1^2 & 1/t_1 & 0 \\ s & 0 & t_1 \end{bmatrix}, \quad \begin{bmatrix} -t_2/t_1^2 & 1/t_1 & 0 \\ -1/t_1^2 & 0 & 0 \\ s & 0 & t_1 \end{bmatrix}, \quad \begin{bmatrix} 1/t_2 & -t_1/t_2^2 & 0 \\ 0 & -1/t_2^2 & 0 \\ 0 & s & t_2 \end{bmatrix}. \tag{1.24}$$

The Jacobian determinant of the single transition mapping listed for $\mathbb{R}P^1$ changes sign on its domain $\mathbb{R}^\times \times \mathbb{R}$, while the three Jacobian determinants listed for $\mathbb{R}P^2$ are all negative. *This seems to suggest that whether the total space of $E_1 \to \mathbb{R}P^n$ is an orientable manifold or not depends on the parity of n.*

2 The line bundles E_d

Still with the setup of the previous section, and noting that tensor products of one-dimensional vector spaces are again one-dimensional, the following generalization becomes natural: let $d > 0$ and assign to each point $L \in \mathrm{P}V$, the tensor power line $L^{\otimes d}$. Write E_d for the total space of such a bundle, so that

$$E_d = \bigsqcup_{L \in \mathrm{P}V} L^{\otimes d} = \bigcup_{L \in \mathrm{P}V} \{L\} \times L^{\otimes d} = \{(L, \Theta) \in \mathrm{P}V \times V^{\otimes d} \mid \Theta \in L^{\otimes d}\}, \tag{2.1}$$

and $\pi\colon E_d \to \mathrm{P}V$ for the projection given by $\pi(L,\Theta) = L$. Clearly $E_d = (E_1)^{\otimes d}$, so the structure of E_d is derived from the one in E_1. Similarly, one may define E_{-1} by assigning to each point $L \in \mathrm{P}V$ the dual line L^*, thus making sense of E_d for $d < 0$. Namely, the fiber of E_d over L when $d < 0$ is simply $[L^*]^{\otimes |d|}$. When $d = 0$, we obtain the trivial line bundle $\mathrm{P}V \times \mathbb{R} \to \mathrm{P}V$ as $L^{\otimes 0} = \mathbb{R}$ by default.

For $d > 0$ and $L \in \mathrm{P}V$, note that if $x \in L \smallsetminus \{0\}$, then $x^{\otimes d} \in L^{\otimes d} \smallsetminus \{0\}$, so we may consider dth tensor power $f^{\otimes d}$ of any linear functional $f \in V^* \smallsetminus \{0\}$ with $f[L] = \mathbb{R}$, characterized by $f^{\otimes d}(x^{\otimes d}) = f(x)^d$, inducing an isomorphism between $L^{\otimes d}$ and $\mathbb{R}$.

When $d < 0$, replace L with L^* and switch the roles of f and x in the previous paragraph, regarding x as an element of L^{**} instead.

With the setup of the previous section, it now follows that the transition maps $g_{fh}\colon U_f \cap U_h \to \mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$ are given by

$$g_{fh}(L) = \frac{f^{\otimes d}(x^{\otimes d})}{h^{\otimes d}(x^{\otimes d})} = \frac{f(x)^d}{h(x)^d} = \left(\frac{f(x)}{h(x)}\right)^d, \tag{2.2}$$

where $x \in L \setminus \{0\}$ is chosen at will, as usual.

As a toy problem, we consider $E_d \to \mathbb{R}\mathrm{P}^1$. When is the manifold E_d orientable? Does the answer depend on d? Mimicking what was done in (1.13) and incorporating dth powers on (1.18), we have that

$$\psi_0 \circ \psi_1^{-1}(t,s) = \psi_0([t:1],(ts,s)^{\otimes d}) = \left(\frac{1}{t}, t^d s\right), \tag{2.3}$$

so that

$$\mathrm{d}(\psi_0 \circ \psi_1^{-1})(t,s) = \begin{bmatrix} -1/t^2 & 0 \\ dt^{d-1}s & t^d \end{bmatrix} \tag{2.4}$$

has determinant equal to $-t^{d-2}$. So, whenever d is even, the sign of such determinant is constant (so that E_d is orientable), but changes signs when d is odd (so that E_d is non-orientable).

3 And on Grassmannians?

Consider instead the Grassmannian manifold of k-dimensional subspaces of V, $\mathrm{Gr}_k(V)$. There is a *tautological vector bundle of rank k* over $E_1 \to \mathrm{Gr}_k(V)$, whose fiber over a point $W \in \mathrm{Gr}_k(V)$ is W itself. If $d \in \mathbb{Z}$, one may again consider $E_d \to \mathrm{Gr}_k(V)$ by assigning to W the vector space $W^{\otimes d}$ (where for $d < 0$ we understand that W is replaced with W^* and d with $-d$). What can be said about the total space of such a bundle?