TIME EVOLUTION OF RIEMANNIAN OBJECTS AND THE EINSTEIN-HILBERT FUNCTIONAL

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Fix a smooth manifold M. Let's start recalling one general fact about connections:

Proposition. *Let $E \to M$ be a vector bundle equipped with an affine connection ∇^*, and A be a $\operatorname{End}(E)$-valued 1-form. Set $\nabla \doteq \nabla^* + A$. Then:*

(a) ∇ is also an affine connection on E and, conversely, every affine connection on E is of this form, for some A.

(b) the relation between the curvature tensors of ∇^* and ∇ (as in the above item) is

$$R^{\nabla} = R^{\nabla^*} + \mathrm{d}^{\nabla^*}A + \frac{1}{2}[A, A],$$

where d^{∇^*} is the covariant exterior derivative acting on bundle-valued forms, induced by ∇^*, and $[\cdot,\cdot]$ is the "wedge product" on $\operatorname{End}(E)$-valued forms induced by the commutator of bundle morphisms as underlying operation.

(c) When $E = TM$, the relation between the torsion tensors is

$$\tau^{\nabla}(\boldsymbol{X}, \boldsymbol{Y}) = \tau^{\nabla^*}(\boldsymbol{X}, \boldsymbol{Y}) + A_{\boldsymbol{X}}\boldsymbol{Y} - A_{\boldsymbol{Y}}\boldsymbol{X},$$

and if g is a pseudo-Riemannian metric on M, then we have

$$(\nabla_{\boldsymbol{X}}\mathrm{g})(\boldsymbol{Y}, \boldsymbol{Z}) = (\nabla^*_{\boldsymbol{X}}\mathrm{g})(\boldsymbol{Y}, \boldsymbol{Z}) + \mathrm{g}(A_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{Z}) + \mathrm{g}(\boldsymbol{Y}, A_{\boldsymbol{X}}\boldsymbol{Z}),$$

for all $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathfrak{X}(M)$.

Remark. Item (c) says that adding a symmetric A to the connection ∇^* does not affect its torsion, and thus the only way to make ∇ torsion-free is by choosing $A = -\tau^{\nabla^*}/2$ (plus something symmetric). Moreover, if ∇^* parallelizes g, then ∇ will parallelize g as well if and only if $A_{\boldsymbol{X}}$ is g-skew-adjoint for all $\boldsymbol{X} \in \mathfrak{X}(M)$.

Proof: For now, an exercise. □

Now, let g be a *curve* of pseudo-Riemannian metrics on M. The velocity vectors $\dot{\mathrm{g}}$ of this curve are all *symmetric* $(0,2)$-tensor fields. For each instant of time, the corresponding g has a Levi-Civita connection ∇, and the velocity $\dot{\nabla}$ of this curve of connections consists of *symmetric* $(1,2)$-tensor fields (this is exactly what $\dot{\tau} = 0$ means, where τ is the torsion of g). The metric compatibility of the Levi-Civita connections, in turn, is slightly less straightforward to translate into a property of $\dot{\nabla}$.

Lemma. $(\nabla_{\boldsymbol{X}}\dot{\mathrm{g}})(\boldsymbol{Y},\boldsymbol{Z}) = \mathrm{g}(\dot{\nabla}_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}) + \mathrm{g}(\boldsymbol{X},\dot{\nabla}_{\boldsymbol{X}}\boldsymbol{Z})$.

Proof: Apply dot on both sides of $\boldsymbol{X}\mathrm{g}(\boldsymbol{Y},\boldsymbol{Z}) = \mathrm{g}(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}) + \mathrm{g}(\boldsymbol{Y},\nabla_{\boldsymbol{X}}\boldsymbol{Z})$ to get that

$$\boldsymbol{X}\dot{\mathrm{g}}(\boldsymbol{Y},\boldsymbol{Z}) = \dot{\mathrm{g}}(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}) + \dot{\mathrm{g}}(\boldsymbol{Y},\nabla_{\boldsymbol{X}}\boldsymbol{Z}) + \mathrm{g}(\dot{\nabla}_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}) + \mathrm{g}(\boldsymbol{Y},\dot{\nabla}_{\boldsymbol{X}}\boldsymbol{Z}).$$

Now use that $\boldsymbol{X}\dot{\mathrm{g}}(\boldsymbol{Y},\boldsymbol{Z}) = (\nabla_{\boldsymbol{X}}\dot{\mathrm{g}})(\boldsymbol{Y},\boldsymbol{Z}) + \dot{\mathrm{g}}(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}) + \dot{\mathrm{g}}(\boldsymbol{Y},\nabla_{\boldsymbol{X}}\boldsymbol{Z})$ and cancel everything possible with $\dot{\mathrm{g}}$. □

With this in place, we can actually compute what $\dot{\nabla}$ is.

Proposition. $2\mathrm{g}(\dot{\nabla}_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}) = (\nabla_{\boldsymbol{X}}\dot{\mathrm{g}})(\boldsymbol{Y},\boldsymbol{Z}) + (\nabla_{\boldsymbol{Y}}\dot{\mathrm{g}})(\boldsymbol{X},\boldsymbol{Z}) - (\nabla_{\boldsymbol{Z}}\dot{\mathrm{g}})(\boldsymbol{X},\boldsymbol{Y})$.

Proof: Apply dot on both sides of the Koszul formula

$$2\mathrm{g}(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}) = (\mathscr{L}_{\boldsymbol{Y}}\mathrm{g})(\boldsymbol{X},\boldsymbol{Z}) + \mathrm{d}(\boldsymbol{Y}_\flat)(\boldsymbol{X},\boldsymbol{Z})$$

to get

$$2\dot{\mathrm{g}}(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}) + 2\mathrm{g}(\dot{\nabla}_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}) = (\mathscr{L}_{\boldsymbol{Y}}\dot{\mathrm{g}})(\boldsymbol{X},\boldsymbol{Z}) + \mathrm{d}(\boldsymbol{Y}_{\dot{\flat}})(\boldsymbol{X},\boldsymbol{Z}),$$

where $\boldsymbol{Y}_\flat = \mathrm{g}(\boldsymbol{Y},\cdot)$ and $\boldsymbol{Y}_{\dot{\flat}} = \dot{\mathrm{g}}(\boldsymbol{Y},\cdot)$. A short computation expanding the definitions on the right side and using symmetry of $\dot{\nabla}$ to gather terms, together with the previous lemma, leads to the result. □

Next stop, curvature. Each Levi-Civita connection ∇ has its own curvature tensor R, and what is the velocity $\dot{R}$ of the curve of curvature tensors? For this, it helps to recall that the space of linear connections is an affine space, with translation vector space consisting of type $(1,2)$-tensor fields. This means that if we fix a "reference" connection ∇^* (which may be arbitrary, and even one connection in the curve ∇), we may write $\nabla = \nabla^* + A$, where A is a curve of type $(1,2)$-tensor fields with $\dot{A} = \dot{\nabla}$.

Proposition. $\dot{R} = \mathrm{d}^\nabla\dot{\nabla}$.

Proof: Apply dot on both sides of curvature relation

$$R = R^* + \mathrm{d}^{\nabla^*}A + \frac{1}{2}[A,A],$$

bearing in mind that $[A,\dot{\nabla}] = [\dot{\nabla},A]$, to obtain $\dot{R} = \mathrm{d}^{\nabla^*}\dot{\nabla} + [A,\dot{\nabla}] = \mathrm{d}^\nabla\dot{\nabla}$. The last equality follows from a straightforward computation. □

For the Ricci curvature, a general lemma comes in handy:

Lemma. *If S is a $\mathrm{End}(TM)$-valued 1-form and tr^1_1 stands for the trace taken relative to the first upper and lower indices of S (relative to some coordinate system – this is invariant), then*

$$\mathrm{tr}(\boldsymbol{X} \mapsto (\mathrm{d}^\nabla S)(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z}) = \mathrm{div}(S)(\boldsymbol{Y},\boldsymbol{Z}) - (\nabla_{\boldsymbol{Y}}\mathrm{tr}^1_1(S))\boldsymbol{Z}.$$

Note that $\mathrm{tr}^1_1(S)$ is a bona fide 1-form on M.

Proof: By definition of $\mathrm{d}^\nabla S$, the components of such trace evaluated at (∂_j, ∂_k) are just given by $(\mathrm{d}^\nabla S)^i_{ijk} = S^i_{jk;i} - S^i_{ik;j} = \mathrm{div}(S)_{jk} - \mathrm{tr}^1_1(S)_{k;j}$, as required. □

Corollary. $\dot{\mathrm{Ric}} = \mathrm{div}(\dot\nabla) - \nabla \mathrm{tr}^1_1(\dot\nabla)$.

Proof: The previous lemma applies to the expression $\dot R = \mathrm{d}^\nabla \dot\nabla$, as time derivative commutes with traces. □

As for the scalar curvature, the expression $\mathrm{s} = \mathrm{tr}_{\,\mathrm{g}}\mathrm{Ric}$ is slightly more complicated to deal with, as the time-dependence appears both on g and Ric. One convenient notation device to proceed will be to consider the fiber metric $\langle\cdot,\cdot\rangle$ induced by g on tensor bundles over TM. In particular, if h is any type $(0,2)$-tensor field on M, we have that $\langle \mathrm{g}, h\rangle = \mathrm{tr}_{\mathrm{g}} h$.

Lemma. *Let S be an $\mathrm{End}(TM)$-valued 1-form, and α be a 1-form on M. If $\mathrm{tr}_{\,1,2}$ stands for the* g*-trace on the lower two indices of S (relative to the vector field arguments – so that $\mathrm{tr}_{1,2}(S)$ is a vector field on M), then:*

(i) $\langle \mathrm{g}, \mathrm{div}(S)\rangle = \mathrm{div}(\mathrm{tr}_{1,2}(S))$.

(ii) $\langle \mathrm{g}, \nabla\alpha\rangle = \mathrm{div}(\alpha^\sharp)$.

Proof: Locally:

(i) $\langle \mathrm{g}, \mathrm{div}(S)\rangle = g^{ij} S^k_{ij;k} = (g^{ij}S^k_{ij})_{;k} = \mathrm{tr}_{1,2}(S)^k{}_{;k} = \mathrm{div}(\mathrm{tr}_{1,2}(S))$.

(ii) $\langle \mathrm{g}, \nabla\alpha\rangle = g^{ij}\alpha_{i;j} = (g^{ij}\alpha_i)_{;j} = \alpha^j{}_{;j} = \mathrm{div}(\alpha^\sharp)$.

□

Proposition. $\dot{\mathrm{s}} = -\langle \dot{\mathrm{g}}, \mathrm{Ric}\rangle + \mathrm{div}(\mathrm{tr}_{\,1,2}\dot\nabla) - \mathrm{div}((\mathrm{tr}^1_1(\dot\nabla)^\sharp)$

Proof: Apply dot on both sides of $\mathrm{s} = \langle \mathrm{g}, \mathrm{Ric}\rangle$ to get

$$\begin{aligned}\dot{\mathrm{s}} &= -\langle \dot{\mathrm{g}}, \mathrm{Ric}\rangle + \langle \mathrm{g}, \dot{\mathrm{Ric}}\rangle \\ &= -\langle \dot{\mathrm{g}}, \dot{\mathrm{Ric}}\rangle + \langle \mathrm{g}, \mathrm{div}(\dot\nabla) - \nabla\mathrm{tr}^1_1(\dot\nabla)\rangle \\ &= -\langle \dot{\mathrm{g}}, \mathrm{Ric}\rangle + \mathrm{div}(\mathrm{tr}_{\,1,2}\dot\nabla) - \mathrm{div}((\mathrm{tr}^1_1(\dot\nabla)^\sharp).\end{aligned}$$

□

Remark. The negative sign in $\langle \dot{\mathrm{g}}, \mathrm{Ric}\rangle$ requires an explanation. Very briefly, it is because the time-dependence of $\langle \mathrm{g}, \mathrm{Ric}\rangle$ appears not only on g and Ric, but also on $\langle\cdot,\cdot\rangle$ itself, so that the formula for the derivative $\mathrm{d}\iota_A(H) = -A^{-1}HA^{-1}$ of the inversion map ι on any Lie group (here taken to be $\mathrm{GL}(n,\mathbb{R})$, once we pass to coordinates) applies.

To apply what was established so far in a more interesting situation, we will also need to know the time evolution of the volume forms ν_{g}.

Proposition. $(\nu_{\rm g})^{\boldsymbol{\cdot}} = \dfrac{{\rm tr}_{\rm g}\dot{\rm g}}{2}\,\nu_{\rm g}$

Proof: A local computation, using that ${\rm d}(\det)_A(H) = \det(A){\rm tr}(A^{-1}H)$, for all matrices $A \in {\rm GL}(n,\mathbb{R})$ and $H \in \mathfrak{gl}(n,\mathbb{R})$, together with the chain rule. We have that

$$\left(|\det {\rm g}|^{1/2}\right)^{\boldsymbol{\cdot}} = \frac{1}{2|\det {\rm g}|^{1/2}}|\det {\rm g}|\,{\rm tr}({\rm g}^{-1}\dot{\rm g}) = \frac{{\rm tr}_{\rm g}\dot{\rm g}}{2}\,|\det {\rm g}|^{1/2},$$

and the result follows from multiplying both sides by ${\rm d}x^1 \wedge \cdots \wedge {\rm d}x^n$ (and bringing it inside the time derivative on the left side). □

The Einstein-Hilbert functional

Assume that M is *compact*. For each pseudo-Riemannian metric g on M, write

$$\mathscr{E}[{\rm g}] \doteq \int_M {\rm s}[{\rm g}]\,\nu_{\rm g},$$

where s[g] stands for the scalar curvature of g. This functional $\mathscr{E}$ defined above is called the *Einstein-Hilbert functional* on M. If $\phi\colon M \to M$ is any diffeomorphism, it holds that $\mathscr{E}[\phi^*{\rm g}] = \mathscr{E}[{\rm g}]$, so that $\mathscr{E}$ may be thought of as being defined on the moduli space of pseudo-Riemannian metrics on M. Let's compute the first variation of $\mathscr{E}$, by letting g depend on time and applying what was done in the previous section. We now go back to writing s for s[g].

$$\begin{aligned} {\rm d}\mathscr{E}_{\rm g}(\dot{\rm g}) &= \int_M (\dot{\rm s}\,\nu_{\rm g} + {\rm s}(\nu_{\rm g})^{\boldsymbol{\cdot}}) \\ &= \int_M \left(-\langle \dot{\rm g}, {\rm Ric}\rangle + {\rm div}({\rm tr}_{1,2}(\dot\nabla) - {\rm tr}_1^1(\dot\nabla)^\sharp) + \frac{\rm s}{2}\langle \dot{\rm g}, {\rm g}\rangle\right)\nu_{\rm g} \\ &= -\int_M \left\langle \dot{\rm g}, {\rm Ric} - \frac{\rm s}{2}{\rm g}\right\rangle \nu_{\rm g}, \end{aligned}$$

where in the last step we have used Stokes' theorem to get rid of the divergence terms. This means that g is a critical point of $\mathscr{E}$ if and only if

$${\rm Ric} - \frac{\rm sg}{2} = 0,$$

which is equivalent to ${\rm Ric} = 0$, provided that $n = \dim M > 2$.

The situation becomes more interesting if we consider restrictions of $\mathscr{E}$. To motivate one such restriction, note that $\mathscr{E}$ is not scale-invariant. For instance, take a constant $\lambda > 0$ and consider the metric $\lambda{\rm g}$. The Levi-Civita connection remains unchanged under homotheties, and hence so does the type $(1,3)$ curvature tensor, as well as the Ricci tensor, leading to ${\rm s}[\lambda{\rm g}] = \lambda^{-1}{\rm s}[{\rm g}]$. A coordinate computation also shows that the relation between volume forms is $\nu_{\lambda{\rm g}} = \lambda^{n/2}\nu_{\rm g}$. Thus

$$\mathscr{E}[\lambda{\rm g}] = \int_M {\rm s}[\lambda{\rm g}]\nu_{\lambda{\rm g}} = \int_M \lambda^{-1}{\rm s}[{\rm g}]\lambda^{n/2}\nu_{\rm g} = \lambda^{(n-2)/2}\mathscr{E}[{\rm g}].$$

How does one rescale $\mathscr{E}$ to a new functional, say, $\hat{\mathscr{E}}$, which is scale-invariant? The idea is to use the *volume functional*, obviously defined by

$$\mathscr{V}[\mathrm{g}] = \int_M \nu_{\mathrm{g}}.$$

Just like $\mathscr{E}$, $\mathscr{V}$ is a *Riemannian functional*, in the sense that if $\phi\colon M \to M$ is a diffeomorphism, then $\mathscr{V}[\phi^*\mathrm{g}] = \mathscr{V}[\mathrm{g}]$. However, it is now easy to see that $\mathscr{V}[\lambda\mathrm{g}] = \lambda^{n/2}\mathscr{V}[\mathrm{g}]$ for all $\lambda > 0$. Write $\hat{\mathscr{E}}[\mathrm{g}] = \mathscr{E}[\mathrm{g}]/\mathscr{V}[\mathrm{g}]^p$, where the power p is to be found. We have that

$$\hat{\mathscr{E}}[\lambda\mathrm{g}] = \hat{\mathscr{E}}[\mathrm{g}] \implies \frac{\mathscr{E}[\lambda\mathrm{g}]}{\mathscr{V}[\lambda\mathrm{g}]^p} = \frac{\mathscr{E}[\mathrm{g}]}{\mathscr{V}[g]^p} \implies \frac{\lambda^{(n-2)/2}\mathscr{E}[\mathrm{g}]}{\lambda^{np/2}\mathscr{V}[\mathrm{g}]^p} = \frac{\mathscr{E}[\mathrm{g}]}{\mathscr{V}[g]^p}$$

Arbitrariety of g allows us to cancel $\mathscr{E}$ and $\mathscr{V}$ everywhere, and use $\lambda^0 = 1$ to conclude that

$$\frac{n-2}{2} = \frac{np}{2} \implies p = \frac{n-2}{n}.$$

In short, we conclude that the following are equivalent:

(i) g is a critical point of the functional $\hat{\mathscr{E}}$ given by $\hat{\mathscr{E}}[\mathrm{g}] = \dfrac{\mathscr{E}[\mathrm{g}]}{\mathscr{V}[\mathrm{g}]^{(n-2)/n}}$;

(ii) g is a critical point of the restriction of the functional $\mathscr{E}$ to the space of metrics for which $\mathscr{V}[\mathrm{g}]$ is a constant (which, without loss of generality, we'll take to be 1).

Option (ii) is more fruitful to pursue, as we have one last technique up our sleeve: Lagrange multipliers. Since we clearly have that

$$\mathrm{d}\mathscr{V}_{\mathrm{g}}(\dot{\mathrm{g}}) = \frac{1}{2}\int_M \langle \dot{\mathrm{g}}, \mathrm{g}\rangle\, \nu_{\mathrm{g}},$$

we conclude that critical points of $\mathscr{E}$ subject to $\mathscr{V}[g] = 1$ must satisfy

$$\int_M \left\langle \dot{\mathrm{g}}, -\mathrm{Ric} + \frac{\mathrm{s}}{2}\mathrm{g} - \lambda\mathrm{g}\right\rangle \nu_{\mathrm{g}} = 0, \qquad \text{for all } \dot{\mathrm{g}},$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier. However, the condition

$$-\mathrm{Ric} + \frac{\mathrm{s}}{2}\mathrm{g} - \lambda\mathrm{g} = 0$$

is equivalent to g being an Einstein metric. We have proved the:

Theorem. *The critical points of the Einstein-Hilbert functional on a compact manifold are Ricci-flat metrics. When restricting this functional to the space of unit volume metrics, the critical points are Einstein metrics.*