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Computational Techniques for the Graphic Simulation of Quadric Surfaces

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Curvature is an aspect of three-dimensional structure that has long been neglected in perceptual psychology. This article describes practical solutions to a number of computational problems that arise when studying the perception of curved surfaces. Procedures are described for (a) representing a quadric surface at any desired position and orientation in space, (b) determining which point on the surface projects to a given picture location, (c) computing the surface orientation at that point, and (d) finding the point on a surface with a given surface orientation. These and related procedures enable experimenters to study perception of curved surfaces in new and powerful ways.

Although the ability of human observers to perceive three-dimensional form from a two-dimensional image has fascinated researchers since the Italian Renaissance, a number of recent developments have given the study of form perception a new impetus. Of particular importance in this regard is the increasing availability of laboratory computer graphics systems. It is now possible, for example, to create visual displays of unprecedented realism through the use of computer animation (Braunstein, 1966; Green, 1961), the manipulation of shading (Pentland, 1982; Todd & Mingolla, 1983), or the generation of optical texture (Braunstein, 1968; Todd & Mingolla, 1984).

Another important development in the study of form perception is the growing number of theoretical analyses in the literature. Researchers within both psychology and artificial intelligence have devised a variety of formal algorithms for determining an object's structure based on motion (Koenderink & van Doorn, 1977; Todd, 1982; Ullman, 1979), shading (Horn, 1977; Koenderink & van Doorn, 1980, 1982), or texture (Braunstein & Payne, 1969; Purdy, 1958; Stevens, 1981) in its visual projection.

Many of these new techniques for simulating an object with a computer graphics display or performing a mathematical analysis of the resulting image are specifically designed for smoothly curved surfaces. Curvature is an aspect of three-dimensional structure that has generally been neglected by perceptual psychology. With few exceptions (e.g., Braunstein, Andersen, & Riefer, 1982; Lappin, Doner, & Kottas, 1980; Todd & Mingolla, 1983), research on form perception has been confined to isolated planar surfaces, plane-faced polyhedra, or configurations of disconnected elements. With the advent of these new methodological tools, however, it is now possible to employ curved surfaces as stimuli in perceptual experiments.

One promising technique for evaluating an observer's perception of curved surfaces in three-dimensional space has recently been described by Mingolla (1983). An observer is presented with a computer-generated image of an object on which a single point is designated by a blinking light. The observer's task is to estimate the surface orientation at that point in terms of the *slant* and *tilt* of the normal at that point. (See Stevens, 1981, 1983.) If many such estimates are obtained at different points on an image, it is possible to determine how the perceived three-dimensional structure of the surface relates to the actual structure under various experimental conditions.

Although this technique is conceptually straightforward, there are several computational problems that can arise in the process of implementation. First, the surface must be described in a formally precise manner. Surfaces known in mathematics as *quadrics* are particularly useful in this regard because they can be described with a simple second-degree equation in three variables (i.e., one that contains no exponent greater than two). The quadric surfaces include a number of familiar objects such as spheres, cones, and cylinders, as well as a few that are not so familiar such as elliptic hyperboloids of one sheet. Because they include such a wide range of shapes and are easy to manipulate in experimental displays, quadric surfaces make excellent stimuli for the study of human perception. However, in order to completely describe one of these surfaces at a particular position and orientation in space, a researcher must be able to generate an appropriate equation.

Once an object in three-dimensional space has

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been specified, the next step in the simulation process is to compute its optical projection. If a surface is depicted using patterns of shading, it is necessary to insure that every addressable element over the entire display screen is assigned an appropriate intensity value. (See Todd & Mingolla, 1983; and Todd & Mingolla, 1984, for a similar requirement in generating images with patterns of texture.) This requirement can be satisfied by performing an inverse projective transformation. That is, for each picture element on the display screen, one determines a corresponding point on the simulated surface to which it is projectively related. In addition, it is necessary to determine the orientation of the surface at that point because the patterns of shading and texture within an image are each dependent on surface orientation.

The final step of our proposed procedure is to select the particular points on an object for which orientation judgments will be obtained. Suppose, for example, that an observer is asked to judge the surface orientations of points on several different objects. To facilitate comparisons between the perceived forms of these objects, it is useful to sample the same orientations on each one. To achieve this level of control, however, one must be able to determine the specific point on each surface that has the desired orientation.

The various computational problems described above can be annoying stumbling blocks to a researcher investigating the perception of curved surfaces. Although the solutions to these problems are known, they are buried in textbooks on solid analytic geometry and are not easily accessible to perceptual psychologists. Thus, in an effort to facilitate future experimentation in this area, the present article discusses several practical solutions that have proven to be useful in our own research. In particular, we present specific procedures for (a) representing a quadric surface at any desired position and orientation in space, (b) determining which point on the surface projects to a given screen location, (c) computing the surface orientation at that point, and (d) finding the point on a surface with a given orientation.

Representing the Shape and Position of Quadric Surfaces

Equation 1 describes a general quadric surface (Dresden, 1930, p. 159):

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0. \qquad (1)$$

The values of the first three coefficients determine the type and shape of the surface. For example, if all three are positive and equal to each other, a sphere results; if they are all positive but unequal, an ellipsoid is described. One negative and two positive coefficients produce a hyperboloid of one sheet, and so on (Dresden, 1930, p. 230). Variation of these coefficients also determines the shape of surface regions within a category from pointed to rounded or flat. Likewise, varying the coefficients a_{12}, a_{13}, and a_{23} produces rotations of the surface relative to the reference coordinate system; a_{14}, a_{24}, and a_{34} describe translations of the surface from the origin, and a_{44} is a scaling factor.

An experimenter can set out to display surfaces of particular shapes and sizes as follows. If a_{44} is set to -1, the first three coefficients can be conveniently scaled to a particular screen resolution in pixel units. Thus, choosing a_{11}, a_{22}, and a_{33} equal to $(1/80)^2$, $(1/60)^2$, and $(1/50)^2$, respectively, produces an ellipsoid 160 units long and 100 units wide at its narrowest diameter. Similarly, rather than arbitrarily entering values of a_{12}, a_{13}, and a_{23}, the experimenter can select angular values of *rotations* of the surface about the x-, y-, or z-axes from a standard position relative to the display screen. The values of a_{12}, a_{13}, and a_{23} that specify the desired rotation can then be found as follows. Imagine that the desired orientation is such that the three principal axes of the surface align with some coordinate system, called the *u-v-w system.* The surface's equation in this system is

$$b_{11}u^2 + b_{22}v^2 + b_{33}w^2 = 1. \qquad (2)$$

Suppose further that the x-y-z- and u-v-w-axes share a common origin and differ only in that the x-y-z-axes would align perfectly with the u-v-w-axes if the former were rotated $\theta°$ clockwise about the z-axis. The equations relating the two coordinate systems are

$$\begin{aligned} u &= \cos(\theta)x - \sin(\theta)y \\ v &= \sin(\theta)x + \cos(\theta)y \\ w &= z. \end{aligned} \qquad (3)$$

Substitution of Equation 3 into Equation 2 produces an equation of the desired form in the x-y-z system, having a new set of coefficients related to the old by

$$\begin{aligned} a_{11} &= \cos^2(\theta)b_{11} + \sin^2(\theta)b_{22} \\ a_{22} &= \sin^2(\theta)b_{11} + \cos^2(\theta)b_{22} \\ a_{33} &= b_{33} \\ a_{12} &= \cos(\theta)\sin(\theta)(b_{22} - b_{11}). \end{aligned} \qquad (4)$$

Translation of the surface in pixel units can similarly be achieved by substituting the values $u =$

$x + k$, $v = y + l$, $w = z + m$, for constants k, l, and m. In this way, the form of Equation 1 can be constructed out of intuitively meaningful lengths, rotations, and translations.

Computing Which Surface Point Corresponds to a Given Screen Pixel

In order to display a quadric surface on a computer-generated display, a programmer must first determine which quadric surface point is on a line of sight through a given picture element (pixel) of the display screen. Although a display screen is naturally specified by two-dimensional rectangular coordinates, the modeled surface is three dimensional. By specifying the spatial relationships of eye, object, and display screen as shown in Figure 1, a programmer can determine at which surface point (x_s, y_s, z_s), if any, the line of sight through the display screen point (x_p, y_p, z_p) intersects the surface.

The line of sight through (x_p, y_p, z_p) is described by these parametric equations:

$$x = x_p t, \quad y = y_p t, \quad z = z_p t, \tag{5}$$

where z_p is constant over the entire display screen. Substituting Equation 5 into Equation 1 describes the locus of points that are both on the surface and in the line of sight. There can be two, one, or no such points, depending on the number of real roots of the equation described by that substitution (Dresden, 1930, p. 150):

$$[a_{11}x_p^2 + a_{22}y_p^2 + a_{33}z_p^2 + 2a_{12}x_p y_p + 2a_{13}x_p z_p + 2a_{23}y_p z_p]t^2 + 2(a_{14}x_p + a_{24}y_p + a_{34}z_p)t + a_{44} = 0. \tag{6}$$

Equation 6 is a quadratic of the form $at^2 + bt + c = 0$, and its discriminant is $b^2 - 4ac$. Because z_p is constant for a given display, the discriminant is tested for each value of x and y on the screen. If the discriminant is negative, the surface is not drawn at that point. If the discriminant is positive, the smaller resulting value of t allows the coordinates of the closer, visible point on the surface to be calculated directly from Equation 5.

Computing the Surface Normal at a Given Point

The heart of all display algorithms for smoothly curved surfaces is the computation of unit surface normals. An introduction to the generation of shaded graphic displays from surface normals can be found in Todd and Mingolla (1983), and a procedure for generating textured surfaces is described in Todd and Mingolla (1984). (See Blinn, 1977, for details, and Newman & Sproull, 1979, chapter 25, for a more extensive survey of shaded graphics techniques.)

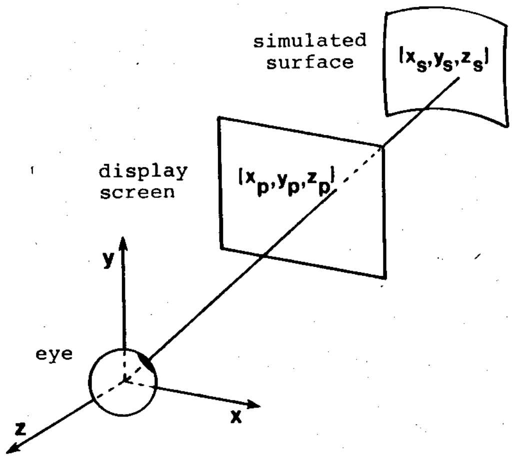

Figure 1. This scheme relates object and screen coordinates. Rectangular coordinates at the point of observation are oriented so that the x-axis and y-axis are parallel to the horizontal and vertical screen dimensions, respectively. The programmer must explicitly relate the scales of the screen and of the eye-centered coordinates, as described in the text.

To compute surface normals, consider partial derivatives of the left side of Equation 1 as follows: Differentiating the left side of Equation 1 with respect to x, y, and z, respectively, yields three quantities, which will be designated Q_x, Q_y, and Q_z:

$$Q_x = 2(a_{11}x + a_{12}y + a_{13}z + a_{14})$$
$$Q_y = 2(a_{21}x + a_{22}y + a_{23}z + a_{24})$$
$$Q_z = 2(a_{31}x + a_{32}y + a_{33}z + a_{34}). \tag{7}$$

(By definition, $a_{21} = a_{12}$, $a_{31} = a_{13}$, and $a_{32} = a_{23}$.) Consider a vector $\vec{N}$ that is aligned with the surface normal at a point P. The quantities Q_x, Q_y, and Q_z are the projected lengths of this vector along the x-, y-, and z-axes, respectively. If the coordinates of the point P are (A, B, C), then a free vector $\vec{N}$ is specified by substituting the coordinate values A, B, and C for x, y, and z, respectively, in Equations 7 to yield (Dresden, 1930, p. 161)[1]

$$\vec{N} = (Q_x, Q_y, Q_z). \tag{8}$$

It is important to note that $\vec{N}$ is not necessarily of unit length and must therefore be normalized before use in graphics computations. (The normalized values of Q_x, Q_y, and Q_z, designated a, b, and c, are the direction cosines of $\vec{N}$.)

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[1] A *free* vector denotes a sense of orientation in space, without being attached to any particular place. (Moon & Spencer, 1965, p. 111.)

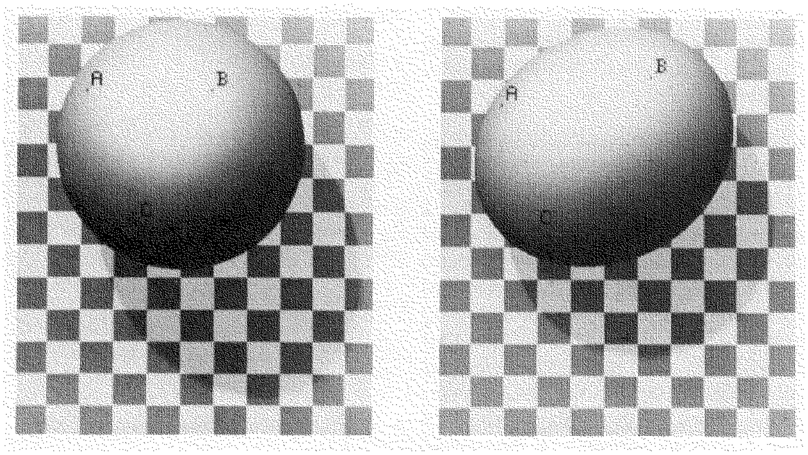

Figure 2. The sphere and ellipsoid were generated using the procedures described in the text. The letters denote points having orientations specified by an experimenter; points labeled with corresponding letters have the same local surface orientation.

Locating a Surface Point With a Given Surface Normal

One approach to probing a subject's impressions of a surface's form or local orientation is to ask the subject to indicate the direction of a surface normal at a point. The experimenter selects a set of surface orientations to be probed and varies the shape or orientation of surfaces to be displayed. For each test surface, the specific points having the selected orientations must then be found. (See Figure 2.)

The components of a unit surface normal in rectangular coordinates are the direction cosines of its orientation. That is, they are the cosines of the angles formed by the normal and by rays parallel to the coordinate axes at a point. However, few individuals find the direction cosine specification of surface normals a natural one for indicating judged surface orientations, and a more natural scheme is given by slant and tilt (Stevens, 1981, 1983). Slant is defined as the angle formed at a surface point by the normal of the visible side of the surface and by a ray pointing to the eye. If the display screen plane is taken to be perpendicular to the line of sight, tilt is specified as a radial direction in that plane, such as "toward 11 o'clock" on an analog watch face. The experimenter can in any case readily convert either normal representation to the other.[2]

Assume then that an experimenter seeks to locate the point on a surface with a given surface normal specified in rectangular coordinates. The components of the surface normal are the direction cosines a, b, and c. These components can readily be thought of as specifying the orientation of a plane perpendicular to that surface normal.[3] Just as an infinite family of parallel rays in space exists having the same orientation as the given surface normal, so also an infinite family of parallel planes exists having an orientation perpendicular to those rays. Locating the surface point with a given normal can thus be approached through the dual procedure of locating the point of tangency on the surface with one of the infinite family of planes perpendicular to the normal direction. Since the orientation of the desired plane is given, all that must be found is a single parameter value, R, specifying where the plane must be placed in order to just graze the surface. Derivation of the proce-

[2] Given a rectangular coordinate system, a line of sight, and a display screen aligned as shown in Figure 1, conversion formulas between rectangular normal components and slant and tilt components are

$$x = \sin(\sigma)\cos(\tau), \quad y = \sin(\sigma)\sin(\tau), \quad z = \cos(\sigma),$$

$$\text{and} \quad \sigma = \cos^{-1}(z), \quad \tau = \tan^{-1}(y/x),$$

where σ denotes slant and τ tilt.

[3] The numbers denoting the components of a free vector (**a**, **b**, **c**) can just as easily be taken to denote the set of all parallel planes satisfying the equation (Dresden, 1930, p. 253):

$$(x - A)a + (y - B)b + (z - C)c = 0,$$

where (A, B, C) denotes a point on the plane. For this reason, finding the point of tangency of one such plane to the surface is equivalent to locating a point with a given surface normal. The shortcut procedure described in this section, however, requires that the surface not be rotated relative to the coordinate axes (i.e., $a_{12} = a_{13} = a_{23} = 0$). If the surface under consideration has been rotated, *the normal orientation must be equivalently rotated before proceeding.*

dure can be found in Dresden (1930, pp. 253–263), but if a surface has a point with a given surface normal at all, a real-valued R can be found by solving

$$D_1 = R^2 D_2. \tag{9}$$

In Equation 9, D_1 is the determinant of the matrix of coefficients of Equation 1,

$$D_1 = \det \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix},$$

where $a_{ij} = a_{ji}$, by definition. D_2 is also defined as a determinant:

$$D_2 = \det \begin{vmatrix} a_{11} & a_{12} & a_{13} & a \\ a_{21} & a_{22} & a_{23} & b \\ a_{31} & a_{32} & a_{33} & c \\ a & b & c & 0 \end{vmatrix}.$$

For each type of surface, computational shortcuts equivalent to Equation 9 are described in Dresden (1930, pp. 256–263). For example, for ellipsoids Equation 9 is equivalent to

$$R^2\left(\frac{a^2}{a_{11}} + \frac{b^2}{a_{22}} + \frac{c^2}{a_{33}}\right) = 1. \tag{10}$$

If Equation 9 has real roots, the surface has one or two points with the desired orientation. In the latter case, the visible point in the image corresponds to the positive root. Simultaneous equations involving partial derivatives from Equation 7 can then be solved to find the desired point coordinates:

$$2(a_{11}x + a_{12}y + a_{13}z + a_{14}) = 2Ra$$

$$2(a_{21}x + a_{22}y + a_{23}z + a_{24}) = 2Rb$$

$$2(a_{31}x + a_{32}y + a_{33}z + a_{34}) = 2Rc \tag{11}$$

The values of x, y, and z that satisfy Equation 11 specify the coordinates of the point on the surface whose surface normal is (a, b, c).

Summary

The present article provides some practical solutions to a number of computational problems that arise when studying the perception of curved surfaces. In particular, we have described procedures for (a) representing a quadric surface at any desired position and orientation in space, (b) determining which point on the surface projects to a given picture location, (c) computing the surface orientation at that point, and (d) finding the point on a surface with a given surface orientation. These procedures are especially useful for depicting curved surfaces with shading (see Todd & Mingolla, 1983) or texture (see Todd & Mingolla, 1984) and for obtaining observers' orientation judgments at different points on a surface to evaluate its perceived shape (see Mingolla, 1983).

There are several other computational problems a researcher is likely to confront when manipulating curved surfaces that are beyond the scope of the present article. For example, when one is examining an observer's ability to discriminate between two objects of different three-dimensional shape based on shading or texture, it is useful to adjust their orientations in space so that their projected outer contours in the picture plane are identical. A related problem can also occur when attempting to determine the projected boundaries of cast shadows. For the solutions to these and other problems, the reader should consult Mingolla (1983).

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