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Exocentric pointing in depth

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Abstract

An exocentric pointing task was used to compare the indicated pointing directions under exchange of target and pointer. Such a pair of pointing directions, together with the pointer and target locations, specifies a unique cubic arc. Such an arc may assume one of two qualitatively different shapes, namely a "C-arc" (constant sign of curvature) or an "S-arc" (containing a point of inflection between the endpoints). We show that human observers most often produce S-curves. This is of fundamental importance, since—in case one interprets the curve as an empirically determined "pregeodesic" ("shortest connection", or "straight" connection in visual space)—it would imply that "visual space" in the strict geometrical sense is a non-entity. The experiments were performed in the outside environment, under normal daylight conditions, for distances ranging from one to over thirty meters. The implications of these data are discussed and possible ways to extend the restricted notion of "visual space" (*e.g.*, as advocated by Luneburg) such as to allow one to account for the present results are suggested. Such extensions of the visual space concept include the local adjustment of geometrical structure in regions adjacent to the fixation direction.

Keywords: Visual space; Exocentric direction; Consistency of visual space

1. Introduction

"Visual Space" still figures prominently in the literature of human vision. Yet the initial euphoria detectible in the literature immediately following Luneburg's (1947) work (*e.g.*, Blank's (1953, 1957, 1961) papers of the fifties and sixties) has largely dwindled. It has become clear that visual space cannot be a *homogeneous* space as Luneburg, and Blank after him took it for granted (Foley, 1972; Suppes, 1977; Wagner, 1985; Zajaczkowska, 1956). One of the central findings that make it difficult to pin down the concept of "visual space" is that the geometrical properties assessed through different paradigms are often inconsistent with one another (Battro, di Pierro Netto, & Rozestraten, 1976; Battro, Reggini, & Karts, 1978; Indow & Watanabe, 1988; Indow, 1990, 1991; Koenderink, van Doorn, Kappers, & Lappin, 2002b). This perplexing conclusion is perhaps still somewhat premature because an extensive and homogeneous corpus of data cannot be said to exist. However, the existing data definitely point into that direction. Several authors have made a strong case for a "contextual geometry" (Ehrenstein, 1977; Foley, 1972; Suppes, 1977) or even for a "momentary geometry" (Schelling, 1956) in which the geometry is fixation dependent (Haubensak, 1970, see also Trommershäuser, Maloney, & Landy, 2003) and thus varies from moment to moment. One also finds task dependencies (Ehrenstein, 1977; Foley, 1972; Indow & Watanabe, 1988; Indow, 1990, 1991; Koenderink et al., 2002b) that suggest that no single formal geometrical structure might suffice throughout. If it turns out to be the case that data relating to directions cannot be predicted from data relating to positions—and this appears likely—then there exists no geometry in the conventional sense. If this is indeed the conclusion, then the expression "Visual Space" had perhaps better be avoided altogether, at least if "space" is intended in the conventional sense of "space

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as a container", and certainly if intended in the sense of the homogeneous spaces (with the implication of a tight projective structure, *e.g.*, see Coxeter (1961)) as contemplated by Luneburg (1947) and as even axiomatized by Blank (1953, 1957, 1958, 1959, 1961). "Space" would have to be understood not in the mathematical sense, as a fixed framework, but in the sense of physics (Suppes, 1977), where space is understood as thoroughly relational and determined by its contents. In physics the discussion concerning the notions of (Newton's) "absolute space" versus space as a nexus of relations became well known through the Leibniz–Clarke correspondence between 1715–1716 (Alexander, 1956), in psychology the discussions on the issue remain in a muddled state.

In this paper we address the problem of the existence of a unique straight connecting arc for any given pair of distinct points $\mathcal{A}$ and $\mathcal{B}$. (Note on terminology: We use "arc" for any finite segment of a curve. "Straight" means that an arc is geodesic, a general arc being "curved". Notice that a "geodesic" need not look "straight" to the Euclidian eye, *e.g.*, think of the meridians of the globe.) We approach this problem via the analysis of the pair of directions $\mathcal{AB}$ and $\mathcal{BA}$. Such directions can easily be operationalized through the psychophysical method of "exocentric pointing". The method was suggested to us through the repeated finding (Cuijpers, Kappers, & Koenderink, 2001; Doumen, Kappers, & Koenderink, 2005) that such a pair of directions typically fails to define a simple arc, that is an arc which curvature does not change sign (*i.e.*, "C"-shaped rather than "S"-shaped) (see Fig. 1).

We have set us to the task of collecting a uniform body of data over a range of distances (greatly extending the range of existing measurements). These data are checked for the existence of simple arcs. We check this via a numerical criterion especially developed for this task.

2. Methods

The method (Koenderink & van Doorn, 1998) used is that of "exocentric pointing". This is implemented in the following way: The observer sees a pointing device and a target and is required (*e.g.*, by remote control) to put the pointer in such an attitude as to apparently be directed at the target. The resulting attitude of the pointing device is interpreted as the observer's "perceived direction of the pointer towards the target". (Whether such an interpretation is indeed reasonable is discussed in our conclusions.) The pointer, target and observer exist in physical space, and thus the notion of a "true" or "veridical" direction from the pointer towards the target is well defined. Experience indicates that the (in the above sense operationally defined) "perceived direction" and the "veridical direction" typically disagree. The deviation depends in a systematic fashion on the locations of pointer, target and observer. Of course the method also generates random deviations and components of the deviation might be due to various factors as the structure of the environment, design of pointer and target, viewing conditions, and so forth.

In the present experiment, we performed the settings in the natural, outdoors environment, a lawn in front of our laboratory of about fifty by fifty meters roughly squarish area. The lawn was bounded by buildings and rows of trees but was itself virtually featureless (just well kept grass). The observer's eyes, centers of target and pointer were put at the same height of 1.5 m above the ground. The observer was placed upon an adjustable chair in order to achieve this, the pointer and target were placed on top of cylindrical poles (see Fig. 3). The observer had a free view of the environment and thus could see the points where the poles supporting target and pointer reached the ground. Conceptually target, pointer and egocenter are located in a single plane, the observer's horizon, that is the horizontal plane at eye height. Thus the geometrical configuration involves a triangle with the egocenter, pointer and target as vertices (see Fig. 2).

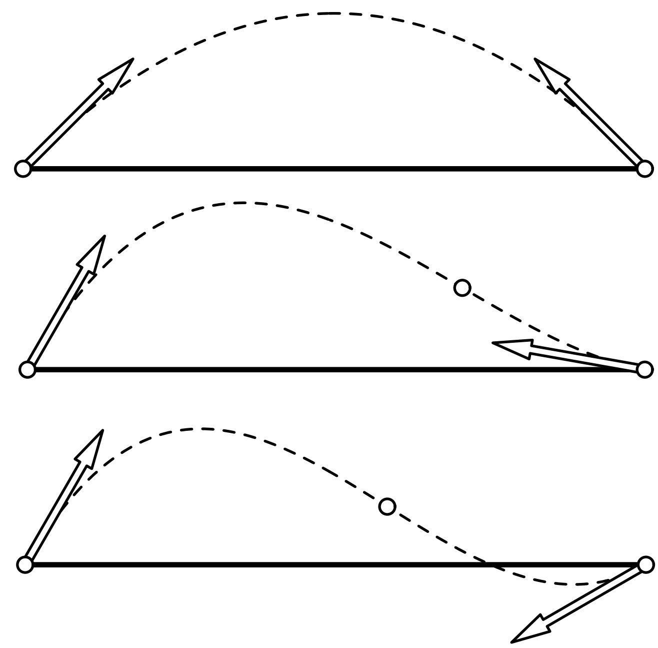

Fig. 1. Examples of cubic curves passing through two points in specified directions. The top example is a "simple arc", *i.e.*, a curve without inflection. The other two examples represent curves that fail to be simple arcs since there exists an inflection point between the endpoints of the arc. In the lower example this is indeed obvious, in the center example perhaps less so. Given the direction at one end there is a limited range of directions for the other end that lead to a "simple arc". This is the condition we test in the experiment.

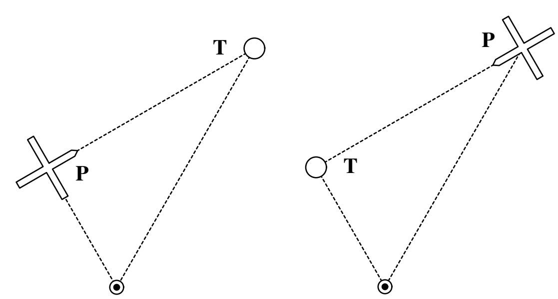

Fig. 2. The geometry used in this experiment. Pointer ***P*** and target ***T*** could be placed at either the far or close position. The mirror reversed configurations were also used. Distances ranged from 1 to 32 m by factors of two. Distance ratios used were two and four. The angle between the directions from the observer towards target and pointer always subtended 60°. This figure is not to scale. The angular sizes of the target and the diameter of the disk of the pointer as well as the length of the pointer's arrow (see Fig. 3) as seen from the observer's position were 1.8°. Distances from the observer to pointer or target were 1, 2, 4, 8, 16 or 32 meters, in the ratio of either two or four.

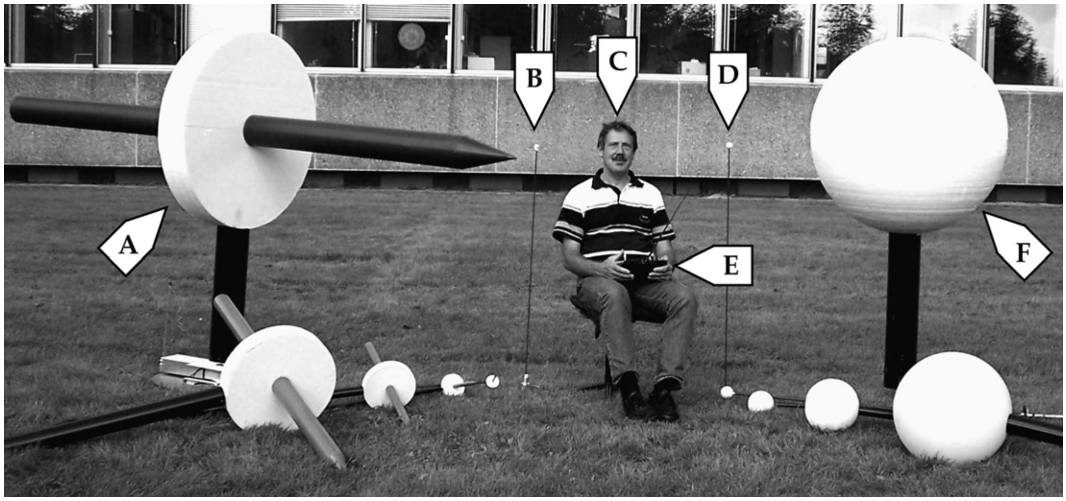

Fig. 3. At the center an observer C, holding the transmitter E for radio-control, flanked by pointers A through B and targets D through F. Notice the range of sizes. The largest pointer (A) and target (F) are 32 times the size of the smallest pointer (B) and target (D) which are barely visible in this photograph. Pointers and targets, when mounted, are at the same height as the observer's eyes (A, F and C).

The design of the target was very simple. We used matte white spheres placed on black, cylindrical poles of much smaller diameters than those of the spheres. We kept the angular size fixed at 1.8°, thus the targets varied a lot in size, the largest (at 32 m) being 1 m in diameter, the smallest (at 1 m) being only about 3 cm in diameter.

There is much freedom in the design of pointers. We aimed at a design that allows one to obtain a vivid impression of spatial attitude in any pose (thus a bad design from this perspective would be a thin cylindrical rod for instance). We used a white, thick, circular disc pierced with a bright red-dish-orange, thick cylindrical, pointed arrow sticking out equally at either side of the disk. The diameter of the disk subtended the same visual angle as the target at the corresponding target positions. We placed the pointers on black cylindrical poles, the same diameters as the corresponding poles supporting the targets. The pointers were placed on a rotatable platform that could be radio-controlled by the observer. Only rotations about the vertical were considered.

Thus we achieve identical visual projections for geometrically similar configurations of various sizes. Of course that is not at all the visual impression (Gilinsky, 1951). The far target looks huge and the near target tiny even though their angular subtends are identical.

Before the actual experiment we performed the necessary geodesy and calibrations. This was done via standard surveying instruments, most notably a theodolite and a 50 m measuring tape. Locations were marked on the lawn in such a way that the observers remained unaware of them. In the course of an experiment targets and pointers were placed at these marked locations. This involved work of four people, one of them being in charge of the operation.

The interior angle at the egocenter was fixed at 60°, the two sides abutting at the egocenter had lengths in the ratio two or four. The distances used were 1, 2, 4, 8, 16 and 32 m, resulting in the pairs 1/2, 2/4, 4/8, 8/16, 16/32 and 1/4, 2/8, 4/16, 8/32. Any pair was presented in two mirror symmetric configurations, thus 4/8 means either "4 on the left, 8 on the right" or "8 on the left, 4 on the right". For each configuration we have two target-pointer combinations. Apart from these constraints we presented each possible configuration three times. Thus a full experiment involves $(5+4)\times 2\times 2\times 3=108$ settings. The sequence was only semi-random because we had to minimize the time involved in displacing targets and pointers. This time was indeed considerable given the distances involved and the sizes (and thus resulting weights) of the targets and pointers. A full experiment involved about $2\frac{1}{2}$ hs of work, of which only a fraction was spent in the observer doing the actual settings. As the configuration was being changed the observer looked the opposite direction, thus the observer was only confronted with the actual configuration. On each new occasion the initial attitude of the pointer was randomized.

A total of seven paid persons participated in the experiments. They were of either sex and of various ages. All were tested for binocular stereopsis and trichromacy. During the conduction of the experiments it was evident that two persons could not be regarded as serious observers. A "serious" observer in this type of experiment should (in our view) at least:

— look back and forth between target and pointer a few times before even starting to do a setting;
— adjust the pointer back and forth a few times before being satisfied with a setting.

The discarded observers failed on one or both of these counts and their data were not taken into account.

3. Results

A typical result is shown in Fig. 4. These are measurements for all observers in which an arc was sought between two points that were 60° apart in the visual field and at egocentric distances differing by a factor of two. The distances used were 2/4 and 4/2 m. The figures represent averages over mirror symmetric configurations and three repeats. The observer directed a pointer from the vertex at shorter distance so as to apparently point to a target at the larger distance and vice versa. The average results of both pointings are shown in the figure. It is immediately obvious that these pointings represent directions that fail to line up with the straight line connecting the two vertices. There appears to be a difference between the cases where the observer pointed towards himself (from the more remote (P) to the more near (T) vertex in Fig. 2 *right*; henceforth referred to as "to-pointings") and the cases where the observer pointed away from himself (from the more near (P) to the more remote (T) vertex in Fig. 2 *left*; henceforth referred to as "fro-pointings") in that in the "to-pointings" the observer points too far towards himself and in the "fro-pointings" the observer mostly points too far away from

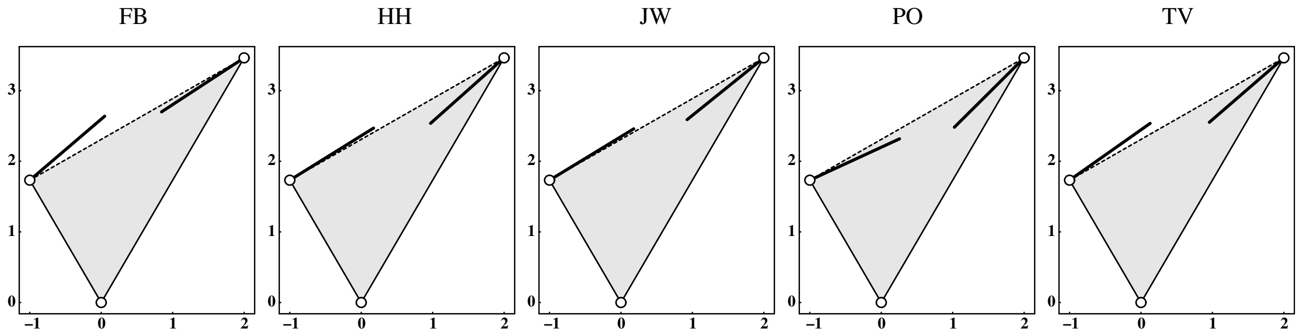

Fig. 4. Examples of exocentric pointing results. These are the averages for the conditions 2/4 and 4/2 m for all five observers.

himself. This is indeed the overall pattern we have encountered over and over again in the past.

A similar general pattern is found for all observers, though looking in detail at the deviations one detects marked individual differences. There does not seem to appear a marked distance dependence. The variance at the nearest locations is somewhat in excess of that at farther locations, an effect that is indeed to be expected because the calibration and methodological random errors decrease with increasing distance. Thus there is no reason to assume that the variance in the observer's settings is a function of distance. The standard error in the average settings as assessed over the repeated settings and mirror symmetric configurations (total of six instances) is between 1.9° and 2.8°. In almost all instances the deviation from the veridical direction is significant at the 5% level (two sided test of the mean with null hypothesis 0, based on a normal distribution).

4. Analysis

The aim of the experiment is to check for possible violations of the conditions that pertain to the existence of simple arc connections between two points. Such simple arcs would qualify as candidates for *pregeodesics* and their non-existence would rule out large classes of geometries as applying to visual space. In order to enable this check we seek for a representation of the data that abstracts from those individual differences that are irrelevant to the issue. When the transformed data are in a format that immediately reflects the relevant properties it might be expected to appear rather more uniform.

The settings of the observer define two directions, one at each endpoint of the segment defined by the near and far target. For the sake of analysis we consider these directions to be tangent to a smooth curve connecting these points. Given two points and two tangent directions, there exists a unique cubic curve that passes through the two points with a tangent direction equal to the given directions at those points. Such a cubic curve is arguably the "simplest" candidate as any smooth, analytic curve is well approximated by it, especially if (as is indeed the case) the curve is nearly straight. Such curves are characterized by *curvature* and "*spirality*", where the latter indicates a gradual progression of curvature along the curve. These two numbers are "differential invariants" in that they specify the intrinsic shape of the curve, independent of its overall orientation or location. A "simple arc" connecting two points is a curve that is largely characterized by its (shallow) curvature. Simple arcs have no points of inflection. Whether there is an inflection depends on the direction of the curve at the end points. Our analysis shows that the ratio of the curvature and the spirality yields a number that allows one to decide on whether the two directions indicated by the observer define a simple arc or not (see Appendix A). This is going to be the criterion used for the analysis.

In the case the data would be like a straight line segment one would say that the settings of the observer were "veridical". This is compatible with a Euclidian structure of visual space. In general we find non-veridical results though. Consider the cases illustrated in Fig. 1. From top to bottom the first instance illustrates a "simple arc". Such a result would be expected in any reasonable model of the Luneburg type. The other instances illustrate cases that fail to be simple arcs. Such cases might be encountered as pregeodesic arcs in inhomogeneous (Riemannian) spaces, but only incidentally, they would not be generically expected. We will consider these cases as violating the assumption of a proper "Visual Space".

We find that in the large majority of cases the data rule out the simple arc hypothesis. Almost all arcs (91%) turn out to be of the S-type rather than the C-type (see Table 1). Either the method is not fit to reveal pregeodesics of visual space (see Section 5) or such pregeodesics—and thus visual space itself—are non-entities.

For the distance ratio of four the results are qualitatively similar to those for the distance ratio of two. Notice that in the extreme cases of distance ratio one and very large distance ratios one does not expect to see non-simple arcs at all, in the former case because of left–right symmetry, in the latter case because all pointings have to be roughly in the radial direction. S-shape curves—if they occur at all—are only to be expected for moderate distance ratios like two. We included the larger (thus less interesting) distance ratio partly to provide the observers with a variety of tasks in order to prevent them to memorize specific config-

Table 1
For the egocentric distance ratio in the top row and all observers marked in the leftmost column the cases where the data were incompatible with the existence of a simple arc are marked •, the simple arcs marked ○

	1/2	2/4	4/8	8/16	16/32	1/4	2/8	4/16	8/32
FB	•	•	•	•	•	•	•	○	•
HH	•	•	•	○	•	•	•	•	•
JW	•	•	•	•	•	•	•	•	•
PO	•	•	•	○	•	•	•	•	○
TV	•	•	•	•	•	•	•	•	•

urations. We base our conclusions (as originally planned) on the data for the distance ratio of two. However, it is encouraging to see that the data for ratio four reveal the same trends.

5. Conclusions

The hypothesis we set out to test in this experiment is that the pregeodesics as indicated by exocentric pointing generally fail to be simple arcs. This hypothesis was suggested to us by a wealth of data gathered in both reduced and natural settings, though all in "near space" (less than five meters exocentric distance) because gathered in indoor environments. Since experiments were done binocularly, this makes it virtually impossible to extrapolate from the indoor environment to large outdoor spaces. In the latter case monocular cues are likely to dominate in many cases.

The present setting was a natural outdoors environment including exocentric distances up to 32 m. The outcome of the analysis of the data is non-ambiguous. The results from the previous experiments are fully confirmed. This is remarkable because many of the previous data were obtained under rather reduced conditions, indeed closely resembling the settings of the early experiments designed to test Luneburg's theory. Remarkably, an outdoors setting with anything in the environment in clear view yields the same data as a setting in reduced laboratory conditions. Moreover, and unexpectedly, the data at near distances (few meters) are not qualitatively different from the data at much more remote distances (tens of meters).

The latter fact is perhaps surprising because binocular stereopsis is certainly important at the near, but hardly at the far distances. The result is also unexpected in view of our earlier results (Koenderink, van Doorn, & Lappin, 2000) with pointings in the frontoparallel. In retrospect, a reason might be that in the latter experiment the targets were scaled in size with their distance from the observer, whereas the pointing device (for technical reasons) was not. In the present experiment we took pains (see Fig. 3) to scale both. That this may well be an important point will be discussed below in the context of some hypothetical explanations for the present finding.

The main fact established in the present experiments is that exocentric pointing results generically violate the assumption of the existence of a simple geodesic arc between any two points. This is a brute fact that is of considerable conceptual interest because it easily leads one to conclude that it is generically not the case that a unique simple arc connects any two points and thus that one cannot define pregeodesics.

If there is no unique connecting geodesic arc many geometries are automatically ruled out as candidates for the structure of visual space, Luneburg's homogeneous spaces being cases in point. A familiar example of a space lacking unique to-and-fro connections is that of a typical European inner city from the perspective of a car driver (Krause, 1987). Here one way traffic regulations commonly make routes from $\mathcal{A}$ to $\mathcal{B}$ quite different from those from $\mathcal{B}$ to $\mathcal{A}$. Clearly such a situation is not naturally modeled by the Euclidian plane, quite unlike the case of a Utah salt flat. It is almost inconceivable that (especially empty) "visual space" could be anything like that though.

The "non-desarguesian planes" that have been constructed in mathematics might be thought to yield sufficient generalization to explain our present results. Well known examples are the Hilbert, Moulton and Veblen-Wedderburn (or Hughes) planes. In such models one starts with a desarguesian plane and redistributes the points of the colinear sets according to certain rules. Thus one obtains geodesics that look (to the Euclidian eye) as "broken" straight lines. This will not work though, because such planes still satisfy the basic axioms of a projective structure, notably any two distinct points are incident with a unique line. This effectively rules out such models as descriptive of our findings. Moreover, we do not expect the desarguesian property to fail empirically, since Pappus' Theorem fails in the non-desarguesian planes whereas we have shown that the Pappus property holds for visual space (Koenderink, van Doorn, Kappers, & Todd, 2002a). That the desarguesian structure holds in visual space is also suggested by the experiments of Foley (1964), though Foley's experiment bears more directly on the Veblen-Young (also known as Pasch) axiom (on the existence of planes in 3D-space) than the Deseargues configuration as such.

Is there a way out, that is to say, can "Visual Space" be saved (at least for the moment)? One would have to drop the notion of "space as a container". The space would have to be "contextual", *i.e.*, dependent of what is in the space, or "momentary", *i.e*, dependent on where the observer is fixating in the space, or task dependent, or perhaps all of these. Such "spaces"—in the sense as used by physicists, rather than the formal geometries contemplated by mathematicians—are certainly conceivable. Whether one chooses to keep the epithet "space" in such cases is a matter of taste. In physics one uses "space" for an entity that is categorically different from a formal (mathematical) geometry.

One way to look at the problem is to assume that "visual space" might depend upon the momentary viewing geometry. Of course such an assumption comes dangerously close to claiming that visual space is a non-entity, that certainly applies if "space" is used in the formal sense. However, such an assumption allows one to devise models of a geometrical nature that have predictive power, so the assumption is not altogether damaging. One would generalize the concept of visual space, rather than discarding it altogether.

The notion that "visual space" might depend upon the momentary viewing geometry is perhaps not all that surprising given the well documented non-uniformity of the visual field, the anisotropic structure of the oculomotor system, and the bilateral symmetry of head and body. Indeed, an isotropic and homogeneous visual space can only be conceived as being constructed over time, including variations over various viewing directions and head and body postures, evidently involving memory mechanisms. That space might thus be "momentary" has been suggested by a number of authors (Haubensak, 1970; Schelling, 1956; Suppes, 1977 see also Trommershäuser et al., 2003).

If we assume a dependence on the fixation direction, results would become dependent on whether the observer tends to favor the pointer or the target when looking towards or away from the pointer during the setting. From our informal observations it seems to be the case that observers look at the pointer (though they alternate between pointer and target for some period) when finalizing their setting. If so, then various models would predict asymmetries between far-to-near and near-to-far pointing, and such asymmetries might well account for the present data.

All such models somehow "deform" visual space in accordance with the present fixation point. The extremes are a purely radial and a purely angular distortion (Richards, 1968, 1971). In the former case one assumes the distance–depth relation to depend upon the angular distance from the fixation direction. By shrewd choice of such a function it is indeed possible to "predict" the data. In the latter case one assumes the distance metric in the visual field to be a function of the angular distance from the fixation direction. For instance, one may assume the visual field to expand about the fixation direction, which expansion would have to be compensated for by a contraction elsewhere. Such assumptions also lead to fixation dependent pointings. Of course one may consider any type of combination of such mechanisms.

The crux of such models is the assumption that the structure of visual space is not fixed, but depends upon the point that happens to be fixated. This destroys the possibility of a fixed geometry or "space as a container". It results in a situation in which the arcs $\mathcal{AB}$ and $\mathcal{BA}$ appear to be different, of course this implies that the (unique) arc $\mathcal{AB}$ does not exist. Thus the consequences of these simple assumptions imply that "visual space" cannot be described by any (even inhomogeneous) Riemannian structure.

Thus the upshot of this work is that the pregeodesics as indicated by exocentric pointing generally fail to be simple arcs. The conclusion from this is that the notion of a "visual space" (in the sense implied by Luneburg say) has to be discarded or suitably amended. In any case this indicates that different methods to approach the problem of "visual space" are needed (Ehrenstein, 1977). Methods that address structural relations may prove to be more revealing and stable than absolute judgments like in the present experiment. For instance in the experiment of Koenderink et al. (2002a) we find evidence for a coherent projective structure (the Pappus property implicating the desarguesian property) and the experiment by Todd, Oomes, Koenderink, and Kappers (2001) has provided evidence for a coherent affine structure. Such puzzling task dependencies deserve close attention.

Appendix A. A measure of confidence for the existence of a "straight connection" between two points

A "pregeodesic" is a curve such that all tangents at the curve are parallel in the sense of the geometry. Thus the intuition is that the pregeodesics are "straight". Thus the meridians of the globe are pregeodesics whereas latitude circles (generically) not. A "pregeodesic" differs from a "geodesic" in that it is a projective, rather than a metrical property. "Geodesics" are the arcs of "shortest distance", they conceptually carry a distance scale (technically: "Are parameterized by arclength"), whereas the pregeodesics are defined by mere "straightness". The meridians of the globe are "straight" because they do not deviate either way and thus divide the globe into two equal halves, which is where the term "geodesic" derives from. In many geometrical contexts one does not distinguish between the projective ("straightness") and metrical ("shortest distance") origins and speaks simply of "geodesics". Such sloppiness will not do in the case of visual space. Here we are singularly concerned with straightness.

A segment of a pregeodesic in some parameter space will typically be curved. Examples include the images of the meridians in maps of the globe. Points of inflection may occur, but only at isolated points.

A pregeodesic of visual space ("apparent straight line") mapped in physical space is likewise expected to be curved, with perhaps occasional points of inflection. In experiments it is these images in physical space of the pregeodesics of visual space that are determined by the experiments. One expects shallow curvatures. Typical theories (*e.g.*, Luneburg's) indeed typically predict curvatures but no inflection points.

In the present experiments pregeodesics are implied by two measurements of the direction of the tangent at different locations. We need a measure of "how reasonable" the estimates are as candidates for pregeodesics. We use the tendency to inflect for this. The measurements are performed at many different distances. We seek a measure that is approximately invariant with respect to such distance variations.

A simple way to achieve this is the following. Consider a Cartesian coordinate system, the X-coordinate from left to right in the frontoparallel direction, the Y-coordinate in the forward direction. Let the pointer and target be located at $\{x_1, y_1\}$ and $\{x_2, y_2\}$. As the observer points from $\{x_1, y_1\}$ to $\{x_2, y_2\}$ we record a direction φ_1, as the observer points the other way a direction φ_2 (say). We look for a curve that passes through $\{x_1, y_1\}$ and $\{x_2, y_2\}$ and has a slope $\mathrm{d}y/\mathrm{d}x|_{x_i} = \tan\varphi_i (i = 1, 2)$ at these locations. The cubic curve

$$y(x) = c_0 + c_1 x + \frac{1}{2}c_2 x^2 + \frac{1}{6}c_3 x^3,$$

is a good candidate because the coefficients $\{c_0, c_1, c_2, c_3\}$ are uniquely determined by the boundary conditions, whereas it should be an ample approximation of virtually any smooth, shallow curve. The curve has a point of inflection at $x_I = -c_2/c_3$ (one easily checks that $\mathrm{d}^2y/\mathrm{d}x^2|_{x_I} = 0$). In terms of the boundary conditions you have

$$x_I = \frac{(x_1 - x_2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}|_2(2x_1 + x_2) + \frac{\mathrm{d}y}{\mathrm{d}x}|_1(x_1 + 2x_2)\right) - 3(x_1 + x_2)(y_1 - y_2)}{3\left(\left(\frac{\mathrm{d}y}{\mathrm{d}x}|_1 + \frac{\mathrm{d}y}{\mathrm{d}x}|_2\right)(x_1 - x_2) - 2(y_1 - y_2)\right)}.$$

Thus the condition for the existence of a simple curve is

$$(x_I \leqslant x_1) \vee (x_I \geqslant x_2),$$

for if this condition is FALSE the inflexion point lies on the segment $x_1 x_2$ and the curve is not simple (a "C-curve"), but inflected (an "S-curve"). Algebraic simplification yields an especially simple form for this condition:

$$-\frac{s_1}{2} < s_2 < -2s_1,$$

where

$$s_0 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right),$$

denotes the slope of the straight line segment connecting the endpoints and

$$s_{1,2} = \frac{\mathrm{d}y}{\mathrm{d}x}|_{1,2} - \left(\frac{y_2 - y_1}{x_2 - x_1}\right),$$

denote the slopes of the pointing directions, corrected for the slope of this straight connection.

Thus the final condition is simply that the ratio of slopes at either end should not exceed a factor of two. This is illustrated by the examples in Fig. 1. In the top figure the ratio is one (the minus sign in the condition means that you flip the arrow at the righthand side by 180°) and you have a perfect (that is symmetrical) "simple arc". In the center figure the slope on the righthand side is less than a factor of one-half. The condition is violated and you lose the simple arc and obtain an inflect S-curve. If you consider the left–right reversal of this figure you have the case of a factor exceeding two, which again violates the condition and leads to an S-shaped curve. In the lower figure the violation is extreme, this case also violates the condition. It is perhaps the most intuitive example of such a violation.

The condition obviously works for any size of the configuration.

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