VARIATIONS OF VOLUME

Ivo Terek

1 Setup

Let (M, g) be a Riemannian manifold, and $\iota\colon \Sigma \hookrightarrow M$ be an embedded submanifold. We will often write $\langle \cdot, \cdot \rangle$ instead of g and identify $\iota(\Sigma)$ with Σ, as it is usual practice. We write ∇ and ∇^Σ for the Levi-Civita connections of (M, g) and $(\Sigma, \iota^*\mathrm{g})$. We have

$$\text{(a) } \nabla_X Y = \nabla^\Sigma_X Y + \mathrm{II}(X, Y), \qquad \text{(b) } \nabla^M_X \xi = -A_\xi(X) + \nabla^\perp_X \xi, \tag{1.1}$$

for all $X, Y \in \Gamma(T\Sigma)$ and $\xi \in \Gamma(T\Sigma^\perp)$, where II denotes the second fundamental form of Σ in (M, g), A_ξ is the shape operator associated with ξ, and $\nabla^\perp$ is the normal connection of Σ. Shape operators are related to the second fundamental form via the relations

$$\langle \mathrm{II}(X, Y), \xi \rangle = \langle A_\xi(X), Y \rangle. \tag{1.2}$$

The mean curvature vector $\boldsymbol{H}_\Sigma \in \Gamma(T\Sigma^\perp)$ is the g-trace of II, and may be computed as

$$\boldsymbol{H}_\Sigma = g^{ij}\mathrm{II}\left(\partial_i, \partial_j\right) = \sum_{i=1}^k \mathrm{II}(E_i, E_i), \tag{1.3}$$

where $k = \dim \Sigma$ and $(x^1, \ldots, x^k)$ are local coordinates on Σ, or $(E_1, \ldots, E_k)$ is a local orthonormal frame tangent to Σ. When not working with orthonormal frames, we use Einstein's summation convention[1].

With the aid of a metric, one may compute the divergence of vector fields and tensor fields, in general. In our setup, we have a divergence operator $\mathrm{div}_{\mathrm{g}}$ associated with (M, g), and $\mathrm{div}_{\iota^*\mathrm{g}}$ associated with $(\Sigma, \iota^*\mathrm{g})$. The latter has an obvious extension to vector fields that are tangent to M along Σ (as opposed to acting just on vector fields that are tangent to Σ).

Definition 1. The **tangential divergence** $\mathrm{div}_\Sigma\colon \Gamma(TM|_\Sigma) \to C^\infty(\Sigma)$ is defined by

$$\mathrm{div}_\Sigma(X) = \mathrm{tr}_{\iota^*\mathrm{g}}((Y, Z) \mapsto \langle \nabla_Y X, Z \rangle) = g^{ij}\left\langle \nabla_{\partial_i} X, \partial_j \right\rangle = \sum_{i=1}^k \langle \nabla_{E_i} X, E_i \rangle, \tag{1.4}$$

where $(x^1, \ldots, x^k)$ are local coordinates on Σ, or $(E_1, \ldots, E_k)$ is a local orthonormal frame tangent to Σ.

[1]Whose true power consists in keeping a consistent index balance, not in omitting summation signs.

For any vector field X tangent to M along Σ, we write $X = X^\top + X^\perp$ according to the direct sum decomposition $TM|_\Sigma = T\Sigma \oplus T\Sigma^\perp$. Here is what we need to know about this tangential divergence operator:

Proposition 2 (Main properties of $\operatorname{div}_\Sigma$)**.**

(a) $\operatorname{div}_\Sigma(fX) = X^\top(f) + f\operatorname{div}_\Sigma(X)$, *for all* $X \in \Gamma(TM|_\Sigma)$ *and* $f \in C^\infty(\Sigma)$.

(b) $\operatorname{div}_\Sigma(X) = \operatorname{div}_{\iota^*g}(X)$, *for all* $X \in \Gamma(T\Sigma)$.

(c) $\operatorname{div}_\Sigma(X) = \operatorname{div}_\Sigma(X^\top) - \langle \boldsymbol{H}_\Sigma, X\rangle$, *for all* $X \in \Gamma(TM|_\Sigma)$.

Remark. In particular, the Laplacian $\triangle_\Sigma f$ of a smooth function $f\colon \Sigma \to \mathbb{R}$ (defined as $\operatorname{div}_{\iota^*g}(\nabla^\Sigma f)$), may be computed as $\operatorname{div}_\Sigma(\nabla^\Sigma f)$.

Proof: Noting that $X^\top = g^{ij}\langle X, \partial_i\rangle\partial_j$ whenever $(x^1,\ldots,x^k)$ is a local coordinate system for Σ, we directly compute

$$\begin{aligned}\operatorname{div}_\Sigma(fX) &= g^{ij}\langle \nabla_{\partial_i}(fX), \partial_j\rangle = g^{ij}\langle (\partial_i f)X + f\nabla_{\partial_i}X, \partial_j\rangle \\ &= g^{ij}\langle X, \partial_j\rangle \partial_i f + f g^{ij}\langle \nabla_{\partial_i}X, \partial_j\rangle = X^\top(f) + f\operatorname{div}_\Sigma(X).\end{aligned} \tag{1.5}$$

This proves (a). For (b), use (1.1-a): note that $\langle \nabla_{\partial_i}X, \partial_j\rangle = \langle \nabla^\Sigma_{\partial_i}X, \partial_j\rangle$ and apply g^{ij}. Finally, as $\operatorname{div}_\Sigma$ is additive, it suffices to again use (1.1-a) to obtain

$$\begin{aligned}\operatorname{div}_\Sigma(X^\perp) &= g^{ij}\langle \nabla_{\partial_i}(X^\perp), \partial_j\rangle = -g^{ij}\langle X^\perp, \nabla_{\partial_i}\partial_j\rangle \\ &= -g^{ij}\langle X^\perp, \mathrm{II}(\partial_i, \partial_j)\rangle = -\langle X^\perp, \boldsymbol{H}_\Sigma\rangle \\ &= -\langle X, \boldsymbol{H}_\Sigma\rangle,\end{aligned} \tag{1.6}$$

as required. □

In a similar manner to what was done for $\operatorname{div}_\Sigma$, we may also consider a "partial" Ricci tensor, acting on vector fields tangent to M along Σ. Our sign convention for the Riemann curvature tensor is $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

Definition 3. The **tangential Ricci tensor** $\operatorname{Ric}^\top_\Sigma\colon \Gamma(TM|_\Sigma) \times \Gamma(TM|_\Sigma) \to C^\infty(\Sigma)$ is defined by

$$\operatorname{Ric}^\top_\Sigma(Y,Z) = \operatorname{tr}_{\iota^*g} R(\cdot, Y, Z, \cdot) = g^{ij}R\left(\partial_i, Y, Z, \partial_j\right) = \sum_{i=1}^k R(E_i, Y, Z, E_i), \tag{1.7}$$

for all vector fields Y and Z tangent to M along Σ, where $(x^1,\ldots,x^k)$ are local coordinates on Σ, or $(E_1,\ldots,E_k)$ is a local orthonormal frame tangent to Σ.

Remark. When Y and Z above happen to be tangent to Σ, $\operatorname{Ric}^\top_\Sigma(Y,Z)$ still does not agree with the Ricci tensor of Σ itself evaluated at Y and Z. This is only guaranteed to happen when Σ is totally geodesic in (M, g), that is, when $\mathrm{II} = 0$. In addition, when Σ is a 2-sided hypersurface[2] in M and ν is a unit normal field along Σ, we have the relation $\operatorname{Ric}^\top_\Sigma(Y,Z) = \operatorname{Ric}(Y,Z) - R(\nu, Y, Z, \nu)$, and hence $\operatorname{Ric}^\top_\Sigma(\nu, \cdot) = \operatorname{Ric}(\nu, \cdot)$.

[2] 2-sided means that the normal bundle of Σ in M is trivial, that is, that there is a globally defined unit normal field along Σ. When there is no metric present, one may define the notion of being 2-sided by using the quotient line bundle $TM|_\Sigma / T\Sigma$ instead.

2 Variation formulas and examples

Let (M, g) be a Riemannian manifold, and Σ be an embedded submanifold, possibly with boundary. The **volume of** Σ is defined by

$$\mathrm{vol}(\Sigma) = \int_\Sigma 1 \, \mathrm{d}\mu_\Sigma, \tag{2.1}$$

where $\mathrm{d}\mu_\Sigma$ stands for the volume form[3] of the induced metric on Σ. When $\dim \Sigma$ equals 1 or 2, "volume" means *arclength* or *area*, respectively. To understand variations of this volume functional, we will need to consider variations of submanifolds:

Definition 4. A **variation of** Σ **in** M is a smooth map $F\colon \Sigma \times (-\varepsilon, \varepsilon) \to M$ such that $F(x, 0) = x$ for every $x \in \Sigma$. Then:

(a) F is **compactly supported** if there is a compact subset $K \subseteq \Sigma$ such that $F(x,t) = x$ for all $(x,t) \in (\Sigma \smallsetminus K) \times (-\varepsilon, \varepsilon)$.

(b) F is **boundary-fixing** if $F(x,t) = x$ for all $(x,t) \in \partial\Sigma \times (-\varepsilon, \varepsilon)$.

(c) For each fixed $t \in (-\varepsilon, \varepsilon)$, the t**-stage of the variation** is $\Sigma_t = F(\Sigma, t)$ (in other words, Σ_t is defined as the image of Σ under the partial map $F_t = F(\cdot, t)\colon \Sigma \to M$. In particular, $\Sigma_0 = \Sigma$.

(d) The **variational vector field** V of F is defined as $V = V_0$, where in general we define V_t, for $t \in (-\varepsilon, \varepsilon)$, by

$$V_t(x) = \frac{\mathrm{d}}{\mathrm{d}t} F(x,t) \in T_{F(x,t)}M, \tag{2.2}$$

for every $x \in \Sigma$.

(e) F is a **normal variation of** Σ if $V_t(x) \in [T_{F(x,t)}\Sigma_t]^\perp$ for all $(x,t) \in \Sigma \times (-\varepsilon, \varepsilon)$.

When F is not explicitly needed, one calls $\{\Sigma_t\}_{t \in (-\varepsilon,\varepsilon)}$ a variation of Σ instead.

Remark. When F is either boundary-fixing, or compactly supported with Σ noncompact, we necessarily have $V|_{\partial\Sigma} = 0$.

Given any variation $F\colon \Sigma \times (-\varepsilon, \varepsilon) \to M$ of Σ has been fixed, local coordinates $(x^1, \ldots, x^k)$ on an open subset $U \subseteq \Sigma$ induce, for each $t \in (-\varepsilon, \varepsilon)$, local coordinates $(x_t^1, \ldots, x_t^k)$ on the open subset $U_t = F(U, t)$ of Σ_t via the composition $x_t^i = x^i \circ (F_t)^{-1}$. Namely, as $F(\cdot, 0) = \mathrm{Id}_\Sigma$ is a diffeomorphism, so is $F_t = F(\cdot, t)\colon \Sigma \to \Sigma_t$ for t small enough, so that $(F_t)^{-1}$ makes sense. Such coordinates satisfy the relations

$$\mathrm{d}F_{(x,t)}\left(\left.\frac{\partial}{\partial x^i}\right|_x \right) = \left.\frac{\partial}{\partial x_t^i}\right|_{F(x,t)} \quad \text{and} \quad \mathrm{d}F_{(x,t)}\left(\left.\frac{\partial}{\partial t}\right|_t \right) = V_t(x), \tag{2.3}$$

for all $(x,t) \in \Sigma \times (-\varepsilon, \varepsilon)$. In particular, the Lie bracket $[V_t, \partial/\partial x_t^i]$ makes sense and *vanishes* due to naturality of the Lie bracket, as $\partial/\partial x^i$ and $\partial/\partial t$ commute as vector

[3] We assume that Σ is orientable, for simplicity. If not, treat $\mathrm{d}\mu_\Sigma$ as a density instead.

fields on $\Sigma \times (-\varepsilon, \varepsilon)$. When doing coordinate computations, we will always assume that the coordinates on each stage Σ_t are related to coordinates on Σ via the above construction. For economy of notation, we will also write[4] $\partial_{i,t}$ for $\partial/\partial x^i_t$.

Each Σ_t has its own volume form $\mathrm{d}\mu_{\Sigma_t}$, so we may write $(F_t)^*(\mathrm{d}\mu_{\Sigma_t}) = v(t)\,\mathrm{d}\mu_\Sigma$, for a suitable smooth function $v(t)\colon \Sigma \to \mathbb{R}$, with $v(0) = 1$. Smoothness of F also ensures smoothness of $v(t)(x)$ in the variable t. Noting that

$$\mathrm{vol}(\Sigma_t) = \int_{\Sigma_t} \mathrm{d}\mu_{\Sigma_t} = \int_{F_t(\Sigma)} \mathrm{d}\mu_{\Sigma_t} = \int_\Sigma (F_t)^*\mathrm{d}\mu_{\Sigma_t} = \int_\Sigma v(t)\,\mathrm{d}\mu_\Sigma, \tag{2.4}$$

we are justified in calling v the **volumetric density of the variation** F.

Proposition 5. *For any variation $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ of Σ, the first and second derivatives of the volumetric density v are given by:*

(a) $\dfrac{\mathrm{d}v}{\mathrm{d}t}(t) = \mathrm{div}_{\Sigma_t}(V_t)\,v(t)$

and, assuming that the variation is normal,

(b) $\dfrac{\mathrm{d}^2 v}{\mathrm{d}t^2}(t) = \left(\|\nabla^\perp V_t\|^2 - \mathrm{Ric}^\top_{\Sigma_t}(V_t, V_t) - \|A_{V_t}\|^2 - \langle \boldsymbol{H}_{\Sigma_t}, \nabla_{V_t} V_t\rangle + \langle \boldsymbol{H}_{\Sigma_t}, V_t\rangle^2\right) v(t)$

Proof: The first step is to describe $v(t)$ in coordinates. By evaluating both sides of $(F_t)^*\mathrm{d}\mu_{\Sigma_t} = v(t)\,\mathrm{d}\mu_\Sigma$ at the coordinate vector fields $(\partial_1, \ldots, \partial_k)$ and using (2.3), we see that $\sqrt{\det g(t)} = v(t)\sqrt{\det g(0)}$, where $g(t) = [g_{ij}(t)]_{i,j=1}^k$ is the matrix of components $g_{ij}(t) = \langle \partial_{i,t}, \partial_{j,t}\rangle$. Differentiating and applying *Jacobi's formula*[5], we have

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\sqrt{\det g(t)}}{\sqrt{\det g(0)}} \\ &= \frac{1}{\sqrt{\det g(0)}}\frac{1}{2}(\det g(t))^{-1/2}\frac{\mathrm{d}}{\mathrm{d}t}\det g(t) \\ &= \frac{1}{\sqrt{\det g(0)}}\frac{1}{2}(\det g(t))^{-1/2}\det g(t)\,\mathrm{tr}\left(g(t)^{-1}\frac{\mathrm{d}g}{\mathrm{d}t}(t)\right) \\ &= \frac{1}{2}\,\mathrm{tr}\left(g(t)^{-1}\frac{\mathrm{d}g}{\mathrm{d}t}(t)\right)\,v(t). \end{aligned} \tag{2.5}$$

However, we may compute this trace as

$$\begin{aligned} \mathrm{tr}\left(g(t)^{-1}\frac{\mathrm{d}g}{\mathrm{d}t}(t)\right) &= g^{ij}(t)\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \partial_{i,t}, \partial_{j,t}\right\rangle \\ &= g^{ij}(t)\left\langle \nabla_{V_t}\partial_{i,t}, \partial_{j,t}\right\rangle + g^{ij}(t)\left\langle \partial_{i,t}, \nabla_{V_t}\partial_{j,t}\right\rangle \\ &= 2g^{ij}\left\langle \nabla_{V_t}\partial_{i,t}, \partial_{j,t}\right\rangle \\ &= 2g^{ij}\left\langle \nabla_{\partial_{i,t}} V_t, \partial_{j,t}\right\rangle \\ &= 2\,\mathrm{div}_{\Sigma_t}(V_t), \end{aligned} \tag{2.6}$$

[4] We do **not** use the convention of writing derivatives as subscripts with commas or semi-colons.

[5] Namely, $\mathrm{d}(\det)_A(H) = \det A\,\mathrm{tr}(A^{-1}H)$ for all $A \in \mathrm{GL}(k)$ and $H \in \mathbb{R}^{k\times k}$. Any Lie group homomorphism $\varphi\colon G_1 \to G_2$ has $\varphi \circ \mathrm{L}_g = \mathrm{L}_{\varphi(g)} \circ \varphi$ for all $g \in G_1$, and so $\mathrm{d}\varphi_g = \mathrm{d}(L_{\varphi(g)})_e \circ \mathrm{d}\varphi_e \circ (\mathrm{d}(\mathrm{L}_g)_e)^{-1}$ by the chain rule. Applying this principle to the Lie group homomorphism $\det\colon \mathrm{GL}(k) \to \mathbb{R} \smallsetminus \{0\}$ together with the easily verified identity $\mathrm{d}(\det)_{\mathrm{Id}} = \mathrm{tr}$ yields the Jacobi formula.

and (a) follows. Differentiating (a) and reusing it leads to

$$\frac{\mathrm{d}^2 v}{\mathrm{d}t^2}(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{div}_{\Sigma_t}(V_t)) + (\mathrm{div}_{\Sigma_t}(V_t))^2\right) v(t), \tag{2.7}$$

and so it remains to compute the t-derivative of $\mathrm{div}_{\Sigma_t}(V_t)$. From here on we assume that, for each t, V_t is normal to Σ_t. Starting from

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{div}_{\Sigma_t}(V_t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}g^{ij}(t)\right)\left\langle \nabla_{\partial_{i,t}} V_t, \partial_{j,t}\right\rangle + g^{ij}(t)\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \nabla_{\partial_{i,t}} V_t, \partial_{j,t}\right\rangle, \tag{2.8}$$

we will compute each term in the right side separately. As

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} g_{k\ell}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \partial_{k,t}, \partial_{\ell,t}\rangle \\ &= \langle \nabla_{V_t}\partial_{k,t}, \partial_{\ell,t}\rangle + \langle \partial_{k,t}, \nabla_{V_t}\partial_{\ell,t}\rangle \\ &= \left\langle \nabla_{\partial_{k,t}} V_t, \partial_{\ell,t}\right\rangle + \left\langle \partial_{k,t}, \nabla_{\partial_{\ell,t}} V_t\right\rangle \\ &= -\langle A_{V_t}(\partial_{k,t}), \partial_{\ell,t}\rangle - \langle \partial_{k,t}, A_{V_t}(\partial_{\ell,t})\rangle \\ &= -2\langle A_{V_t}(\partial_{k,t}), \partial_{\ell,t}\rangle,\end{aligned} \tag{2.9}$$

it follows that

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}g^{ij}(t)\right)\left\langle \nabla_{\partial_{i,t}} V_t, \partial_{j,t}\right\rangle &= -g^{ik}(t)\left(\frac{\mathrm{d}}{\mathrm{d}t} g_{k\ell}(t)\right) g^{\ell j}(t)\left\langle \nabla_{\partial_{i,t}} V_t, \partial_{j,t}\right\rangle \\ &= 2g^{ik}(t)\langle A_{V_t}(\partial_{k,t}), \partial_{\ell,t}\rangle g^{\ell j}(t)\left\langle \nabla_{\partial_{i,t}} V_t, \partial_{j,t}\right\rangle \\ &= -2g^{ik}(t)g^{\ell j}(t)\langle A_{V_t}(\partial_{k,t}), \partial_{\ell,t}\rangle\langle A_{V_t}(\partial_{i,t}), \partial_{j,t}\rangle \\ &= -2\|A_{V_t}\|^2.\end{aligned} \tag{2.10}$$

In addition, observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \nabla_{\partial_{i,t}} V_t, \partial_{j,t}\right\rangle &= \left\langle \nabla_{V_t}\nabla_{\partial_{i,t}} V_t, \partial_{j,t}\right\rangle + \left\langle \nabla_{\partial_{i,t}} V_t, \nabla_{V_t}\partial_{j,t}\right\rangle \\ &= R(V_t, \partial_{i,t}, V_t, \partial_{j,t}) + \left\langle \nabla_{\partial_{i,t}}\nabla_{V_t} V_t, \partial_{j,t}\right\rangle + \left\langle \nabla_{\partial_{i,t}} V_t, \nabla_{\partial_{j,t}} V_t\right\rangle,\end{aligned} \tag{2.11}$$

and thus

$$g^{ij}(t)\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \nabla_{\partial_{i,t}} V_t, \partial_{j,t}\right\rangle = -\mathrm{Ric}^{\top}_{\Sigma}(V_t, V_t) + \mathrm{div}_{\Sigma_t}(\nabla_{V_t} V_t) + \|A_{V_t}\|^2 + \|\nabla^{\perp} V_t\|^2. \tag{2.12}$$

Here, we have used self-adjointness of A_{V_t} to conclude that $\|A^2_{V_t}\| = \|A_{V_t}\|^2$, as well as the Pythagorean relation $\|\nabla V_t\|^2 = \|A_{V_t}\|^2 + \|\nabla^{\perp} V_t\|^2$. Finally, as $\langle A_{V_t}(\cdot), V_t\rangle = 0$ implies that[6] $\langle \nabla_{V_t} V_t, Z_t\rangle = 0$ whenever $Z_t \in \Gamma(T\Sigma_t)$, we have that

$$\begin{aligned}\mathrm{div}_{\Sigma_t}(\nabla_{V_t} V_t) &= g^{ij}(t)\left\langle \nabla_{\partial_{i,t}}\nabla_{V_t} V_t, \partial_{j,t}\right\rangle \\ &= g^{ij}(t)\partial_{i,t}\langle \nabla_{V_t} V_t, \partial_{j,t}\rangle - g^{ij}(t)\left\langle \nabla_{V_t} V_t, \nabla_{\partial_{i,t}}\partial_{j,t}\right\rangle \\ &= 0 - g^{ij}(t)\langle \nabla_{V_t} V_t, \mathrm{II}(\partial_{i,t}, \partial_{j,t})\rangle \\ &= -\langle \nabla_{V_t} V_t, \boldsymbol{H}_{\Sigma_t}\rangle.\end{aligned} \tag{2.13}$$

Putting (2.10), (2.12), (2.13), and Proposition 2 together, (b) follows from (2.7). □

[6]Namely: $\langle \nabla_{V_t} V_t, \partial_{i,t}\rangle = -\langle V_t, \nabla_{V_t}\partial_{i,t}\rangle = -\langle V_t, \nabla_{\partial_{i,t}} V_t\rangle = \langle V_t, A_{V_t}(\partial_{i,t})\rangle = 0$.

Corollary 6. *For any compactly-supported variation* $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ *of* Σ*, we have:*

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(\Sigma_t) = \displaystyle\int_{\Sigma_t} \mathrm{div}_{\Sigma_t}(V_t)\,\mathrm{d}\mu_{\Sigma_t}$,

and, assuming that the variation is normal,

(b) $\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{vol}(\Sigma_t) = \displaystyle\int_{\Sigma_t} \|\nabla^\perp V_t\|^2 - \mathrm{Ric}^\top_{\Sigma_t}(V_t,V_t) - \|A_{V_t}\|^2 - \langle \boldsymbol{H}_{\Sigma_t}, \nabla_{V_t}V_t\rangle + \langle \boldsymbol{H}_{\Sigma_t}, V_t\rangle^2\,\mathrm{d}\mu_{\Sigma_t}$

Proof: As the variation has compact support, we may differentiate (2.4) under the integral sign and apply Proposition 5. The original definition of $v(t)$ allows us to rewrite the resulting quantities as integrals over Σ_t (as opposed to integrals over Σ). □

Definition 7. An embedded submanifold Σ of (M, g) is called **minimal** if

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathrm{vol}(\Sigma_t) = 0 \tag{2.14}$$

for every compactly supported and boundary-fixing variations $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ of Σ.

Example 8. No closed submanifold of Euclidean space $\mathbb{R}^n$ is minimal. More precisely, if $\Sigma \subseteq \mathbb{R}^n$ is k-dimensional and closed, consider the compactly supported variation $\{(1+t)\Sigma\}_{t\in(-\varepsilon,\varepsilon)}$ and note that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathrm{vol}((1+t)\Sigma) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(1+t)^k\mathrm{vol}(\Sigma) = k\,\mathrm{vol}(\Sigma) > 0, \tag{2.15}$$

as a consequence of the general relation $\mathrm{vol}(\lambda\Sigma) = \lambda^k\mathrm{vol}(\Sigma)$, valid for $\lambda \in (0,\infty)$.

Example 9. A k-dimensional submanifold Σ of Euclidean space $\mathbb{R}^n$ is minimal if and only if all coordinate projections $x^r|_\Sigma\colon \Sigma \to \mathbb{R}$ are harmonic functions. Abbreviating $x^r|_\Sigma$ simply to x^r, assume that Σ is minimal, and let $\eta \in C^\infty_c(\Sigma)$ be arbitrary. Writing $(e_1,\ldots,e_n)$ for the canonical basis of $\mathbb{R}^{n+1}$ and regarding all such vectors as constant fields on $\mathbb{R}^n$, we have that

$$\begin{aligned}\mathrm{div}_\Sigma(\eta e_r) &= g^{ij}\langle \nabla_{\partial_i}(\eta e_r), \partial_j\rangle = g^{ij}\langle (\partial_i\eta)e_r, \partial_j\rangle \\ &= g^{ij}\langle e_r, \partial_j\rangle\partial_i\eta = e_r^\top(\eta) \\ &= \langle e_r^\top, \nabla^\Sigma\eta\rangle = \langle \nabla^\Sigma x^r, \nabla^\Sigma\eta\rangle,\end{aligned} \tag{2.16}$$

where $\nabla^\Sigma f \in \Gamma(T\Sigma)$ denotes the gradient field of any smooth function $f \in C^\infty(\Sigma)$, and we use that the gradient $\nabla^\Sigma x^r$ is obtained by projecting onto $T\Sigma$ the full gradient $\nabla x^r = e_r$. In addition, Proposition 2 together of the definition of Laplacian $\triangle_\Sigma$ gives us that $\mathrm{div}_\Sigma(\eta\nabla^\Sigma x^r) = \langle \nabla^\Sigma\eta, \nabla^\Sigma x^r\rangle + \eta\triangle_\Sigma x^r$. Applying the first variation formula for both fields ηe_r and $\eta\nabla^\Sigma x^r$ gives us that

$$\int_\Sigma \langle \nabla^\Sigma x^r, \nabla^\Sigma\eta\rangle\,\mathrm{d}\mu_\Sigma = 0 \quad\text{and}\quad \int_\Sigma \langle \nabla^\Sigma\eta, \nabla^\Sigma x^r\rangle + \eta\triangle_\Sigma x^r\,\mathrm{d}\mu_\Sigma = 0, \tag{2.17}$$

leading to

$$\int_\Sigma \eta \triangle_\Sigma x^r \, \mathrm{d}\mu_\Sigma = 0. \tag{2.18}$$

Arbitrariety of η now implies that $\triangle_\Sigma x^r = 0$, as required. Conversely, assume that $V \in \Gamma(TM|_\Sigma)$ is compactly supported with $V|_{\partial\Sigma} = 0$, and written as $V = \sum_{r=1}^n V^r e_r$. Then $V^r \in C_c^\infty(\Sigma)$ for all $r = 1, \ldots, n$, and thus (2.16) for $\eta = V^r$ yields

$$\begin{aligned} \int_\Sigma \operatorname{div}_\Sigma(V) \, \mathrm{d}\mu_\Sigma &= \int_\Sigma \operatorname{div}_\Sigma \left(\sum_{r=1}^n V^r e_r \right) \mathrm{d}\mu_\Sigma \\ &= \sum_{r=1}^n \int_\Sigma \operatorname{div}_\Sigma(V^r e_r) \, \mathrm{d}\mu_\Sigma \\ &= \sum_{r=1}^n \int_\Sigma \langle \nabla^\Sigma V^r, \nabla^\Sigma x^r \rangle \, \mathrm{d}\mu_\Sigma \\ &= -\sum_{r=1}^n \int_\Sigma V^r \triangle_\Sigma x^r \, \mathrm{d}\mu_\Sigma = 0, \end{aligned} \tag{2.19}$$

as required. As one consequence of this equivalence, it now follows that if Σ is minimal, then it is contained in the convex hull of its boundary, which is defined as the intersection

$$\operatorname{Conv}(\partial\Sigma) = \bigcap \{ H \mid H \text{ is a half-space in } \mathbb{R}^n \text{ with } \partial\Sigma \subseteq H \}. \tag{2.20}$$

Indeed, let H be a half-space of $\mathbb{R}^n$, written as $H = \{x \in \mathbb{R}^n \mid \varphi(x) \leq b\}$ for suitable $\varphi \in (\mathbb{R}^n)^*$ and $b \in \mathbb{R}$. Since all the coordinate functions are harmonic on Σ, so is the restriction $\varphi|_\Sigma$. By the maximum principle for harmonic functions, there is $x_0 \in \partial\Sigma$ for which $\varphi|_\Sigma(x_0)$ is maximum. Now, it follows that if $\partial\Sigma \subseteq H$, then $\Sigma \subseteq H$ as well: let $x \in \Sigma$, and estimate $\varphi(x) \leq \varphi(x_0) \leq b$, so that $x \in H$. This proves that $\Sigma \subseteq \operatorname{Conv}(\partial\Sigma)$.

Example 10. We may generalize the first part of Example 9, considering now submanifolds of the sphere $\mathbb{S}^n$ and of hyperbolic space $\mathbb{H}^n$. To treat them simultaneously, fix a parameter $c \in \{1, -1\}$ and consider in $\mathbb{R}^{n+1}$ the scalar product

$$\langle v, w \rangle_c = v^1 w^1 + \ldots + v^n w^n + c v^{n+1} w^{n+1}, \tag{2.21}$$

for all $v = (v^1, \ldots, v^{n+1})$ and $w = (w^1, \ldots, w^{n+1})$ in $\mathbb{R}^{n+1}$. When $c = 1$ we have classical Euclidean space, and when $c = -1$ we have Minkowski space. This means that the space form

$$\mathbb{M}^n(c) = \{ p \in \mathbb{R}^{n+1} \mid \langle p, p \rangle_c = c \} \tag{2.22}$$

is the sphere for $c = 1$, and hyperbolic space for $c = -1$. We claim that a submanifold $\Sigma \subseteq \mathbb{M}^n(c)$ is minimal if and only if all coordinate functions $x^r \colon \Sigma \to \mathbb{R}$ satisfy the eigenvalue equation $\triangle_\Sigma x^r + ckx^r = 0$, where $k = \dim \Sigma$.

All formulas seen so far remain valid when the ambient manifold (M, g) is pseudo-Riemannian and with indefinite metric signature, provided that squared norms are suitably interpreted (e.g., $\|A\|^2 = \langle A, A \rangle$, which may now vanish even when $A \neq 0$).

The difficulty now is that the canonical basis $(e_1, \ldots, e_{n+1})$ of $\mathbb{R}^{n+1}$ is not in general tangent to $\mathbb{M}^n(c)$. Denoting by P the position vector field of $\mathbb{M}^n(c)$ in $\mathbb{R}^{n+1}$ (i.e., given by $P(x) = x \in T_x(\mathbb{R}^{n+1})$, for every $x \in \mathbb{M}^n(c)$), we consider the orthogonal projections $\widetilde{e}_r(x) = e_r - c\langle e_r, P\rangle P$, that is, $\widetilde{e}_r(x) = e_r - cx^r P$. The Levi-Civita connection ∇ of $\mathbb{M}^n(c)$ is given by $\nabla_X Y = \mathrm{d}Y(X) - c\langle \mathrm{d}Y(X), P\rangle P$, and thus

$$\begin{aligned} \nabla_X \widetilde{e}_j &= \mathrm{d}\widetilde{e}_j(X) - c\langle \mathrm{d}\widetilde{e}_j(X), P\rangle P \\ &= -c\,\mathrm{d}x^r(X)P - cx^r X - c\langle -c\,\mathrm{d}x^r(X)P - cx^r X, P\rangle P \\ &= -c\,\mathrm{d}x^r(X)P - cx^r X + c\,\mathrm{d}x^r(X)P + 0 \\ &= -cx^r X. \end{aligned} \tag{2.23}$$

Repeating (2.16) with $\widetilde{e}_r$ instead of e_r and (2.23) instead of $\nabla_X e_j = 0$ leads to

$$\mathrm{div}_\Sigma(\eta \widetilde{e}_r) = \langle \nabla^\Sigma x^r, \nabla^\Sigma \eta\rangle - ck\eta x^r, \tag{2.24}$$

for all $\eta \in C_c^\infty(\Sigma)$. This means that, if Σ is minimal, (2.17) now reads

$$\int_\Sigma \langle \nabla^\Sigma x^r, \nabla^\Sigma \eta\rangle - ck\eta x^r \,\mathrm{d}\mu_\Sigma = 0 \quad \text{and} \quad \int_\Sigma \langle \nabla^\Sigma \eta, \nabla^\Sigma x^r\rangle + \eta \triangle_\Sigma x^r \,\mathrm{d}\mu_\Sigma = 0, \tag{2.25}$$

so that

$$\int_\Sigma \eta(\triangle_\Sigma x^r + ckx^r)\,\mathrm{d}\mu_\Sigma = 0, \tag{2.26}$$

and thus $\triangle_\Sigma x^r + ckx^r = 0$ by arbitrariety of η. Conversely, if $\triangle_\Sigma x^r + ckx^r = 0$ holds for $r = 1, \ldots, n+1$, then Σ must be minimal: (2.19) boils down to

$$\int_\Sigma \mathrm{div}_\Sigma(V)\,\mathrm{d}\mu_\Sigma = -\sum_{r=1}^{n+1} \int_\Sigma V^r(\triangle_\Sigma x^r + ckx^r)\,\mathrm{d}\mu_\Sigma = 0, \tag{2.27}$$

for every compactly supported $V \in \Gamma(TM|_\Sigma)$.

Remark. The above example is easily modified to provide the obvious analogous conclusions for other indefinite signature space forms, such as the de Sitter space $\mathbb{S}_1^n = \{p \in \mathbb{R}^{n+1} \mid \langle p, p\rangle_{-1} = 1\}$, etc. (here, it becomes $\triangle_\Sigma x^r + kx^r = 0$, just as in the sphere $\mathbb{S}^n$).

Due to the general relation

$$\int_\Sigma \mathrm{div}_\Sigma V\,\mathrm{d}\mu_\Sigma = -\int_\Sigma \langle \boldsymbol{H}_\Sigma, V\rangle\,\mathrm{d}\mu_\Sigma + \int_{\partial\Sigma} \langle V, N\rangle\,\mathrm{d}\mu_{\partial\Sigma}, \tag{2.28}$$

valid as a consequence of the divergence theorem (with N being the unit outward conormal to $\partial\Sigma$ along Σ), we see that:

(i) Σ is minimal if and only if[7] $\boldsymbol{H}_\Sigma = 0$.

(ii) purely tangential compactly supported variations produce no variation.

With item (i) in mind, the next example should not be surprising:

[7] Considering only boundary-fixing variations allows us to ignore the boundary term in (2.28), while the L^2 inner product between compactly supported vector fields tangent to M along Σ is nondegenerate.

Example 11. Consider again the space forms $\mathbb{M}^n(c)$ from Example 10, and also set $\mathbb{M}^n(0) = \mathbb{R}^n$ so we can discuss all three of $\mathbb{R}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$ together. Letting Σ be a submanifold of $\mathbb{M}^n(c)$ and writing P for the position vector field of Σ, we claim that $\triangle_\Sigma P + ckP = \boldsymbol{H}_\Sigma$ (this is usually written just as $\triangle_\Sigma x + ckx = \boldsymbol{H}_\Sigma$), where the Laplacian $\triangle_\Sigma P$ is defined componentwise.

To see this, we will repeatedly use the following general fact: whenever (M, g) is a pseudo-Riemannian manifold and N is a nondegenerate submanifold of M, the formula $(\mathrm{Hess}_M f)(X,Y) = (\mathrm{Hess}_N(f|_N))(X,Y) - \mathrm{d}f(\mathrm{II}(X,Y))$ holds for $X, Y \in \Gamma(TN)$, with II denoting the second fundamental form of N in M.

The second fundamental form of $\mathbb{M}^n(c)$ in $\mathbb{R}^{n+1}$ (set to zero when $c = 0$) is given by $(X,Y) \mapsto c\langle \mathrm{d}Y(X), P\rangle P = -c\langle X, Y\rangle P$ (as seen by differentiating $\langle Y, P\rangle = 0$ in the direction of X), so that $(\mathrm{Hess}_{\mathbb{M}^n(c)} x^r)(X,Y) = \mathrm{d}x^r(-c\langle X,Y\rangle P) = -cx^r\langle X, Y\rangle$, and thus

$$(\mathrm{Hess}_\Sigma x^r) = -cx^r\langle X,Y\rangle + \mathrm{d}x^r(\mathrm{II}(X,Y)) \tag{2.29}$$

for all $X, Y \in \Gamma(T\Sigma)$, with II denoting the second fundamental form of Σ in $\mathbb{M}^n(c)$. Hence

$$\begin{aligned} \triangle_\Sigma x^r &= g^{ij}(\mathrm{Hess}_\Sigma x^r)(\partial_i, \partial_j) \\ &= g^{ij}(-cx^r g_{ij} + \mathrm{d}x^r(\mathrm{II}(\partial_i, \partial_j)) \\ &= -ckx^r + \mathrm{d}x^r(\boldsymbol{H}_\Sigma), \end{aligned} \tag{2.30}$$

leading to $\triangle_\Sigma x^r + ckx^r = \mathrm{d}x^r(\boldsymbol{H}_\Sigma)$, as required.

Item (ii) is the reason we have focused on normal variations in the previous result. Observe that evaluating item (b) of Corollary 6 at $t = 0$ yields that the second derivative of $\mathrm{vol}(\Sigma_t)$ at $t = 0$ equals

$$\int_\Sigma \|\nabla^\perp V\|^2 - \mathrm{Ric}^\top_\Sigma(V,V) - \|A_V\|^2 - \left\langle \boldsymbol{H}_\Sigma, \nabla_{V_t} V_t\big|_{t=0}\right\rangle + \langle \boldsymbol{H}_\Sigma, V\rangle^2 \, \mathrm{d}\mu_\Sigma. \tag{2.31}$$

However, the term $\nabla_{V_t} V_t\big|_{t=0}$ depends not only on the vector field V, but also on the variation F itself. In general, the value $(\nabla_X Y)_p$ depends on X_p and on the values of Y along a small piece of the integral curve of X passing through p, and this means that $\nabla_V V$ is not even well-defined in our setting. More precisely, evaluating $\nabla_{V_t} V_t$ at $t = 0$ does define a vector field tangent to M along Σ, but also eliminates transverse information that V by itself cannot recover.

This issue dissapears in the case where Σ is minimal, as (2.31) gets simplified to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \mathrm{vol}(\Sigma_t) = \int_\Sigma \|\nabla^\perp V\|^2 - \mathrm{Ric}^\top_\Sigma(V,V) - \|A_V\|^2 \, \mathrm{d}\mu_\Sigma. \tag{2.32}$$

The right side of (2.32) depends only on V and not on the variation itself. For this reason, it is usually denoted by $I_\Sigma(V,V)$, and I_Σ is called the **index form** of Σ. This motivates the following definition:

Definition 12. A minimal submanifold Σ of (M, g) is **stable** if

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \mathrm{vol}(\Sigma_t) \geq 0 \tag{2.33}$$

for every boundary-fixing normal variation $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ of Σ or, equivalently, if

$$\int_\Sigma \|\nabla^\perp V\|^2 \mathrm{d}\mu_\Sigma \geq \int_\Sigma \mathrm{Ric}_\Sigma^\top(V,V) + \|A_V\|^2\,\mathrm{d}\mu_\Sigma \tag{2.34}$$

for every $V \in \Gamma(T\Sigma^\perp)$ with $V|_{\partial\Sigma} = 0$.

3 The hypersurface case and the Gauss formula

Let (M, g) be a Riemannian manifold, and assume this time that $\iota\colon \Sigma \hookrightarrow M$ is an embedded 2-sided hypersurface in M, with a fixed choice of a unit normal field $\nu \in \Gamma(T\Sigma^\perp)$. Writing A for the shape operator associated with ν, we have the Gauss formula[8]

$$R^\Sigma(X,Y,Z,W) = R(X,Y,Z,W) + \langle AY, Z\rangle\langle AX, W\rangle - \langle AX, Z\rangle\langle AY, W\rangle, \tag{3.1}$$

valid for all $X, Y, Z, W \in \Gamma(T\Sigma)$. Writing the mean curvature vector as[9] $\boldsymbol{H} = H\nu$ and $(\iota^*\mathrm{g})$-tracing in the variables X and W yields

$$\mathrm{Ric}^\Sigma(Y,Z) = \mathrm{Ric}_\Sigma^\top(Y,Z) + H\langle AY, Z\rangle - \langle AY, AZ\rangle \tag{3.2}$$

which, rewritten without $\mathrm{Ric}_\Sigma^\top$, reads

$$\mathrm{Ric}^\Sigma(Y,Z) = \mathrm{Ric}(Y,Z) - \mathrm{Ric}(\nu,\nu) + H\langle AY, Z\rangle - \langle AY, AZ\rangle. \tag{3.3}$$

Taking the $(\iota^*\mathrm{g})$-trace yet again, we obtain

$$\mathrm{s}^\Sigma = \mathrm{s}^M - 2\,\mathrm{Ric}(\nu,\nu) + H^2 - \|A\|^2, \tag{3.4}$$

where s^M and s^Σ are the scalar curvatures of (M,g) and $(\Sigma, \iota^*\mathrm{g})$, respectively. Here, $\|A^2\| = \|A\|^2$ by self-adjointess of A. Note also that $\|A\|^2 = \|\mathrm{II}\|^2$ vanishes if and only if Σ is totally geodesic in (M,g). A convenient rearrangement of (3.4) is

$$\mathrm{Ric}(\nu,\nu) = \frac{1}{2}(\mathrm{s}^M - \mathrm{s}^\Sigma + H^2 - \|A\|^2). \tag{3.5}$$

Now, every $V \in \Gamma(T\Sigma^\perp)$ may be written as $V = \eta\nu$ for some $\eta \in C^\infty(\Sigma)$, with $V|_{\partial\Sigma} = 0$ if and only if $\eta|_{\partial\Sigma} = 0$, and V compactly supported if and only if η is. As $\nabla^\perp\nu = 0$ (since differentiating $\langle\nu,\nu\rangle = 1$ in the direction of X leads to $\langle\nabla_X^\perp\nu,\nu\rangle = 0$ and thus $\nabla_X^\perp\nu = 0$), we have that $\|\nabla^\perp V\|^2 = \|\nabla^\perp(\eta\nu)\|^2 = \|\mathrm{d}\eta\otimes\nu\|^2 = \|\nabla^\Sigma\eta\|^2$. Also note that $A_V = A_{\eta\nu} = \eta A$. Putting all of this together, we have that

$$I_\Sigma(\eta,\eta) = \int_\Sigma \|\nabla^\Sigma\eta\|^2 - (\mathrm{Ric}(\nu,\nu) + \|A\|^2)\eta^2\,\mathrm{d}\mu_\Sigma. \tag{3.6}$$

[8] Conveniently expressed in general codimension as $R^\Sigma = R + \mathrm{II} \owedge \mathrm{II}$, where $\owedge$ is the Kulkarni-Nomizu product between TM-valued twice-covariant symmetric tensor fields: it is defined by $2(T\owedge S)(X,Y,Z,W) = \langle T(Y,Z), S(X,W)\rangle - \langle T(X,Z), S(X,W)\rangle + \langle S(Y,Z), T(X,W)\rangle - \langle S(X,Z), T(X,W)\rangle$.

[9] Here, H is the **mean curvature scalar associated with** ν. Replacing ν with $-\nu$ causes H to be replaced with $-H$ (as the vector $\boldsymbol{H}$ must remain invariant).

Integrating by parts and using that $V|_{\partial\Sigma} = 0$, we also obtain

$$\begin{aligned} I_\Sigma(\eta,\eta) &= \int_\Sigma -\eta\triangle_\Sigma\eta - (\mathrm{Ric}(\nu,\nu) + \|A\|^2)\eta^2 \, \mathrm{d}\mu_\Sigma \\ &= -\int_\Sigma \eta L_\Sigma \eta \, \mathrm{d}\mu_\Sigma, \end{aligned} \tag{3.7}$$

where the **stability operator of** Σ acting on functions is given by

$$L_\Sigma\eta = \triangle_\Sigma\eta + (\mathrm{Ric}(\nu,\nu) + \|A\|^2)\eta. \tag{3.8}$$

Observe that L_Σ is elliptic (its principal symbol equals the one of $\triangle_\Sigma$) and self-adjoint. It is a well-known fact that the first Dirichlet eigenvalue[10] $\lambda_1(L_\Sigma)$ is non-negative, and expressed via a Rayleigh quotient:

$$\lambda_1(L_\Sigma) = \min_\eta \left\{ -\int_\Sigma \eta L_\Sigma \eta \, \mathrm{d}\mu_\Sigma \mid \int_\Sigma \eta^2 \, \mathrm{d}\mu_\Sigma = 1 \right\} \geq 0. \tag{3.9}$$

Proposition 13. *If (M, g) has $\mathrm{Ric} > 0$, then M does not contain any closed stable minimal 2-sided hypersurfaces.*

Proof: If Σ were such a hypersurface, one could set $\eta = 1$ in the stability condition

$$\int_\Sigma \|\nabla^\Sigma \eta\|^2 \, \mathrm{d}\mu_\Sigma \geq \int_\Sigma (\mathrm{Ric}(\nu,\nu) + \|A\|^2)\eta^2 \, \mathrm{d}\mu_\Sigma \tag{3.10}$$

to obtain

$$0 \geq \int_\Sigma \mathrm{Ric}(\nu,\nu) + \|A\|^2 \, \mathrm{d}\mu_\Sigma \geq \int_\Sigma \mathrm{Ric}(\nu,\nu) \, \mathrm{d}\mu_\Sigma > 0, \tag{3.11}$$

a contradiction. □

A small modification of the above argument gives:

Proposition 14. *If (M, g) has $\mathrm{Ric} \geq 0$, then any closed stable minimal 2-sided hypersurface Σ of M is totally geodesic. In addition, $\mathrm{Ric}(\nu,\nu) = 0$ (for any unit normal field ν along Σ) and $\mathrm{s}^M|_\Sigma = \mathrm{s}^\Sigma$. When $\dim M = 3$, the surface Σ (when connected) must be homeomorphic to a sphere or isometric to a flat torus.*

Proof: Choosing $\eta = 1$, we see that the strict inequality in (3.11) now becomes weak and leads to

$$\int_\Sigma \mathrm{Ric}(\nu,\nu) + \|A\|^2 \, \mathrm{d}\mu_\Sigma = 0. \tag{3.12}$$

As each term in the above integrand is non-negative, continuity leads to $\mathrm{Ric}(\nu,\nu) = 0$ and also to $\|A\|^2 = 0$ (and hence $A = 0$). The relation between scalar curvatures now follows from (3.4). When $\dim M = 3$, we may apply the Gauss-Bonnet theorem to Σ, using the relation $\mathrm{s}^\Sigma = 2K^\Sigma$, with K^Σ being the Gaussian curvature of Σ. Namely, as $\mathrm{s}^\Sigma \geq 0$, we have that

$$\chi(\Sigma) = \frac{1}{2\pi}\int_\Sigma K^\Sigma \, \mathrm{d}\mu_\Sigma \geq 0 \implies \chi(\Sigma) = 0 \text{ or } \chi(\Sigma) = 1, \tag{3.13}$$

[10] By a Dirichlet eigenvalue of an operator L we mean a real number λ such that there is a non-zero function f with $f|_{\partial\Sigma} = 0$ and $Lf + \lambda f = 0$.

and the dichotomy follows from the classification of closed surfaces. In the case where Σ is a torus, $\chi(\Sigma) = 0$ implies that $\int_\Sigma \mathrm{s}^\Sigma \, \mathrm{d}\mu_\Sigma = 0$, so that $\mathrm{s}^\Sigma \geq 0$ implies that $\mathrm{s}^\Sigma = 0$ by continuity.

□

The assumption that $\mathrm{Ric} \geq 0$ in the previous result may be somewhat weakened when $\dim M = 3$.

Proposition 15. *If (M, g) has $\mathrm{s}^M \geq 0$ and $\dim M = 3$, then any stable minimal torus Σ is totally geodesic and flat, with $\mathrm{Ric}(\nu,\nu) = 0$ for any chosen unit normal field ν along Σ.*

Proof: In general, substituting (3.5) into (3.10) gives us a rephrased stability condition

$$\int_\Sigma \|\nabla^\Sigma \eta\|^2 \, \mathrm{d}\mu_\Sigma \geq \frac{1}{2}\int_\Sigma \mathrm{s}^M - \mathrm{s}^\Sigma - H^2 + \|A\|^2 \, \mathrm{d}\mu_\Sigma. \tag{3.14}$$

We know that $\int_\Sigma \mathrm{s}^\Sigma \, \mathrm{d}\mu_\Sigma = 0$ by the Gauss-Bonnet theorem, as Σ is a torus. If Σ is also minimal and stable, then setting $\eta = 1$ in the above yields

$$0 \geq \frac{1}{2}\int_\Sigma \mathrm{s}^M + \|A\|^2 \, \mathrm{d}\mu_\Sigma \geq 0, \tag{3.15}$$

as $\mathrm{s}^M \geq 0$. Therefore $\mathrm{s}^M = 0$ along Σ and $\|A\|^2 = 0$, so Σ is totally geodesic. We may now integrate (3.5) to obtain $\int_\Sigma \mathrm{Ric}(\nu,\nu) \, \mathrm{d}\mu_\Sigma = 0$. This means that the constant function $\mathrm{vol}(\Sigma)^{-1/2}$ is an eigenfunction for L_Σ and that $\lambda_1(L_\Sigma) = 0$ is the first eigenvalue, in view of (3.9). More precisely, we know that the minimum of the integrals $-\int_\Sigma \eta L_\Sigma \eta \, \mathrm{d}\mu_\Sigma$ is nonnegative, but that the value zero was indeed realized by the constant function $\mathrm{vol}(\Sigma)^{-1/2}$, so such minimum in fact equals zero. This means that $\mathrm{Ric}(\nu,\nu) = L_\Sigma(1) = 0$ (this spectral argument is crucial in passing from "average zero" to "pointwise zero"). In any case, (3.5) now implies that $\mathrm{s}^\Sigma = 0$ as well. □

For our next result, we'll describe (still in the hypersurface case) how the mean curvature of the stages of a variation evolves. First, note that

$$\boldsymbol{H} = H\nu \implies H = -\mathrm{div}_\Sigma \nu, \tag{3.16}$$

by a direct computation:

$$\mathrm{div}_\Sigma \nu = g^{ij}\langle \nabla_{\partial_i}\nu, \partial_j\rangle = -g^{ij}\langle \nu, \nabla_{\partial_i}\partial_j\rangle = -g^{ij}\langle \nu, \mathrm{II}(\partial_i,\partial_j)\rangle = -\langle \nu, \boldsymbol{H}\rangle = -H. \tag{3.17}$$

Thus:

Proposition 16. *Let $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ be a normal variation of Σ, and write the variational vector fields as $V_t = \eta_t \nu_t$, where ν_t is an unit normal field along Σ_t and η_t is a suitable smooth function. Denoting by A_t and H_t the shape operator and mean curvature scalar of Σ_t associated with ν_t, respectively, we have*

$$\eta_t \frac{\mathrm{d}}{\mathrm{d}t} H_t = -\|\nabla^{\Sigma_t}\eta_t\|^2 + (\mathrm{Ric}(\nu_t,\nu_t) + \|A_t\|^2)\eta_t^2 \tag{3.18}$$

Proof: From (2.10), (2.12), (2.13), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{div}_{\Sigma_t}(V_t) = -\|A_{V_t}\|^2 - \operatorname{Ric}^{\top}_{\Sigma_t}(V_t, V_t) - \langle \nabla_{V_t} V_t, \boldsymbol{H}_{\Sigma_t}\rangle + \|\nabla^{\perp} V_t\|^2. \tag{3.19}$$

Substituting $V_t = \eta_t \nu_t$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{div}_{\Sigma_t}(\eta_t \nu_t) = -\|A_t\|^2\eta_t^2 - \operatorname{Ric}(\nu_t, \nu_t)\eta_t^2 - V_t(\eta_t)H_t + \|\nabla^{\Sigma_t}\eta_t\|^2. \tag{3.20}$$

However, due to Proposition 2 and the product rule, the left side above reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{div}_{\Sigma_t}(\eta_t \nu_t) = \frac{\mathrm{d}}{\mathrm{d}t}(-\eta_t H_t) = -V_t(\eta_t)H_t - \eta_t \frac{\mathrm{d}}{\mathrm{d}t} H_t. \tag{3.21}$$

Hence, (3.18) follows. □

Example 17. For $a > 0$ and a smooth function $f\colon [a,\infty) \to (0,\infty)$, consider in the product $M = [a,\infty) \times \mathbb{S}^{n-1}$ the Riemannian metric

$$\mathrm{g} = \frac{\mathrm{d}r^2}{f(r)} + r^2 \mathrm{g}^{\circ}, \tag{3.22}$$

where g° is the standard round metric in $\mathbb{S}^{n-1}$. Observe that if $\theta_2, \ldots, \theta_n$ is an orthonormal coframe for the sphere (i.e., $\mathrm{g}^{\circ} = \theta_2 \otimes \theta_2 + \cdots + \theta_n \otimes \theta_n$), then the volume form of $\mathbb{S}^{n-1}$ equals $\mathrm{d}\mu_{\mathbb{S}^{n-1}} = \theta_2 \wedge \cdots \wedge \theta_n$. As $f(r)^{-1/2}\mathrm{d}r, r\theta_2, \ldots, r\theta_n$ is an orthonormal coframe for (M, g), the same principle yields

$$\mathrm{d}\mu_M = \frac{\mathrm{d}r}{f(r)^{1/2}} \wedge (r\theta_2) \wedge \cdots \wedge (r\theta_n) = \frac{r^{n-1}}{f(r)^{1/2}}\mathrm{d}r \wedge \mathrm{d}\mu_{\mathbb{S}^{n-1}}. \tag{3.23}$$

For a fixed value of $r \in [a,\infty)$, we will compute the mean curvature scalar of the slice $\Sigma = \{r\} \times \mathbb{S}^{n-1}$ associated with the unit normal field $\nu = f(r)^{1/2}\partial_r$. To do so, consider the variation $F\colon \Sigma \times (-\varepsilon, \varepsilon) \to M$ given by $F((r,x),t) = (r + tf(r)^{1/2}, x)$, so that $V = \nu$. To do so, we will use the (valid, as $\partial\Sigma = \varnothing$) formula

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \operatorname{vol}(\Sigma_t) = -\int_{\Sigma} \langle \boldsymbol{H}_{\Sigma}, V\rangle \, \mathrm{d}\mu_{\Sigma} \tag{3.24}$$

which, due to symmetry reasons, reduces to

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \operatorname{vol}(\Sigma_t) = -H \operatorname{vol}(\Sigma). \tag{3.25}$$

To explicitly compute $\operatorname{vol}(\Sigma_t)$, observe that $\mathrm{d}\mu_{\Sigma} = \iota_{f(r)^{1/2}\partial_r}\mathrm{d}\mu_M = r^{n-1}\,\mathrm{d}\mu_{\mathbb{S}^{n-1}}$ as a consequence of (3.23), so that replacing r with $r + tf(r)^{1/2}$ yields

$$\mathrm{d}\mu_{\Sigma_t} = (r + tf(r)^{1/2})^{n-1}\mathrm{d}\mu_{\mathbb{S}^{n-1}} = \left(1 + \frac{tf(r)^{1/2}}{r}\right)^{n-1} \mathrm{d}\mu_{\Sigma}, \tag{3.26}$$

and thus, differentiating the newfound relation

$$\mathrm{vol}(\Sigma_t) = \left(1 + \frac{tf(r)^{1/2}}{r}\right)^{n-1} \mathrm{vol}(\Sigma), \tag{3.27}$$

we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{vol}(\Sigma_t) = \frac{(n-1)f(r)^{1/2}}{r}\mathrm{vol}(\Sigma). \tag{3.28}$$

Hence, $H = -(n-1)f(r)^{1/2}/r$.

We can proceed further by observing that, for geometric reasons, Σ is totally umbilic in M. Hence all the eigenvalues of A are equal to $H/(n-1) = -f(r)^{1/2}/r$, meaning that $\|A\|^2 = (n-1)f(r)/r^2$. Using that $\mathrm{s}^\Sigma = (n-1)(n-2)/r^2$, relation (3.4) becomes

$$\frac{(n-1)(n-2)}{r^2} = \mathrm{s}^M - 2\mathrm{Ric}(\nu,\nu) + \frac{(n-1)^2 f(r)}{r^2} - \frac{(n-1)f(r)}{r^2}. \tag{3.29}$$

The strength of the approach here is that we may solve for $\mathrm{Ric}(\nu,\nu)$, and then s^M, *without resorting to further computations involving Christoffel symbols or moving frames.* As $\eta_0 = 1$, the formula given in Proposition 16 reduces to

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} H_t = \mathrm{Ric}(\nu,\nu) + \frac{(n-1)f(r)}{r^2}. \tag{3.30}$$

However, by replacing r with $r + tf(r)^{1/2}$, we see that

$$H_t = -\frac{(n-1)f(r+tf(r)^{1/2})^{1/2}}{r+tf(r)^{1/2}}, \tag{3.31}$$

and thus

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} H_t &= \frac{-(n-1)\frac{1}{2}f(r)^{-1/2}f'(r)f(r)^{1/2}r + (n-1)f(r)^{1/2}f(r)^{1/2}}{r^2} \\ &= \frac{-(n-1)\frac{f'(r)}{2}r + (n-1)f(r)}{r^2} \\ &= \frac{(n-1)}{r^2}\left(-\frac{rf'(r)}{2} + f(r)\right).\end{aligned} \tag{3.32}$$

Then (3.30) gives us that

$$\mathrm{Ric}(\nu,\nu) = -\frac{(n-1)rf'(r)}{2r^2}, \tag{3.33}$$

so (3.29) now reads, after factoring out $(n-1)/r^2$,

$$\frac{(n-1)(n-2)}{r^2} = \mathrm{s}^M + \frac{(n-1)}{r^2}\left(rf'(r) + (n-1)f(r) - f(r)\right). \tag{3.34}$$

We conclude that

$$\mathrm{s}^M = \frac{(n-1)}{r^2}((n-2)(1-f(r)) - rf'(r)). \tag{3.35}$$

4 The geodesic case

Let (M, g) be a Riemannian manifold, and $\Sigma \hookrightarrow M$ be an embedded **curve**, that is, $\dim \Sigma = 1$. We assume that Σ is parametrized by a unit speed regular curve $\gamma\colon I \to \Sigma$. It follows from differentiating $\langle \dot\gamma, \dot\gamma \rangle = 1$ that $\langle \nabla_{\dot\gamma}\dot\gamma, \dot\gamma \rangle = 0$, and thus $\nabla_{\dot\gamma}\dot\gamma = \mathrm{II}(\dot\gamma, \dot\gamma)$. Thus, the mean curvature vector is given by $\boldsymbol{H}_\Sigma = \nabla_{\dot\gamma}\dot\gamma$. Hence, Σ is minimal if and only if γ is a geodesic.

Now, let V be a vector field normal to a unit speed geodesic $\gamma\colon I \to \Sigma$, that is, with $\langle V, \dot\gamma \rangle = 0$. Differentiating and using that $\dot\gamma$ is a geodesic, we see that $\langle \nabla_{\dot\gamma} V, \dot\gamma \rangle = 0$, so that $\nabla_{\dot\gamma} V = \nabla^{\perp}_{\dot\gamma} V$ and $A_V = 0$. As $\mathrm{Ric}^{\top}_{\Sigma}(V, V) = R(\dot\gamma, V, V, \dot\gamma)$, the stability condition (2.34) reads

$$\int_I \|\nabla_{\dot\gamma} V\|^2 - R(\dot\gamma, V, V, \dot\gamma)\, \mathrm{d}s \geq 0 \tag{4.1}$$

or, equivalently,

$$-\int_I \langle V, \nabla_{\dot\gamma}\nabla_{\dot\gamma} V - R(\dot\gamma, V)\dot\gamma \rangle \, \mathrm{d}s \geq 0. \tag{4.2}$$

The stability operator L_Σ is given by $L_\Sigma V = \nabla_{\dot\gamma}\nabla_{\dot\gamma} V - R(\dot\gamma, V)\dot\gamma$ and its kernel consists of classical Jacobi fields (as predicted).