Competitive Nonlinear Pricing and Contract Variety*

JIAN SHEN†
HUANXING YANG‡
LIXIN YE§

We analyze markets with both horizontally and vertically differentiated products under both monopoly and duopoly. In the base model with two consumer types, we identify conditions under which entry prompts an incumbent to expand or contract its low end of the product line. Our analysis offers a novel explanation for the widespread use of 'fighting brands' and 'product line pruning.' We also extend our analysis to asymmetric firms and three types of consumers and show that depending on the specific environment, entry may lead the incumbent to expand or contract the middle range of its product line (middle contracts). Our results are mainly driven by interactions between horizontal differentiation (competition) and vertical screening of consumers.

I. INTRODUCTION

In response to entry or increased competition, incumbents often expand their product lines to introduce new products, or contract their product lines to remove some existing products. When the expansion and contraction occur on the low end of the market or product line, they are known as '*fighting brands*' and '*product line pruning,*' respectively (Johnson and Myatt [2003]). Both fighting brands and product line pruning are common in business. A notable fighting brands example is

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†Authors' Affiliation: JPMorgan Chase, Consumer & Community Banking, 1111 Polaris Pkwy., Columbus, OH 43240-2050, USA.
email: karljian2004@gmail.com

‡The Ohio State University, Department of Economics, 1945 N. High St., Columbus, OH 43210-1172, USA.
email: yang.1041@osu.edu

§The Ohio State University, Department of Economics, 1945 N. High St., Columbus, OH 43210-1172, USA.
email: ye.45@osu.edu

Intel's Celeron processor.[1] Intel dominated the microprocessor market with its Pentium chips. However, AMD entered the low-end market with its K6 chips in 1997 and had a considerable impact on the PC market. In order to retain its dominance, Intel launched Celeron, a less powerful version of Pentium, targeted at the budget PC market. As a result, with its so-called three-tier strategy, Intel successfully protected its high-end market.[2] Other cases of fighting brands abound. For example, Qantas (Australia) launched Jetstar to take on Virgin Blue,[3] British Airways (UK) launched GO to take on Ryanair and EasyJet,[4] GM launched Saturn to take on Japanese imports into America.[5] Product line pruning is also widespread. For example, in response to private label brands in the early 1990's, Procter & Gamble removed some weak products from its product line.[6] More recently, Honda decided to eliminate its Element SUV after 2011 due to a mix of internal and external competition.[7] Stating that the new focus is its business customers, the networking giant Cisco abandoned Flip digital video cameras and several other consumer business products in 2011.[8]

Note that incumbent firms do not only respond to competition by adjusting the low end of their product lines; they may adjust the middle range of product lines as well. On the one hand, increased competition may lead to addition of middle products. For example, following the release of TomTom's first GPS series GO in March 2004, the incumbent Garmin introduced the Quest series as a medium-level product which featured a 2.7" 240x160 non-touch-sensitive color screen.[9] On the other hand, increased competition may lead to the removal of some existing middle products. For example, after OpenOffice's entry to release OpenOffice.org 1.0 for free on May 1, 2002, on November 17, 2003, Office 2003 removed two truncated variations of the Professional edition, i.e., Professional with FrontPage and Professional with Publisher, leaving only the Professional edition that contains all the packages and other more basic versions.[10]

The examples above show that incumbent firms may respond to competition by adjusting their product lines either at the low end or in the middle range. In this research, we offer a framework

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[1] 'Chip rivalry cuts PC prices Intel rolls out Celeron line to counter AMD's K6,' *USA Today*, April 9, 1998.

[2] 'Who says Intel's chips are down? Its three-tier strategy keeps sales afloat in an off year,' *Business Week*, December 7, 1998.

[3] 'Jetstar launches in budget market,' *Sydney Morning Herald*, May 24, 2004.

[4] 'British Airways launches low-cost airline to European destinations,' *Associated Press Archive*, May 22, 1998.

[5] 'GM seen announcing Saturn project funding,' *The Wall Street Journal*, January 8, 1985.

[6] 'Procter & Gamble to cut some items,' *The Sun*, August 3, 1992.

[7] 'Say good-bye to the Honda Element,' *Fortune*, December 10, 2010.

[8] 'Cisco kills Flip, cuts 550 workers,' *CNNMoney*, April 12, 2011.

[9] 'TomTom unveils TomTom GO All-in-one navigation device is the easiest to use and most portable car navigation tool ever,' *PR Newswire*, March 22, 2004, and 'Quest(TM): Garmin's new pocket-sized street navigator proof that big things come in small packages,' *PR Newswire*, July 12, 2004.

[10] 'Free office suite reaches milestone release,' *CNET News*, May 2, 2002, and 'Microsoft offers various bundles in its new Office,' *Pittsburgh Post-Gazette* , October 23, 2003.

to analyze how increased competition affects the product line or the variety of contracts offered.[11] Specifically, in our model consumers are both vertically and horizontally differentiated: in the vertical dimension they have different marginal utilities of quality and in the horizontal dimension they have different tastes over firms' products (or brands). Firms' products are horizontally differentiated, and in the vertical dimension each firm offers a range of products with different qualities. Under duopoly, firms compete by offering a menu of contracts (or, equivalently, price-quality schedules). The driving force in our model is the interaction between horizontal differentiation (competition) and the vertical screening (on vertical types).

In the base model, we focus on the case where consumers only have two vertical types, h (high) and l (low). We compare the optimal menu of contracts under monopoly to the equilibrium menu of contracts in the symmetric equilibrium under duopoly. Our main result is that when the degree or extent of horizontal differentiation (measured by the per unit transportation cost k) is low, entry will never lead to product line pruning, but it may lead to fighting brands; on the other hand, when the extent of differentiation is high, entry will never lead to fighting brands, but it may lead to product line pruning. Similar results hold when further entry occurs.[12] If the initial intensity of competition is strong, further entry can only lead to fighting brands; if the initial intensity of competition is weak, further entry can only lead to product line pruning.

The intuition of the above results is based on the interaction between horizontal differentiation/competition and vertical screening, which is captured by the effects of entry on the rent provision for the high-type consumers and the relative importance of serving the low-type consumers. First, entry causes competition between firms for marginal consumers, which tends to increase the rent provision for the high type (the competition effect). Second, entry reduces the incumbent firm's market share of the high type, which tends to reduce the rent provision for the high type as the incumbent firm now has a smaller market to penetrate into (the market share effect on rent provision). Third, the reduced market share of the high type also makes the low-type consumers relatively more important (the market share effect on relative importance of the low type). Among the three effects, the first two effects work against each other and determine the change in rent provision to the higher type. The competition effect tends to increase the rent for the high type, which relaxes the incentive compatibility (IC) constraint along the vertical dimension,[13] making the offer of the low contract (targeting the low type) more likely. The market share effect on rent provision tends to reduce the rent for the high type, which tightens the IC constraint and makes the offer of the low contract less likely. The third effect tends to make the offer of the low contract more likely as

[11] The terms 'product(s)' and 'contract(s)' can be used interchangeably in our model, although we will mostly use the term 'contract(s).'

[12] In Section 2.3, we demonstrate that our analysis of two-type case can be easily translated to the case with n firms, which is standard for a Salop circular city model.

[13] The IC constraint is also referred to as the screening or sorting conditions in this paper.

the low-type consumers become relatively more important. Whether entry leads to the introduction or removal of the low contract depends on which effect dominates. In our model, when the extent of differentiation is low, the competition effect dominates the market share effect on rent provision, leading to higher rent for the high type. Combining with the third effect, this makes the offer of the low contract more likely (fighting brands). When the extent of differentiation is high, however, the market share effect on rent provision dominates the competition effect, leading to lower rent for the high type. When the extent of differentiation is high enough, the market share effect on rent provision is so strong that it even offsets the third effect, leading to the removal of the low contract (product line pruning).

We also extend our analysis to asymmetric firms, where the entrant is technologically inferior to the incumbent in the sense that the upper bound of the quality range of the entrant is lower than that of the incumbent (Section 3). We show that when the extent of differentiation is sufficiently small (so that there is effective competition for the high type), as the maximum quality of the entrant becomes higher, fighting brands become more likely and product pruning becomes less likely. Finally, we extend our analysis to the case with symmetric firms and three vertical types, high (h), middle (m), and low (l) (Section 4). Unlike in the two type case, now we show that entry may lead to the addition or removal of the middle contract targeting at middle type consumers. In both extensions, we demonstrate that the key insight of our analysis is once again the interaction of horizontal differentiation/competition and vertical screening.

While it is fairly common for incumbent firms to respond to competition by adjusting product lines (contract variety), the connection between competition and contract variety has received little attention from economists. Johnson and Myatt [2003] were the first to develop a formal model and offer an explanation for fighting brands and product line pruning. In their analysis, a single firm enters a market originally dominated by a monopolist. The duopolists then compete in quantities, each potentially offering a range of quality-differentiated products. They show that whether the incumbent will choose to extend or contract its product line mainly depends on the shape of the marginal revenue curves in the market. When marginal revenue is decreasing, the incumbent may respond to entry by pruning low-quality products. However, when marginal revenue is increasing in some regions, upon entry an incumbent may find it optimal to introduce a lower-quality product (fighting brand).[14] The base model in our paper offers an alternative explanation for fighting brands and product line pruning: in Johnson and Myatt, whether fighting brands or product pruning will occur mainly depends on the shape of the marginal revenue curve, which in turn depends on the distribution of consumer (vertical) types; while in our model, it is the extent of horizontal differentiation (intensity of competition) that determines whether fighting brands or product pruning will occur.

[14] Johnson and Myatt [2006] extend the two-firm model of Johnson and Myatt [2003] into a setting with n firms engaging in Cournot competition.

Since the seminal work of Mussa and Rosen [1978] and Maskin and Riley [1984] on monopolistic nonlinear pricing, there is a growing literature on nonlinear pricing in competitive settings, see, for example, Spulber [1989], Champsaur and Rochet [1989], Wilson [1993], Gilbert and Matutes [1993], Stole [1995], Verboven [1999], Villas-Boas and Schmidt-Mohr [1999], Armstrong and Vickers [2001, 2006], Rochet and Stole [1997, 2002], Ellison [2005].[15] However, all these papers assume that all the (vertical) types of consumers are served in the market. This full market coverage assumption does greatly simplify the analysis, but precludes the effect of competition on the number of contracts offered on the vertical dimension, which is central to our analysis.

Technically speaking, our approach is most related to Rochet and Stole [2002], who offer a general framework with both horizontally and vertically differentiated products and discrete and continuous types of consumers (on the vertical dimension). This framework covers both monopoly and duopoly cases. Their analysis focuses on the case where all consumer types on the vertical dimension are covered. Our base model with two types of consumers is comparable to their two-type model. They focus on how horizontal differentiation/competition affects quality distortion at the bottom, and in the duopoly case, they find that quality distortions disappear and the equilibrium is characterized by the cost-plus-fee pricing feature (a similar result is obtained in Armstrong and Vickers [2001]). Our model differs from Rochet and Stole in that we impose a minimum quality requirement. With such a requirement, firms may find it optimal to exclude the low type consumers, and the choice of contract variety becomes nontrivial. Without such a requirement (as in Rochet and Stole), under both monopoly and duopoly two contracts targeting high and low types are always offered, and neither fighting brands nor product pruning will arise in our base model.

By allowing for partial market coverage on the vertical dimension, Yang and Ye [2008] provide a complementary analysis of the continuous type model in Rochet and Stole [2002]. By focusing on the case where the lowest type of consumer being served is endogenously determined, they are able to study the effect of varying horizontal differentiation (competition) on the market coverage. However, since Yang and Ye [2008] assume a continuous type space along the vertical dimension, when moving from monopoly to duopoly, the quality range offered is always the same.[16] This prevents an analysis of how increased competition can affect the number of contracts or product lines offered, a main task left for the current research with discrete (vertical) types of consumers.[17]

In common agency games, Martimort and Stole [2009] study a setting where consumers have

[15] In a continuous type model without horizontal differentiation, Champsaur and Rochet [1989] show that two competing firms will choose non-overlapping quality ranges to soften competition. In their model firms choose quality ranges in the first stage and then set prices in the second stage, while in our model firms choose quality ranges and prices at the same time.

[16] The quality range offered is always from 0 to 1 in their model.

[17] In both Rochet and Stole and Yang and Ye, vertical types are assumed to be uniformly distributed. An implication is that results from the discrete-type model cannot be obtained by taking limits from their continuous-type models.

multi-unit demand and each consumer can split her purchase from two competing sellers offering nonlinear pricing schedules. In a case most relevant to ours (the delegated common-agency game), they show that in the case of substitutes, the participation rate is higher in the duopoly market than in the monopoly market, and the quality distortion is lower in the duopoly market. However, they do not consider horizontal differentiation that has a critical impact on whether to cover low type consumers, which is a focus in our paper.

The paper is organized as follows. The next section lays down the base model with two types. Section 3 analyzes the case with asymmetric firms, in which the entrant may be technologically inferior to the incumbent. Section 4 extends the base model to the case with three consumer types. Section 5 concludes. The proofs not provided in the main text are all relegated to the appendix.

II. THE BASE MODEL

We consider a market with both horizontally and vertically differentiated products where consumers' preferences differ in two dimensions. In the horizontal dimension, consumers have different tastes for different products (firms); in the vertical dimension consumers have different marginal utilities over quality. More specifically, in the vertical dimension a consumer is either type h (High) or type l (Low), i.e., the vertical type $\theta \in \{\theta_h, \theta_l\}$, where $\theta_h > \theta_l > 0$. Without loss of generality, we normalize $\theta_h = 1$. The proportions of types h and l are α and $1-\alpha$, respectively. We model the taste dimension as the horizontal 'location' of a consumer representing the ideal product for that consumer: we adopt Salop's circular city model so that in the horizontal dimension, each type of consumer is uniformly distributed on the unit-length circumference. Consumers' vertical and horizontal tastes are independently distributed. The total measure of consumers is 1.

We consider cases with one or two horizontal products or brands. A brand may offer two goods of different qualities q. In the horizontal dimension, each consumer is characterized by d, the distance between his own (ideal) location and the location of a particular product (say, product 1). Each consumer is characterized by a two-dimensional type (θ, d), and has a unit demand for the good. If a type θ consumer consumes a product of quality q which is located away from his own location by distance d and pays a transfer t, his utility is given by

$$u(\theta, q, t, d) = \theta q - t - kd, \tag{1}$$

where k measures the degree or extent of horizontal differentiation: k indicates consumers' willingness to buy a good that is not exactly of his own taste and is the per-unit transportation cost in the standard Hotelling or Salop circular city models.[18]

[18] We assume that both types have the same transportation cost k. Sometimes it is reasonable to think that the low type consumers have a smaller transportation cost, as in Ellison [2005]. Allowing different transportation costs for

If a brand offers two products of different qualities, we assume that they share the same location in the horizontal dimension. Thus, the horizontal differentiation in our model should be thought of as a brand preference, which is common across all varieties (of quality) offered. As such, we rule out the possibility of a brand choosing different horizontal locations when it offers two qualities.

We assume that there is a minimum quality standard so that each firm can only produce $q \geq \underline{q}$, where $\underline{q} \in (0, \theta_l)$. Such a requirement is standard in various industries and is mainly due to government regulations for safety or externality considerations. For example, if the quality of a car is below some threshold level, it might not be safe to drive. Another example is the new fuel-economy standards issued by the U.S. government for the automobile industry.[19] The impacts of the minimum quality requirement on firms' production decisions and consumers' welfare are documented and studied in the literature (see Armstrong and Sappington, 2007, for references).[20]

If a firm sells a product of quality q to a consumer, its profit from that sale is given by

$$v(q,t) = t - \frac{1}{2}q^2,$$

where $\frac{1}{2}q^2$ is the cost of producing a good of quality q.[21]

Neither θ nor d is observable to firms, but as can be seen from (1) the single crossing property is only satisfied in the vertical dimension. As a result firms can only make offers to sort consumers with respect to their vertical types in our model. We are interested in how market structure affects the products offered in the vertical dimension. Specifically, we compare two different scenarios. The first scenario is a monopoly, where a single brand is offered by a single firm. The second scenario is a duopoly, where two brands are offered by two different firms, who evenly split the unit-length circle as illustrated by Figure 1.

II(i). *Monopoly*

A contract is denoted as a quality-price pair, (q, t). The monopolist may offer two separate contracts, (q_h, t_h) and (q_l, t_l), targeting consumers with types h and l, respectively. Associated with each contract, the gross utility of a type i (ignoring the transportation cost incurred), $i = h, l$, who chooses contract (q_i, t_i) is given by $u_i = \theta_i q_i - t_i$. Since it is more convenient to use u_i instead of t_i, we write a contract as a quality-utility pair (q_i, u_i).[22] For a menu of (two) contracts to be incentive compatible, the l type should have no incentive to choose the h contract, or the upward incentive

different vertical types will not change our results qualitatively.

[19]'U.S. sets higher fuel efficiency standards,' The New York Times, August 28, 2012.

[20]An alternative interpretation of the minimum quality standard is that the underlying cost function exhibits increasing returns to quality in the region $(0, \underline{q}]$.

[21]The specific form of the cost function should not affect the general insight of our results, though we do require the cost function to be convex.

[22]Here we follow the lead of Armstrong and Vickers [2001], who model firms as supplying utility directly to consumers.

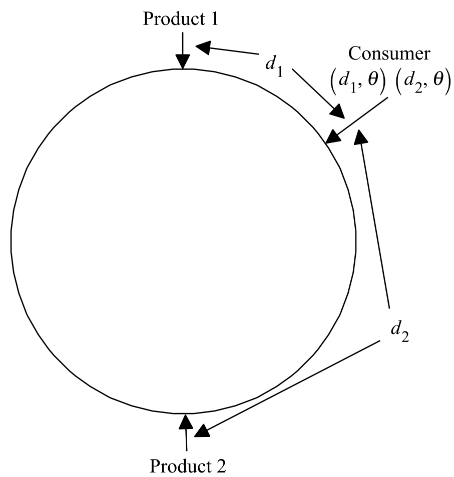

Figure 1: The Salop Circular Model

compatibility constraint (UIC) should be satisfied: $u_l \geq \theta_l q_h - t_h$. Similarly, the h type should have no incentive to choose the l contract, or the downward incentive compatibility constraint (DIC) should be satisfied: $u_h \geq \theta_h q_l - t_l$. These two constraints can be written more compactly as follows (recall that θ_h is normalized to be 1):

$$(1-\theta_l)q_h \geq u_h - u_l \geq (1-\theta_l)q_l, \tag{2}$$

where the first inequality is the UIC, and the second inequality is the DIC.

Given u_i, type i consumers whose location $d_i \leq u_i/k$ will participate. Combining with the boundary condition $d_i \leq 1/2$, the (half) market share for each type, $M(u_i, i)$,[23] is given by

$$M(u_h, h) = \alpha \min\left\{\frac{1}{2}, \frac{u_h}{k}\right\} \text{ and } M(u_l, l) = (1-\alpha)\min\left\{\frac{1}{2}, \frac{u_l}{k}\right\}.$$

So higher u_i, or lower k, implies more market penetration (in the horizontal dimension). It is worth emphasizing that the utilities u_h and u_l play dual roles: in the horizontal dimension, u_i affects the market share for type i; in the vertical dimension, u_h and u_l must satisfy the IC constraints (2).

The monopolist has two options in terms of contract variety: offering one contract or offering two contracts. Without the minimum quality requirement $q \geq \underline{q}$, it can be readily shown that offering two contracts dominates offering only one contract.[24] With the minimum quality requirement, the

[23] To ease exposition, we use half market share throughout the paper.

[24] In particular, the monopolist can always distort quality q_l downward towards 0 so that offering the low contract has no impact on the profitability of offering the high contract, which is the case in Rochet and Stole [2002].

above property no longer holds, as it is sometimes optimal for the monopolist to only offer one contract. Therefore, to make the choice of contract variety nontrivial, we assume a minimum quality requirement in our model. When only one contract is offered, either only type-h agents participate or both types participate (pooling), as the high type has a higher marginal utility of quality. The following lemma establishes that pooling is never optimal.

Lemma 1. Suppose the monopolist offers a single contract with $q \in [\underline{q}, 1]$ and both h and l types of consumers purchase in equilibrium. Then the monopolist can earn a higher profit by offering h and l contracts targeting at the high and low types of consumers, respectively.

The implication from Lemma 1 is that when searching for optimal contract(s), we can focus on offering two contracts or offering h contracts targeting at type h only. When two contracts are offered, the UIC is always slack (see Lemma 1 in Rochet and Stole, 2002, for the details). Given that the UIC is slack (type l does not want to mimic type h), the quality provision for type h should be efficient; i.e., $q_h = 1$. Moreover, we only need to worry about the DIC. Because of horizontal differentiation, the DIC might be binding or slack: although k does not enter the DIC directly, it affects u_h and hence the DIC indirectly.

We first study the case when a single contract targeting type h is offered. In that case, the firm's programming problem is:

$$\max_{u_h} \alpha \min\left\{\frac{1}{2}, \frac{u_h}{k}\right\}\left(\frac{1}{2} - u_h\right)$$

It can be verified that the optimal solution is given by

$$u_h^{fb} = \begin{cases} \frac{1}{4} & \text{if } k \in (\frac{1}{2}, 1] \\ \frac{k}{2} & \text{if } k \in (0, \frac{1}{2}] \end{cases}.$$

The optimal solution is determined by the classical tradeoff between profit margin and market penetration. The resulting market share for type h is either $1/2$ (full market coverage) if $k \leq 1/2$, or $1/(4k)$ (partial market coverage) if $k > 1/2$. For competition to be nontrivial in the duopoly case, we assume $k < 1$, so that the market share for type h under monopoly is more than $1/4$, i.e., type h is fully covered.

We next consider the case when two contracts are offered. The firm's programming problem is:

$$\begin{aligned} &\max_{(u_h, q_l, u_l)} \alpha \min\left\{\frac{1}{2}, \frac{u_h}{k}\right\}\left(\frac{1}{2} - u_h\right) + (1-\alpha)\min\left\{\frac{1}{2}, \frac{u_l}{k}\right\}\left(\theta_l q_l - \frac{1}{2}q_l^2 - u_l\right) \\ &\text{subject to: } u_h \geq u_l + (1-\theta_l)q_l \qquad \text{(DIC)} \\ &\qquad\qquad\quad q_l \geq \underline{q}, \quad u_l \geq 0 \end{aligned}$$

The optimal solution to the above problem is quite complex. As such we will only present the main results, leaving the more technical details to the appendix.

We first note that in the optimal solution, DIC might be slack. Since the h type is more profitable than the l type, in the unconstrained optimal solution the monopolist will offer a higher u_h than u_l in order to penetrate more into the market for the h type. Thus for some parameter values, the unconstrained solution automatically satisfies DIC, in which case offering two contracts is better than offering the h contract alone. Since our main focus is on the number of contracts offered, we will identify conditions under which offering two contracts is optimal and conditions under which offering the h contract alone is optimal. Specifically, we consider two cases below.

The case that $k \le \frac{1}{2}$. If $(1-\theta_l)\underline{q} < k/2$, then offering two contracts is better than offering the h contract alone. To see this, suppose the firm offers the first-best h contract $(1, u_h^{fb})$ alone. Recall that in this case $u_h^{fb} = k/2$. We argue that the monopolist can profitably introduce an l contract without changing the h contract or affecting the DIC. In particular, given that $(1-\theta_l)\underline{q} < k/2$, the firm can profitably offer an l contract (q_l, u_l) with $q_l \ge \underline{q}$, $u_l \ge 0$, and the DIC, $u_l + (1-\theta_l)q_l \le k/2$, is still satisfied. Therefore, offering h contract alone cannot be optimal.

If $(1-\theta_l)\underline{q} \ge k/2$, then the optimal number of contracts could be 1 or 2. In general, it is hard to determine the necessary and sufficient conditions under which offering h contract alone is optimal. For this reason we will identify sufficient conditions only. The result is summarized in Lemma 2 below.

The case that $k \in (\frac{1}{2}, 1]$. Similar to the previous case, if $(1-\theta_l)\underline{q} < 1/4$, then offering two contracts is optimal. The reason is that the firm can always profitably add a low contract without having to raise u_h when the first-best h contract is offered (recall that in this case $u_h^{fb} = 1/4$). Note that $(1-\theta_l)\underline{q} < 1/4$ is always satisfied due to the following inequalities:

$$(1-\theta_l)\underline{q} < (1-\theta_l)\theta_l \le \frac{1}{4}.$$

The first inequality above is due to $\underline{q} < \theta_l$. Therefore, we conclude that when $k \in (1/2, 1]$ two contracts must be offered under monopoly.

The following lemma summarizes the previous analysis.

Lemma 2. (i) The case $k \in (0, 1/2]$. If $(1-\theta_l)\underline{q} < k/2$, it is optimal for the monopolist to offer both h and l contracts. If $(1-\theta_l)\underline{q} \ge k/2$ and the following two conditions are satisfied, then it is optimal for the monopolist to offer one contract targeting type-h consumers only:

$$\frac{k\alpha}{1-\alpha} > \frac{\left(\theta_l - \underline{q}\right)\theta_l^2}{2-\theta_l-\underline{q}}; \tag{3}$$

$$\frac{k\alpha}{2(1-\alpha)} > \theta_l\underline{q} - \frac{\underline{q}^2}{2}. \tag{4}$$

(ii) The case $k \in (1/2, 1]$. It is optimal for the monopolist to offer both h and l contracts.

The result that offering an h contract alone is optimal is due to informational rent considerations (recall the IC condition (2)). If offering a low contract leads to too much informational rent to high types (relative to the profit from low types), then the firm will optimally exclude low types by not offering the l contract. From conditions (3) and (4), we see that exclusion is more likely to occur when α is large and θ_l is close to $\underline{q}$. A bigger α implies that the high type becomes relatively more important. Moreover, when θ_l is close to $\underline{q}$, the low quality cannot be distorted downward by a large amount, which makes the low-type contract more attractive to the high type consumers. This makes the firm more reluctant to offer a low contract. Part (ii) of Lemma 2 shows that the exclusion of low types is only possible when k is small. When k is large, the firm is willing to offer a high rent to type h in order to increase the market share of the high type. As a result, the informational rent consideration becomes less important, and the exclusion of the low type becomes less likely.

II(ii). *Duopoly*

Under duopoly, two firms compete by offering menu of contracts in the form of $(q_i^j, u_i^j)_{i\in\{h,l\}}$ with $j = 1, 2$ denoting firm 1 and firm 2, respectively, and q_i^j and u_i^j denoting the quality and rent provision schedules, respectively, for type i consumer, $i \in \{h, l\}$. We adopt Bertrand-Nash equilibrium as our solution concept. Specifically, $\left\{(q_i^1, u_i^1)_{i\in\{h,l\}}, (q_i^2, u_i^2)_{i\in\{h,l\}}\right\}$ is an equilibrium if given $(q_i^{-j}, u_i^{-j})_{i\in\{h,l\}}$, firm j maximizes its own profit by choosing $(q_i^j, u_i^j)_{i\in\{h,l\}}$, $j = 1, 2$. Given that firms are symmetric, we will focus on symmetric equilibria in which both firms offer the same contract(s), i.e., $q_i^1 = q_i^2$ and $u_i^1 = u_i^2$, $i \in \{h, l\}$.

The result of Lemma 1 can be readily extended to the duopoly setting: there is no equilibrium in which both firms offer one contract and both high and low type consumers are served. In constructing a profitable deviation, we can fix the other firm's contract and let one firm offer another contract targeting either type h or type l, which offers the same utility to the targeting type as the original contract. This means that pooling equilibria do not exist. Therefore, we can concentrate on two possible equilibria. In the first scenario, each firm offers contract h only and only type h consumers are served. In the second scenario, each firm offers two contracts targeting at types h and l separately. Given that $k < 1$, the market for type h will be fully covered in the horizontal dimension. Therefore, the market share for type h of firm 1 becomes $\frac{1}{4} + \frac{u_h^1 - u_h^2}{2k}$. On the other hand, the market for the low type might not be fully covered. As a result, the market share for type l of firm 1 is $\min\left\{\frac{u_l^1}{k}, \frac{1}{4} + \frac{u_l^1 - u_l^2}{2k}\right\}$.

When both firms offer h contracts only, the profit maximization problem for firm 1, given (q_h^2, u_h^2), is as follows:

$$\max_{u_h^1} \alpha \left(\frac{1}{4} + \frac{u_h^1 - u_h^2}{2k}\right)\left(\frac{1}{2} - u_h^1\right), \text{ if } u_h^1 + u_h^2 \quad \geq \quad \frac{k}{2},$$

$$\max_{u_h^1} \alpha \frac{u_h^1}{k}\left(\frac{1}{2}-u_h^1\right), \text{ if } u_h^1+u_h^2 \quad \leq \quad \frac{k}{2}.$$

In the programming problem above, the condition $u_h^1+u_h^2 \geq \frac{k}{2}$ implies that type h is fully covered in the horizontal dimension, and thus two firms are actively competing with each other. On the other hand, when $u_h^1+u_h^2 < \frac{k}{2}$, type h is not fully covered and two firms act as two local monopolists. Note that given u_h^2, firm 1's objective function is not differentiable at $u_h^1 = \frac{k}{2} - u_h^2$, where firms switch from being local monopolists to actively competing with each other. This is the well-known kinked demand curve in the Hotelling framework (see D'Aspremont et. al, 1979). In the symmetric equilibrium, the equilibrium utilities u_h^D are given by

$$u_h^D = \begin{cases} \frac{k}{4} & \text{if } k \in (\frac{2}{3}, 1] \\ \frac{1-k}{2} & \text{if } k \in (0, \frac{2}{3}] \end{cases}.$$

Basically, when the degree of horizontal differentiation k is smaller than 2/3, two firms actively compete for consumers. As k decreases in this range, competition becomes fiercer and consumers' equilibrium utilities increase. However, when $k \in (2/3, 1]$, even though the market for type h is fully covered, each firm has no incentive to steal the other firm's market share. In other words, there is no active competition between two firms. This feature arises due to the kinked demand curve. Specifically, $k \leq 1$ means that the market for type h must be fully covered, as each firm, acting as a local monopolist, will cover more than half of the market for type h.[25] But $k > 2/3$ implies that each firm has no incentive to steal the other firm's market share.[26] It turns out that when $k \in (2/3, 1]$, the equilibrium is at the kink: both firms offer $u_h^D = k/4$, and $u_h^1 + u_h^2 = \frac{k}{2}$.[27] That is, each firm offers just enough utility to cover half of the market and the marginal consumer in equilibrium gets zero utility. As k decreases in the range of $(2/3, 1]$, consumers' equilibrium utilities actually decrease, as it becomes easier for each firm to cover half of the market.

Now suppose that both firms offer h and l contracts. The profit maximization problem for firm 1, given (q_i^2, u_i^2), $i \in \{h, l\}$, is as follows:

$$\max_{(u_h^1, q_l^1, u_l^1)} \alpha\left(\frac{1}{4}+\frac{u_h^1-u_h^2}{2k}\right)\left(\frac{1}{2}-u_h^1\right) + (1-\alpha)\min\left\{\frac{1}{4}+\frac{u_l^1-u_l^2}{2k}, \frac{u_l^1}{k}\right\}\left(\theta_l q_l^1 - \frac{1}{2}(q_l^1)^2 - u_l^1\right)$$

$$\text{subject to: } u_h^1 \geq u_l^1 + (1-\theta_l)q_l^1 \quad \text{(DIC)}$$

[25]The solution to the second case in the programming problem is $u_h^D = 1/4$. Thus the market for type h is fully covered if $2u_h^D \geq k/2$, which is equivalent to $k \leq 1$.

[26]Solving the first case in the programming problem, we have the equilibrium utility $u_h^D = (1-k)/2$. If $k > 2/3$, then $2u_h^D < k/2$, which implies that the market is not fully covered.

[27]To see this, suppose firm 2 offers $u_h^D = k/4$. Then it can be verified that firm 1's profit decreases if u_h^D increases from $k/4$ (the first case), and firm 1's profit decreases as well if u_h^D decreases from $k/4$ (the second case).

$$q_l^1 \geq \underline{q};\ u_l \geq 0$$

Again, we will only present the main results, and the remaining details are relegated to the appendix.
The case that $k \in \left(0, \frac{1}{2}\right]$. In this case, in the duopoly equilibrium firms must offer two contracts. To see this, suppose in equilibrium each firm only offers an h contract. From the previous analysis, the full-information utility $u_h^D = (1-k)/2$. Now given that $k \leq 1/2$, we have

$$u_h^D \geq \frac{1}{4} > (1-\theta_l)\underline{q},$$

and thus firm 1 can profitably offer a low contract (q_l, u_l), with $u_l > 0$, $q_l \geq \underline{q}$, and $(1-\theta_l)\underline{q} + u_l \leq u_h^D$ (the DIC is satisfied). Therefore, when $k \in (0, 1/2]$, in the duopoly equilibrium both firms offer two contracts.
The case that $k \in \left(\frac{1}{2}, 1\right]$. The number of contracts offered in the duopoly equilibrium depends on parameter values. The following lemma characterizes the number of contracts in the duopoly equilibrium.

Lemma 3. (i) The case $k \in (0, 1/2]$. Both firms offer h and l contracts in the duopoly equilibrium.

(ii) The case $k \in (1/2, 1]$. Both h and l contracts will be offered in equilibrium if

$$\underline{q} < q^\dagger \equiv \frac{1}{1-\theta_l} \cdot \max\left\{\frac{1-k}{2}, \frac{k}{4}\right\}.$$

Both firms will offer an h contract alone if $\underline{q} \geq q^\dagger$ and the following two conditions are satisfied:

$$\frac{\alpha}{1-\alpha}\left(\frac{3}{4}k - \frac{1}{2}\right) > \frac{\left(\theta_l - \underline{q}\right)\theta_l^2}{2-\theta_l-\underline{q}}; \tag{5}$$

$$\frac{\alpha}{1-\alpha}\left(\frac{3}{8}k - \frac{1}{4}\right) > \theta_l\underline{q} - \frac{\underline{q}^2}{2}. \tag{6}$$

To understand why in the duopoly equilibrium two contracts must be offered when the degree of horizontal differentiation is low, note that a lower degree of horizontal differentiation under duopoly implies fiercer competition for type h, which leads to a higher rent to type h. This relaxes the incentive compatibility constraint along the vertical dimension. Hence, the informational rent consideration becomes less important as type h secures higher rent due to competition. This implies that offering a contract to low-type consumers has less negative impact on the profitability from the high type. As a result, offering a contract to low-type consumers might be profitable, which turns out to be indeed the case when $k \leqslant 1/2$. For the case of offering only the h contract, the set of sufficient conditions (5) and (6) essentially ensure that it is not profitable for firms to offer contract l in equilibrium.[28]

[28] Note that this set of sufficient conditions implies that $k > 2/3$. When $k \leq 2/3$, conditions (5) and (6) fail to hold

Roughly speaking, when it is too costly to serve the low type consumers (a bigger $\underline{q}$) and there are enough high type consumers (a bigger α), then only one contract will be offered in equilibrium. A more detailed discussion of the intuition and driving forces of the results is provided in the next subsection.

II(iii). *Comparison*

We summarize the results from the previous subsection in the following table:

Table 1: Horizontal Differentiation and Contract Variety

	Low Differentiation	High Differentiation
Monopoly	Single contract (h) with - Large α (high types more important) - Large $\underline{q}$ (l contract more attractive to h types)	Two contracts (h and l)
Duopoly	Two contracts (h and l)	Single contract (h) for each firm with - Large α (high types more important) - Large $\underline{q}$ (l contract more attractive to h types)

Based on this table, we will compare the duopoly equilibrium with the optimal solution under monopoly. Our focus is on how entry affects the product line, or the number of contracts offered. We first point out that when only offering an h contract, for $k < 1/2$ the rent to the h type under duopoly is higher than that under monopoly ($u_h^D > u_h^{fb}$), while for $k \in (1/2, 1]$ the relationship is reversed ($u_h^D < u_h^{fb}$). Combining the results from Lemmas 2 and 3, the following two propositions identify sufficient conditions under which competition leads to an increase or decrease in the number of contracts offered.

Proposition 1. (Low degree of horizontal differentiation) Suppose $k \leq \frac{1}{2}$. Then competition will never lead to product line pruning, as both firms always offer two contracts in the duopoly equilibrium. Competition may lead to fighting brands. Specifically, if $\frac{k}{2} < (1-\theta_l)\underline{q}$ and α is sufficiently large so that conditions (3) and (4) hold, then under monopoly it is optimal to offer an h contract only, while under duopoly both firms offer h and l contracts in equilibrium.

Proposition 2. (High degree of horizontal differentiation). Suppose $k \in (\frac{1}{2}, 1]$. Then competition will never lead to fighting brands, as two contracts are always offered under monopoly. Competition may lead to product line pruning. Specifically, if $k \in (\frac{2}{3}, 1)$, $\underline{q} < q^{\dagger}$, and α is sufficiently large so that conditions (5) and (6) are satisfied, then under monopoly it is optimal to offer two contracts, while in the duopoly equilibrium each firm offers an h contract only.

and results are more sensitive to the specifications of primitives.

The general intuition for Propositions 1 and 2 is clear. As mentioned in the introduction, entry induces three effects: the competition effect, the market share effect on rent provision, and the market share effect on the relative importance of the low type. When $k \leq 1/2$ (Proposition 1) the competition between two firms is intense and hence the competition effect dominates the market share effect on rent provision, leading to higher rent for type h. Higher rent for type h in turn relaxes the IC constraint. This, combined with the increased importance of serving the low type (the market share effect on the relative importance of the low type), leads to the offer of the low contract under duopoly even if it is absent under monopoly (fighting brands). When k is sufficiently large (Proposition 2), however, the competition between two firms is weak and hence the market share effect on rent provision dominates the competition effect, leading to lower rent for type h. Lower rent for type h makes the IC constraint more binding. This effect is so strong that it more than offsets the market share effect on the relative importance of the low type, leading to the removal of the low contract even if it is offered under monopoly (product line pruning). Note that although k does not affect the IC constraint directly, it affects the IC constraint indirectly through its effect on the rent provision for type-h consumers. In this sense the driving force behind Propositions 1 and 2 is the interaction between horizontal differentiation and vertical screening.

From Propositions 1 and 2, we also see that fighting brands or product line pruning are more likely to occur when the proportion of type h is relatively high and θ_l is close to $\underline{q}$. These conditions ensure that exclusion of the low type is more likely (either in monopoly or duopoly).

Note that our explanation for fighting brands and product line pruning is quite different from that offered by Johnson and Myatt [2003]. The difference mainly stems from the difference in modeling: in Johnson and Myatt products are only vertically differentiated and firms compete in quantities,[29] while in our model products are both vertically and horizontally differentiated and firms compete by offering a menu of contracts (equivalently, price-quality schedules). This leads to very different implications. In Johnson and Myatt, whether fighting brands or product line pruning will emerge depends on the production technology (whether the cost function exhibits increasing or decreasing returns to quality) and the shape of the marginal revenue curve (which in turn depends on the distribution of consumer vertical types). In our model, however, it is the extent of horizontal differentiation (intensity of competition) that determines whether fighting brands or product line pruning should occur. Here is one example. In order to have fighting brands in Johnson and Myatt, the production must exhibits increasing returns to quality (i.e. it costs less to produce higher quality) in some region. In our model, however, production exhibits decreasing returns to quality (given the quadratic cost function), which implies that fighting brands would never occur in Johnson and Myatt's framework. Further, Johnson and Myatt show that product pruning will emerge only when the entrant is very much inferior to the incumbent in terms of the technological capabilities

[29] They assume that the set of product qualities available to firms is fixed, which is not the case in our model.

(their Propositions 7 and 8). In our analysis, however, both fighting brands and produce line pruning may emerge when firms are technologically symmetric.

At a broader level, both papers study the following tradeoff of introducing a low contract (low-quality product). On the one hand, the introduction of a low contract increases the market share by capturing additional low type consumers. On the other hand, the introduction of a low contract offers an additional substitution possibility for high type consumers, so the price of high contracts will have to fall. In Johnson and Myatt, the cost of introducing a low contract depends on the production technology and the shape of marginal revenue. In our model, however, the cost of introducing a low contract depends on the degree of horizontal differentiation. Under monopoly, a low degree of horizontal differentiation implies a high cost of introducing a low contract, as it is easy to penetrate into the market for high type consumers. Under duopoly, a low degree of horizontal differentiation implies a low cost of introducing a low contract, as high type consumers will secure a high rent due to the intense competition anyway.

Suppose that under both monopoly and duopoly two contracts are offered, and let the qualities of the low quality products under monopoly and duopoly be q_l^m and q_l^d, respectively. The following lemma illustrates how entry affects quality distortion of the low contract.

Lemma 4. If $k \leq \frac{1}{2}$ and both h and l contracts are offered under monopoly and duopoly, then $q_l^m \leq q_l^d$.

The intuition is consistent with that provided in Yang and Ye [2008]. When k is small, competition in duopoly leads to higher rent for type h. This relaxes the DIC constraint. On the other hand, the market share for type h becomes smaller in duopoly, which makes each firm more willing to increase the rent to the low type. These effects both contribute to a smaller downward quality distortion. However, when $k > 1/2$ and two contracts are offered under both duopoly and monopoly, the quality distortion can be higher or lower under duopoly. This is because with a higher k, the reduction in market share of the h type under duopoly reduces the rent to the h type. Thus the two effects mentioned earlier work against each other, leading to ambiguous results regarding quality distortion.

It turns out that we can also analyze the effect of entry on the welfare. The result is summarized as follows.

Proposition 3. Suppose $k < 1/2$. Compared to the monopoly case, type-h consumers are strictly better off, type-l consumers are better off, and the aggregate consumer welfare is strictly higher under duopoly in the following scenarios: (i) entry leads to fighting brand; (ii) both h and l contracts are offered and DIC is either slack or binds under both monopoly and duopoly.

The intuition for part (i) of Proposition 3 is as follows. In the case of fighting brand, given that the degree of horizontal differentiation k is less than $1/2$, type-h consumers are clearly better off under duopoly. The reason is, even if only the h contract were offered under duopoly, type h would have received a higher utility than under monopoly due to competition, and adding a low contract would only increase the rent to the h type. Type-l consumers are obviously better off as well as the low contract is not offered under monopoly.

When two contracts are offered under both monopoly and duopoly, the welfare comparison is less clear-cut. This is because the DIC is more likely to bind under monopoly. Roughly speaking, when the DIC is either slack or binding under both duopoly and monopoly (part (ii) of Proposition 3), competition for type h under duopoly implies that type h is better off under duopoly. In particular, when the DIC is slack under both monopoly and duopoly (the first subcase), θ_l is rather small relative to k, and the market for the l type is not fully covered even under duopoly. The h type is clearly better off under duopoly, as the full-information rent for type h is higher. The l type gets the same rent under both monopoly and duopoly. When the DIC binds under both monopoly and duopoly (the second subcase), k is rather small and the market for type l is not fully covered even under duopoly, meaning that θ_l is small as well. Given that k is small, entry leads to fierce competition for type-h consumers, who are better off under duopoly. In the mean time, fierce competition for type h relaxes the IC constraint in the vertical dimension. This means that under duopoly firms will offer higher quality and utility to type-l consumers in order to penetrate more into the market of type l. Thus type-l consumers are better off as well under duopoly.

However, in the third subcase that the DIC binds under monopoly but is slack under duopoly (k is close to $1/2$ and θ_l is rather large), entry can either increase or decrease consumer surplus. In this scenario, although the full-information rent for type h is higher under duopoly due to competition, the binding DIC under monopoly tends to increase the rent for type h. In the following example, the aggregate consumer surplus is lower under duopoly. Suppose $k = 0.493$, $\alpha = 0.01$, $\theta_l = 0.86$, and $\underline{q} = 0.8$. Under duopoly, the two contracts are given by $(q_h^d, u_h^d) = (1, 0.2535)$ and $(q_l^d, u_l^d) = (0.86, 0.1233)$. Under monopoly, the two contracts are given by $(q_h^m, u_h^m) = (1, 0.3038)$ and $(q_l^m, u_l^m) = (0.858, 0.1837)$. The aggregate consumer surplus is 0.0315 under duopoly, which is lower than that under monopoly 0.0348. In this example, since the proportion of type h is very small, what matters is type l. Given the parameter values, the competition effect for type l caused by entry is dominated by the market share effect (the cutoff is $\theta_l^2/2$). As a result, type l is better off under monopoly. Moreover, type h is better off under monopoly, as they can enjoy a higher utility by mimicking type l. For $\theta_l = 0.86$, and $\underline{q} = 0.8$, we also solve a numerical example systematically by varying $k \in (0, 1/2]$ and α. Our result indicates that the aggregate consumer surplus is higher under monopoly only in the neighborhood of k being close to $1/2$ and α being close to 0.

When $k \geq 1/2$, the previous results show that entry might lead to product line pruning. In this case, entry makes type-l consumers worse off. However, some of the type-h consumers might be better off as they can save some transportation costs by buying from the entrant. The aggregate consumer welfare can go either way, as the following example illustrates. Suppose $\underline{q} = 0.282$, $\theta_l = 0.29$,

$\alpha = 0.9$, and $k \in [2/3, 0.8]$. Under monopoly two contracts are offered, while under duopoly only the high contract is offered.[30] The aggregate consumer welfare is higher under monopoly when $k \in [2/3, 0.707]$, while it is higher under duopoly when $k \in [0.707, 0.8]$. The underlying reason is that type-h consumers may save more transportation costs under duopoly when k is relatively large.

For duopoly equilibrium, we are also interested in how the menu of contracts offered changes as the degree of horizontal differentiation (k) decreases.

Proposition 4. (i) If $k \in (0, \frac{2}{3}]$, then in the duopoly equilibrium, a decrease in k can only lead to fighting brands; (ii) if $k \in (\frac{2}{3}, 1]$, then in the duopoly equilibrium a decrease in k may result in either fighting brands or product line pruning.

These results again come from the combined effects of entry. When the initial level of k is low, a further decrease in k leads to fiercer competition for the high type, which increases the rent to the high type. Moreover, a decrease in k means that it becomes easier to penetrate into the market for the low type, which makes that market more important. These two effects work in the same direction, relaxing the DIC and making it potentially profitable to introduce low contracts. When the initial level of k is high, the competition effect is absent, but a decrease in k reduces the rent to the high type. This tends to make the DIC constraint more binding. On the other hand, a reduced k implies that the market for the low type becomes more profitable or important. If the first effect dominates, entry will lead to the removal of low contracts. If the second effect dominates, however, entry will lead to the addition of low contracts.

As shown in Stole [1995] and Yang and Ye [2008], a decrease in k under duopoly is equivalent to an increase in n (the number of firms) in an n-firm Salop circular city model, where products evenly split the unit-length circle.[31] In this n-firm model, Proposition 4 implies that whether entry leads to fighting brands or product line pruning depends on the initial degree of competition. When initial competition is fierce (k is small or the initial n is large), then further entry can only lead to fighting brands and a decrease in quality distortion. On the other hand, when initial competition is weak (k is large or the initial n is small), then further entry can lead to either fighting brands or product line pruning. Our results are consistent with the empirical findings of Seim and Viard [2011], who study how entry into local cellular phone market affects the number of calling plans offered by each incumbent firms.[32] When the initial number of firms is small in a local market, entry reduces the number of calling plans offered by incumbents. However, when the initial number of firms is large, incumbent firms respond to entry by increasing the number of calling plans.

[30]The conditions in part (ii) of Lemma 3 are satisfied.

[31]As standard in the Salop circular city model, an increase in n in the n-firm model is equivalent to a decrease in k in the duopoly model, as competition may exist only between two adjacent firms.

[32]Other empirical analysis of the early US cellular phone industry can be found in, e.g., Miravete and Röller [2004].

III. ASYMMETRIC FIRMS

In this section, we study the situation in which the entrant is technologically inferior to the incumbent. More specifically, we assume that the quality range that the entrant is able to produce is $[\underline{q}, \overline{q}]$, where $\overline{q} < 1$. This means that the maximum quality that the entrant is capable of producing is lower than that of the incumbent. We are interested in how a change in $\overline{q}$ affects the incumbent's response to entry. To simplify our analysis, we assume that $\overline{q} > \theta_l$. This assumption implies that, for the entrant, the first-best quality for type h is always strictly higher than that for type l. As a result, Lemma 1 still applies, which means that the entrant will either offer two contracts or offer one contract targeting at type h only.

In terms of the menu of contracts offered, there are four possible outcomes: both firms offer high contracts only, both firms offer two contracts, the incumbent offers two contracts while the entrant offers only a high contract, and the incumbent offers only a high contract while the entrant offers two contracts. The case where the entrant only offers a low contract cannot occur, because high types not covered by the incumbent would take the entrant's low contract, leading to pooling,[33] which is not possible.

Two main results emerge when firms are asymmetric. First, in equilibrium the incumbent offers weakly more contracts than the entrant does. Second, when k is small enough, as $\overline{q}$ becomes lower, fighting brands become less likely and product pruning becomes more likely.

We first specify the first-best solution under full information. We use superscripts I and E to denote the incumbent and entrant, respectively, and that u_h^I and u_h^E are the utilities offered to type h by the incumbent and entrant, respectively. Note that $q_h^I = 1$ and $q_h^E = \overline{q}$. We need to discuss two cases. First, when $u_h^I + u_h^E \leq k/2$ (the market is not fully covered), the problem can be written as follows:

$$\mathrm{I} : \max_{u_h^I} \alpha \frac{u_h^I}{k} \left(\frac{1}{2} - u_h^I \right); \;\; \mathrm{E} : \max_{u_h^E} \alpha \frac{u_h^E}{k} \left(\overline{q} - \frac{1}{2}\overline{q}^2 - u_h^E \right)$$

Second, when $u_h^I + u_h^E \geq k/2$ (the market is fully covered), the problem becomes:

$$\mathrm{I} : \max_{u_h^I} \alpha \left(\frac{1}{4} + \frac{u_h^I - u_h^E}{2k} \right) \left(\frac{1}{2} - u_h^I \right); \;\; \mathrm{E} : \max_{u_h^E} \alpha \left(\frac{1}{4} + \frac{u_h^E - u_h^I}{2k} \right) \left(\overline{q} - \frac{1}{2}\overline{q}^2 - u_h^E \right)$$

The equilibrium under full information is given below:

$$(u_h^I, u_h^E) = \begin{cases} \left(\frac{1-k}{2} - \frac{1}{6}(1-\overline{q})^2, \frac{1-k}{2} - \frac{1}{3}(1-\overline{q})^2 \right) & \text{if} \quad k \leq \frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2]; \\ \left(\frac{k}{2} \frac{1}{2-(1-\overline{q})^2}, \frac{k}{2} \frac{1-(1-\overline{q})^2}{2-(1-\overline{q})^2} \right) & \text{if} \quad \frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2] < k \leq 1 - \frac{1}{2}(1-\overline{q})^2; \\ \left(\frac{1}{4}, \frac{\overline{q}}{2} - \frac{\overline{q}^2}{4} \right) & \text{if} \quad k > 1 - \frac{1}{2}(1-\overline{q})^2. \end{cases}$$

[33] The only situation where the entrant only offers low contracts is when the incumbent covers all high types, but this could be included in our previous four cases.

Naturally, since the maximum quality of the entrant is lower than that of the incumbent, the incumbent offers a higher rent to h type and has a bigger market share. When $\overline{q} = 1$, the equilibrium characterized above is consistent with the previously derived u_h^D, the equilibrium with symmetric firms. To ensure interaction between the incumbent and the entrant, we assume that $k \leq 1 - \frac{1}{2}(1-\overline{q})^2$. Note that when $k \in (\frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2], 1 - \frac{1}{2}(1-\overline{q})^2]$, although the market is fully covered, there is no effective competition for the high type.

The full-information low contracts are the same as those in the symmetric case (as the competition between low contracts remains symmetric).

Now we turn our attention to the case of private information. We first identify a sufficient condition under which both firms offer two contracts in the duopoly equilibrium.

Lemma 5. If $k < \frac{1}{2} - \frac{1}{3}(1-\overline{q})^2 \equiv \widehat{k}_1(\overline{q})$, in the duopoly equilibrium the incumbent offers both h and l contracts. If $k < \frac{1}{2} - \frac{2}{3}(1-\overline{q})^2 \equiv \widehat{k}_2(\overline{q})$, in the duopoly equilibrium both firms offer h and l contracts.

Recall that in the duopoly equilibrium with symmetric firms, the incumbent will offer two contracts if $k \leq 1/2$. Note that $\widehat{k}_1(\overline{q}) < 1/2$. Moreover, $\widehat{k}_1(\overline{q})$ is increasing in $\overline{q}$. This suggests that when the maximum quality of the entrant becomes lower, the condition under which the incumbent will offer two contracts becomes more stringent. In later examples, we will show that when $\widehat{k}_1(\overline{q}) < k < 1/2$, the incumbent might offer h contract only with asymmetric firms. Under private information we use superscripts i and e to denote the incumbent and entrant, respectively.

Proposition 5. In the duopoly equilibrium, (i) if both firms offer h and l contracts, then the quality distortion of the incumbent is smaller, $q_l^i \geq q_l^e$; (ii) if the entrant offers h and l contracts, then the incumbent must also offer h and l contracts.

Part (i) of Proposition 5 tells us that, when both firms offer two contracts, the low quality of the incumbent must be higher than that of the entrant. This result comes from the fact that the entrant is technologically inferior. In equilibrium the incumbent will offer higher rent to the high type than the entrant does. This means that the DIC is more relaxed for the incumbent than for the entrant. If both firms offer two contracts, the quality distortion for the incumbent's low contract must be smaller. For the same reason, if the entrant has an incentive to offer the low contract, then the incumbent must offer two contracts as well, leading to part (ii) of Proposition 5. Note that the result $q_l^i \geq q_l^e$ is consistent with Johnson and Myatt [2003], who show that the incumbent will never offer products that are of quality lower than that of the entrant's lowest-quality product.

Part (ii) of Proposition 5 rules out the equilibrium where the incumbent offers only an h contract and the entrant offers both contracts. However, examples can be constructed in which the incumbent offers two contracts and the entrant offers only an h contract.[34]

[34] Consider the following case: $k = 1/2$, $\theta_l = 0.25$, $\underline{q} = 0.2$, $\overline{q} = 0.3$ and $\alpha = 3/4$. In this case, $u_h^I = 0.168$, $u_h^E = 0.067$,

Proposition 6. Suppose $k \le \frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2]$. Let $\overline{q}' > \overline{q}$. If in the duopoly equilibrium with $\overline{q}$ the incumbent offers h and l contracts, then in the duopoly equilibrium with $\overline{q}'$ the incumbent must offer h and l contracts as well. Moreover, $q_l^{i\prime} \ge q_l^i$.

The intuition for Proposition 6 is similar to what is offered in the previous section. When k is so small that there is effective competition for the high type, as $\overline{q}$ becomes higher, the competition for the high type becomes more fierce. As a result, the competition effect (of entry) dominates the market share effect on rent provision, increasing the incentive for the incumbent to introduce a low contract. If the low contract has been originally offered, the quality for the low contract will be less distorted.

An implication from Proposition 6 is that when k is sufficiently small (so that there is effective competition for the high type), as $\overline{q}$ becomes lower, fighting brands become less likely and product pruning becomes more likely. The following example shows that a decrease in $\overline{q}$ can make product pruning more likely. Let $k = 0.5$, $\theta_l = 0.38$, $\underline{q} = 0.35$, and $\alpha = 3/4$. By Lemma 2, in monopoly the incumbent offers two contracts. If $\overline{q} = 1$, by Lemma 3, under the duopoly equilibrium the incumbent offers two contracts as well. Now suppose $\overline{q} = 0.4$. It can be verified that in the duopoly equilibrium both firms only offer an h contract, with $u_h^i = u_h^I = 0.19$ and $u_h^e = u_h^E = 0.13$.[35] This example thus suggests that when the maximum quality of the entrant becomes sufficiently low, entry leads to product line pruning.

Note that a similar result does not hold when k is relatively large such that there is no effective competition for the high type. In that case, when $\overline{q}$ becomes lower there are two opposing effects. First, the incumbent tends to reduce its rent to the high type. This can be seen from the fact that u_h^I is decreasing in $\overline{q}$ when $k > \frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2]$. This effect tends to make the DIC more stringent. On the other hand, an increase in $\overline{q}$ reduces the incumbent's market share for the high type. This makes the low type relatively more important and tends to relax the DIC. Whether the incumbent' DIC is slackened or not as $\overline{q}$ increases depends on which effect dominates.

As in Subsection II.iii, we can also study the impact of entry on consumer surplus when k is relatively small. It turns out that Proposition 3 continues to hold even when the entrant is

and $(1-\theta_l)\underline{q} = 0.15$. Since $u_h^I > (1-\theta_l)\underline{q}$, the incumbent must offer two contracts under private information. Now we claim that in equilibrium the incumbent offers two contracts with $u_h^i = u_h^I$ and the entrant only offers an h contract with $u_h^e = u_h^E$. From the incumbent's FOC (16) (in the appendix), it is not difficult to see that the LHS (excluding μ^i) is negative if we impose $u_l^i = 0$. Therefore, $u_h^i = u_h^I$. Now inspect the entrant's FOC (18) (also in the appendix) with $u_h^e = 0.15$ and $u_h^i = 0.168$. It can be verified that the LHS (excluding μ^e) is negative if we impose $u_l^e = 0$, which means that it is not profitable for the entrant to offer the low contract. Thus in this example we show that the incumbent will offer two contracts while the entrant only offers one contract.

[35] To show this, first note that $(1-\theta_l)\underline{q} = 0.217 > u_h^I$. Therefore, if two contracts are offered, the DIC must bind for both the incumbent and entrant. We first check the incumbent's incentive given $u_h^e = 0.13$. If the incumbent offers two contracts, then $u_h^i \ge 0.217$. In that case it can be easily verified that the LHS of (16) is less than zero. Thus no u_l^i satisfies (16), and the incumbent has no incentive to offer a low contract. Now consider the entrant's incentive given $u_h^i = 0.19$. When the entrant offers two contracts, then $u_h^e \ge 0.217$, in which case it can be verified that the LHS of (18) is less than zero. Therefore, no u_l^i satisfies (16), and the entrant has no incentive to offer a low contract either.

technologically inferior. Formally we have the following proposition.

Proposition 7. Suppose $k < \widehat{k}_1(\overline{q}) \equiv \frac{1}{2} - \frac{1}{3}(1-\overline{q})^2$ so that the incumbent always offers h and l contracts in duopoly. Compared to the monopoly case, type-h consumers are strictly better off and type-l consumers are better off under duopoly if (i) entry leads to fighting brands, or (ii) two contracts are offered and the incumbent's DIC is either slack or binding under both monopoly and duopoly.

Therefore, competition from a technologically inferior entrant will also improve consumers' welfare. Interestingly, Proposition 7 holds regardless of the number of contracts the entrant offers, suggesting that the improvement in welfare is mainly driven by the competition for type-h consumers.

IV. THREE-TYPE MODEL WITH PARTIAL POOLING

In this section we consider a model with three vertical types. Suppose in the vertical dimension consumers have three types: θ_h, θ_m, and θ_l, where $\theta_h = 1 > \theta_m > \theta_l$. The proportions of types are α_h, α_m, and α_l, respectively ($\alpha_h + \alpha_m + \alpha_l = 1$). All the other assumptions are the same as in the base model.

As in the two-type model, in the three-type model entry might lead to fighting brands or product pruning. Since three contracts can be potentially offered, entry may lead to the introduction (or removal) of a middle quality product, a low quality product, or both (contracts). Although the pattern can be more complicated, they are qualitatively the same as fighting brands and product line pruning in the two-type model, since expansion or contraction of the product lines (contracts) only occurs at the low end. With three types, since pooling of the middle and the low types becomes a possibility, the expansion or contraction of the set of contracts offered might occur for the middle product (contract), such as Garmin's response to entry by releasing the Quest series or Microsoft's response to entry by releasing Office 2003 mentioned in the introduction. This new feature will be the focus of this section.

IV(i). *Entry Leads to The Introduction of A Middle Contract*

Monopoly. We start with the analysis of monopoly. Under monopoly, the full information solution is as follows: $q_i^{fb} = \theta_i$ and

$$u_i^{fb} = \begin{cases} \frac{\theta_i^2}{4} & \text{if } k \in (\frac{\theta_i^2}{2}, 1] \\ \frac{k}{2} & \text{if } k \in (0, \frac{\theta_i^2}{2}] \end{cases} .$$

Under private information, we again have $q_h = 1$. Similarly to the argument in Lemma 1, we can show that type h is never pooled with other two types. Overall, we have four cases to consider: only an h contract is offered, only h and m contracts are offered and type l is excluded, three contracts are offered (full separating), and two contracts are offered, with types m and l pooling at the low contract (partial pooling). We are interested in the last case, as it is qualitatively different from the two-type base model. The relevant ICs are: $u_h - u_m \geq (1-\theta_m)q_m$ (DIC$_{hm}$), $u_m - u_l \geq (\theta_m - \theta_l)q_l$ (DIC$_{ml}$), $u_h - u_l \geq (1-\theta_l)q_l$ (DIC$_{hl}$), and $u_m - u_l \leq (\theta_m - \theta_l)q_m$ (UIC$_{lm}$). Note that when $q_m \geq q_l$, then DIC$_{hl}$ is redundant.

Lemma 6. Suppose $k \leq 1/2$, $\theta_m < 3\theta_l$, and $\frac{k}{2} < \frac{\theta_m^2}{4} + (1-\theta_m)\theta_m$. Provided that the following restrictions regarding the type distribution hold, under monopoly the optimal menu of contracts exhibits partial pooling: two contracts are offered, with the high contract targeting at type h, and types m and l pooled at the low contract with $q_l = \underline{q}$:

$$\text{(7)} \qquad \begin{cases} \frac{\alpha_h}{\alpha_m} & \geq \max\left\{\frac{\theta_m - \underline{q}}{1-\theta_m}, \frac{2}{k}\left(\theta_m \underline{q} - \frac{\underline{q}^2}{2}\right)\right\}, \\ \alpha_h\left(1-\underline{q}\right) & \geq \frac{\alpha_m}{k}(\theta_m - \theta_l)\theta_l^2 + \theta_l - \underline{q}, \end{cases}$$

and

$$\text{(8)} \qquad \left(\theta_l \underline{q} - \frac{1}{2}\underline{q}^2 - \frac{\alpha_h}{1-\alpha_h}\frac{k}{2}\right)^2 > \frac{2k\alpha_h}{1-\alpha_h}\left((1-\theta_l)\underline{q} - \frac{k}{2}\right).$$

Note that condition (8) holds for sure when $\underline{q}$ is small enough such that $(1-\theta_l)\underline{q} \leq k/2$; otherwise the type distribution needs to be more carefully selected so that partial pooling will occur. Specifically, by (7), the proportion of the middle type (α_m) needs to be small compared to that of the high type (α_h). The proportion of the high type cannot be too large either given condition (8). Intuitively, since α_m is small, it is too costly to price discriminate between the middle and low types. Meanwhile, when α_h is not too large or $\underline{q}$ is sufficiently small, offering a second contract targeting at both the middle and low types would be profitable. Therefore partial pooling arises in monopoly equilibrium.[36]

Duopoly. Under full information, the duopoly equilibrium contracts take the following form: $q_i^D = \theta_i$, and

$$u_i^D = \begin{cases} \frac{\theta_i^2}{4} & \text{if } k \in [\theta_i^2, 1] \\ \frac{k}{4} & \text{if } k \in [\frac{2}{3}\theta_i^2, \theta_i^2) \\ \frac{\theta_i^2 - k}{2} & \text{if } k \in (0, \frac{2}{3}\theta_i^2) \end{cases}.$$

We are interested in the case $k \leq 1/2$, and will focus on the duopoly equilibrium in which the menu of contracts is fully separating. The following lemma summarizes the results.

Lemma 7. Suppose $k \leq 1/2$. (i) If $k \leq \frac{2}{3}\theta_l^2$, then in the duopoly equilibrium the full-information solution is feasible: each firm offers three contracts without quality distortion. (ii) If $k \in [\frac{2}{3}\theta_m^2, \theta_m^2)$,

[36] A concrete example will be introduced after the analysis of duopoly.

$2-3k \geq 4(1-\theta_m)\theta_m$, and $(\theta_m-\theta_l)\underline{q} < \frac{k}{4}$, then in the duopoly equilibrium each firm offers three contracts, with $q_m^d = \theta_m$.

Combining Lemma 6 and Lemma 7, we have the following result.

Proposition 8. Let $k \leq 1/2$. If the parameter values are such that all the conditions in Lemma 6 are satisfied, and either $k \leq \frac{2}{3}\theta_l^2$ or the conditions in part (ii) of Lemma 7 are satisfied, then under monopoly two contracts are offered, with the low and middle types pooled at $\underline{q}$, while in the duopoly equilibrium each firm offers three contracts (fully separating).

It is easy to see that there are parameter values such that both conditions in Lemma 6 and Lemma 7 are satisfied. This is because the conditions in Lemma 7 have nothing to do with the distribution of types. So we can choose α's freely to satisfy the conditions in Lemma 6.[37]

Proposition 8 illustrates that entry can expand the incumbent's menu of contracts by converting a partial pooling equilibrium to a fully separating equilibrium. We should emphasize that this scenario is different from fighting brands. Recall that in the case of fighting brands, entry leads to an introduction of a low quality good (contract). However, in the scenario described by Proposition 8, the low quality good (contract) is offered under monopoly, and entry leads to an introduction of a middle quality good (contract).[38] Our analysis thus suggests a new pattern of product line expansion that is different from fighting brands. Such a pattern is consistent with, for example, a finding in Seim and Viard [2011] that with more entry, firms may spread their calling plans more evenly over the usage spectrum.

The driving force behind Proposition 8 is again the interaction between horizontal competition and vertical screening. When k is small, competition for high types after entry leads to higher rent to high types. This relaxes the sorting constraint and makes informational rent consideration along the vertical dimension less important. As a result, the incumbent has less incentive to exclude low types or to pool the low types.

IV(ii). *Entry Leads to The Removal of A Middle Contract*

In this subsection we provide an analysis of the opposite case, which exhibits fully separating under monopoly but partial pooling under duopoly. In effect we will identify conditions under which entry will lead to fewer contracts offered. We restrict attention to the case that $\frac{1}{2} < k < \frac{2}{3}$.

Monopoly. Under monopoly, $u_h^{fb} = \frac{1}{4}$, $u_m^{fb} = \frac{\theta_m^2}{4}$, and $u_l^{fb} = \frac{\theta_l^2}{4}$. When $u_m^{fb} + (1-\theta_m)\theta_m \leq u_h^{fb}$, the DIC_{hm} is slack under full information. When $u_m^{fb} > (\theta_m - \theta_l)\underline{q}$, which is always valid since $\theta_l > \underline{q}$,

[37] The following parameter values satisfy all the conditions in Lemma 6 and the conditions in part (ii) of Lemma 7: $k = 0.3$, $\theta_m = 0.6$, $\theta_l = 0.5$, $\underline{q} = 0.4$, $\alpha_h = 0.19$, $\alpha_m = 0.1$, $\alpha_l = 0.71$.

[38] Note that in the duopoly equilibrium the middle quality is strictly higher and the low quality is weakly higher than the low quality under monopoly. In this sense, entry leads to the addition of the middle contract (quality) instead of the low contract (quality).

it is always profitable to offer a low contract. Overall, we conclude that if $\theta_m \leq \frac{1}{3}$, it is optimal to offer three separate contracts under monopoly.

Duopoly. Under duopoly, $u_h^D = \frac{1-k}{2}$, $u_m^D = \frac{\theta_m^2}{4}$, and $u_l^D = \frac{\theta_l^2}{4}$.[39] When $u_m^D + (1-\theta_m)\theta_m > u_h^D$, DIC$_{hm}$ binds; when $u_m^D < u_l^D + (\theta_m - \theta_l)\theta_l$, the DIC$_{ml}$ also binds. Combining these two conditions, we have that if $(\theta_m - \frac{2}{3})^2 < \frac{6k-2}{9}$ and $\theta_m < 3\theta_l$, both DICs bind.

Proposition 9. When $\frac{1}{2} < k < \frac{2}{3}$, $\theta_m \leq \frac{1}{3}$, and $\theta_m < 3\theta_l$, if conditions (33), (34), (38), and (39) (listed in the appendix) hold, then the firm will offer three separate contracts under monopoly while partial pooling of middle and low types would take place under duopoly.

Proposition 9 shows that if k is relatively large ($k > 1/2$), α_m is sufficiently small (condition (39)), and type m and type l are fairly close to each other ($\theta_m < 3\theta_l$) but rather far away from type h ($\theta_m \leq 1/3$), then the incumbent monopolist responds to entry by removing the middle contract. The rough intuition is as follows. A relatively large k makes the monopolist willing to give the high type a high rent in order to penetrate into its market. This means that the IC constraints in the vertical dimension are relaxed, leading to a fully separating equilibrium under monopoly. On the other hand, a relatively large k under duopoly leads to a lower rent to the high type, which makes the IC constraints in the vertical dimension more stringent. Given that α_m is sufficiently small, and type m and type l are fairly close to each other, entry makes it too costly for firms to offer a separate contract to the middle type, thus the middle contract of the incumbent is removed. We can easily choose α_h to satisfy the conditions in the proposition.[40]

The practice of removing some middle contracts (or middle-ranged quality product line) in response to entry is very common. For example, following the entry of Toyota into North American market, Buick reduced the number of its mid-size models offered from two (Special and Skylark) to one (Skylark) in 1970.[41] Ford also reduced its mid-size model line from two models, Fairlane and Torino, to just one model, Torino.[42] In response to the competition from more and more downsized vehicles, in 1980 Ford cut its largest mid-size car, LTD II, from its product line.[43]

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[39] Given that $\theta_m \leq \frac{1}{3}$ and $k \geq 1/2$, $\theta_m^2 < k$. This implies that both the low and the middle types will not be fully covered in the horizontal dimension.

[40] One such choice is the following combination of parameters: $k = 0.62$, $\theta_h = 1$, $\theta_m = 0.33$, $\theta_l = 0.32$, $\underline{q} = 0.31$, and $\alpha_h = 0.62$, $\alpha_m = 0.10$, $\alpha_l = 0.28$.

[41] J. 'Kelly' Flory, Jr., *American Cars, 1960-1972: Every Model, Year by Year,* (Jefferson, North Carolina: McFarland & Company, Inc., Publishers, 2004), pages 641 and 718.

[42] J. 'Kelly' Flory, Jr., *American Cars, 1960-1972: Every Model, Year by Year* (Jefferson, North Carolina: McFarland & Company, Inc., Publishers, 2004), pages 752 and 830.

[43] 'Plenty of big four car activity scheduled for this fall: autos,' *Chicago Tribune*, April 29, 1979.

V. CONCLUSION

In this paper, we study the effect of entry on the variety of contracts offered in a standard Salop circular city model, with both horizontally and vertically differentiated products. In our base model with two types of consumers, we show that when the extent of horizontal differentiation is small or competition is strong, entry typically leads to the introduction of the lower end product; when the extent of horizontal differentiation is large or competition is weak, however, entry typically leads to the removal of the lower end product. The driving force behind our model is the interaction between horizontal differentiation (competition) and vertical screening. We thus offer a new explanation for fighting brands and product line pruning, which is different from that offered by Johnson and Myatt [2003].

The extension to asymmetric firms and three types of consumers further confirms the general insights obtained from our base model. In particular, our analysis of three types of consumers reveals an interesting pattern between fully separating and partial pooling equilibria and offers an explanation for why incumbent firms also adjust the middle range of a product line in response to competition. Note that this is also different from the analysis of Johnson and Myatt, who show that the changes in product variety always happen at the low-end. Our result thus points to more subtle effects of increased competition on the variety of contracts offered. Our results are potentially testable.

From both theoretical and practical points of view, it would be desirable to work out a more general model allowing for any finite number of types. However, doing so presents some technical difficulty, as the incentive comparability constraints along the vertical dimension become quite complicated. Fully working out a general n-type model will be challenging, but we believe that the main insights obtained from our current analysis should be quite robust. Intuitively, in the n-type model there could be exclusion of low types and bunching. In particular, all the types below some threshold type might be excluded. If bunching occurs, it must be the case that all the types between the lowest type covered and some cutoff type choose the same contract (at the minimum quality level), and the higher types are fully separated. In this setting, entry could affect the set of low types that are excluded, possibly leading to the introduction of multiple fighting brands or the pruning of multiple products. Moreover, entry could also affect the set of bunching types, potentially leading to the addition or removal of middle contracts. We believe that the driving force should again be the interaction between horizontal differentiation/competition and screening in the vertical dimension. If the degree of horizontal differentiation is low, intense competition upon entry will relax the screening condition in the vertical dimension, which could reduce the set of excluded low types (fighting brands) and reduce the possibility of bunching (more middle range products). On the other hand, if the degree of horizontal differentiation is high, then reduced market share due to entry will tighten the screening condition in the vertical dimension, which could increase the set of excluded low types (product pruning) and increase the possibility of bunching (fewer middle range products). Given the technical challenge, the generalization to the n-type model is left for future research.

APPENDIX

Proof of Lemma 1: Let t be the transfer under the single contract. First consider the case $q \in [\underline{q}, 1)$. Suppose the monopolist introduces another contract targeting at type h: $q_h = 1$ and $t_h = t + (1-q)$. By construction, it can be verified that $u_h(q,t) = q - t = q_h - t_h = u_h(q_h, t_h)$. Thus type h will accept contract h and its market coverage does not change. On the other hand, $u_l(q,t) = \theta_l q - t > \theta_l q_h - t_h = u_l(q_h, t_h)$. Hence type l will still buy the original contract and the firm's profit from type l does not change. However, the profit per consumer from type h increases under contract h: under the original contract the profit margin is $t - \frac{1}{2}q^2$, and under contract h it becomes $t + (1-q) - \frac{1}{2}$, which is strictly greater than $t - \frac{1}{2}q^2$ since $q < 1$. Because the market share for type h remains the same, the introduction of contract h strictly increases the firm's profit.

Next consider the case $q = 1$. Suppose the monopolist introduces another contract targeting at type l: $q_l = \theta_l$ and $t_l = t - \theta_l(1-\theta_l)$. By construction, type l is indifferent between the original contract and contract l. Thus type l selects the l contract and the market share for type l does not change. It can be verified that type h prefers the original contract: $(1-t) - (\theta_l - t_l) = (1-\theta_l)^2 > 0$. Thus type h will stick to the old contract and the profit from type h agents does not change. However, the profit margin from type l becomes higher:

$$\left(t_l - \frac{1}{2}\theta_l^2\right) - \left(t - \frac{1}{2}\right) = \frac{1}{2}\theta_l^2 - \theta_l + \frac{1}{2} = \frac{1}{2}(1-\theta_l)^2 > 0.$$

Therefore, the introduction of contract l strictly raises the firm's profit. ■

Solution to the monopoly programming problem when two contracts are offered.
The (full-information) unconstrained solution is $u_h = u_h^{fb}$, $q_l = \theta_l$ and

$$u_l^{fb} = \begin{cases} \frac{\theta_l^2}{4} & \text{if } k \in (\frac{\theta_l^2}{2}, 1] \\ \frac{k}{2} & \text{if } k \in (0, \frac{\theta_l^2}{2}] \end{cases}.$$

We now turn our attention to the case in which types are private information. The unconstrained solution is not feasible if $k \leq \frac{\theta_l^2}{2}$. It is not feasible either when

$$\begin{array}{ll} \frac{k}{2} - \frac{1}{4}\theta_l^2 < (1-\theta_l)\theta_l & \text{if } k \in (\frac{\theta_l^2}{2}, \frac{1}{2}] \\ \frac{1}{4}(1-\theta_l^2) < (1-\theta_l)\theta_l \Leftrightarrow \theta_l > \frac{1}{3} & \text{if } k \in (\frac{1}{2}, 1] \end{array}.$$

Combining the above conditions, the unconstrained solution is not feasible if $k < 2\theta_l - \frac{3}{2}\theta_l^2$ in the case $k \leq 1/2$, and $\theta_l > \frac{1}{3}$ in the case $k \in (\frac{1}{2}, 1]$.

Suppose $k \leq \frac{1}{2}$ and $(1-\theta_l)\underline{q} \geq \frac{k}{2}$. Note that this implies that the unconstrained solution is not feasible, so the DIC must bind.[44] By DIC, this implies that $u_h \geq k/2$ if two contracts are offered. The programming problem becomes:[45]

$$\max_{(u_h, q_l, u_l)} \frac{\alpha}{2}\left[\frac{1}{2} - u_l - (1-\theta_l)q_l\right] + (1-\alpha)\frac{u_l}{k}\left[\theta_l q_l - \frac{1}{2}q_l^2 - u_l\right]$$
$$\text{subject to: } u_l \geq 0, \quad q_l \geq \underline{q}$$

Let the Lagrangian multiplier for the first and second constraints be μ and λ, respectively. The first order conditions are

$$-\frac{\alpha}{2} + \frac{1-\alpha}{k}\left(\theta_l q_l - \frac{1}{2}q_l^2 - 2u_l\right) + \mu = 0;\ \mu \geq 0,\ \mu = 0 \text{ if } u_l > 0; \tag{9}$$

$$-\frac{\alpha}{2}(1-\theta_l) + \frac{1-\alpha}{k}u_l(\theta_l - q_l) + \lambda = 0;\ \lambda \geq 0,\ \lambda = 0 \text{ if } q_l > \underline{q}. \tag{10}$$

Proof of Lemma 2: Suppose two contracts are offered. The first order conditions are given as above, and conditions ((3)) and (4) then imply that only the h contract will be offered. To see this, note that condition (3) implies that the LHS of (10) (excluding λ) is negative. Thus $q_l = \underline{q}$ is binding. Now condition (4) implies that the LHS of (9) is negative. Therefore, $u_l = 0$ and the firm has no incentive to offer an l contract. ■

Details of duopoly equilibrium.

Under full information, the quality for type l is efficient, $q_l^D = \theta_l$, and the equilibrium utility for type l, u_l^D, takes the following form:

$$u_l^D = \begin{cases} \frac{\theta_l^2}{4} & \text{if } k \in [\theta_l^2, 1] \\ \frac{k}{4} & \text{if } k \in [\frac{2}{3}\theta_l^2, \theta_l^2) \\ \frac{\theta_l^2 - k}{2} & \text{if } k \in (0, \frac{2}{3}\theta_l^2) \end{cases}.$$

Competition occurs for type l consumers only when $k \in (0, \frac{2}{3}\theta_l^2)$. When $k \in [\frac{2}{3}\theta_l^2, \theta_l^2)$, although type l consumers are fully covered, there is no competition for type l consumers.

[44] This is because $(1-\theta_l)\underline{q} < (1-\theta_l)\theta_l < \theta_l - \frac{3}{4}\theta_l^2$.

[45] In writing the following programming problem, we implicitly assumed that $u_l \leq k/2$. Note that in the optimal solution $u_l \leq k/2$, since offering u_l more than $k/2$ will lead to a loss in profit.

The case that $k \in (\frac{1}{2}, \frac{2}{3}]$. There are two subcases. First when $k < 1 - 2(1-\theta_l)\underline{q}$, the full-information utility $u_h^D = \frac{1-k}{2}$. Since $\frac{1-k}{2} > (1-\theta_l)\underline{q}$, two contracts will be offered in equilibrium.

Second, when $k \geq 1 - 2(1-\theta_l)\underline{q}$, we have $(1-\theta_l)\underline{q} \geq \frac{1-k}{2}$ and offering two contracts means that the DIC must bind. Note that this condition implies that $k > \theta_l^2$. To see this, suppose $k \leq \theta_l^2$, then we have

$$(1-\theta_l)\underline{q} < (1-\theta_l)\theta_l < \frac{1-\theta_l^2}{2} \leq \frac{1-k}{2},$$

which contradicts the previous condition. When $k > \theta_l^2$, even under full information the market for type l is not fully covered. Hence under private information, the market for type l is not fully covered either. Given (q_i^2, u_i^2), firm 1's programming problem becomes

$$\max_{\{u_h^1, q_l^1, u_l^1\}} \alpha \left[\frac{1}{4} + \frac{u_l^1 + (1-\theta_l)q_l^1 - u_h^2}{2k}\right]\left[\frac{1}{2} - u_l^1 - (1-\theta_l)q_l^1\right] + (1-\alpha)\frac{u_l^1}{k}\left[\theta_l q_l^1 - \frac{1}{2}(q_l^1)^2 - u_l^1\right]$$

subject to: $u_l^1 \geq 0, \quad q_l^1 \geq \underline{q}$

Note that the above characterization is based on the condition that $1 - 2(1-\theta_l)\underline{q}$ does not exceed the upper bound of k in this case: $2/3$. If instead $1 - 2(1-\theta_l)\underline{q} > 2/3$, or equivalently $(1-\theta_l)\underline{q} < 1/6$, then by the fact that $k \leq \frac{2}{3}$, we have

$$\frac{1-k}{2} \geq \frac{1}{6} > (1-\theta_l)\underline{q},$$

which implies that two firms will surely offer two contracts in this case.

The case that $k \in \left(\frac{2}{3}, 1\right]$. In this case, recall that under full information, $u_h^D = \frac{k}{4}$, and although the market for type h is fully covered there is no competition for type h. If $(1-\theta_l)\underline{q} < \frac{k}{4}$, then in equilibrium two contracts must be offered, as offering some low contract will not violate the DIC.

Now suppose $(1-\theta_l)\underline{q} \geq \frac{k}{4}$. Then the DIC must bind if two contracts are offered. Moreover, $k \in (2/3, 1]$ implies that $u_l^D \leq k/4$. This means that in the duopoly equilibrium, $u_l^d \leq k/4$, i.e., there is no competition for type l. Therefore, the programming problem is the same as before.

Proof of Lemma 3: We only need to show conditions under which only h contracts are offered in the duopoly equilibrium, as the rest have been shown in the previous analysis. Suppose two contracts are offered for both firms. Given the programming problem listed in the text, we write down the first order conditions as follows. Let the Lagrangian multipliers of constraints $u_l^1 \geq 0$ and $q_l^1 \geq \underline{q}$ be μ_D and λ_D, respectively. The symmetric equilibrium is characterized by the following first order conditions:

$$(11)\ \alpha\left[\frac{1}{2k}(\frac{1}{2} - u_h^d) - \frac{1}{4}\right] + \frac{1-\alpha}{k}\left[\theta_l q_l^d - \frac{1}{2}(q_l^d)^2 - 2u_l^d\right] + \mu_D \quad = \quad 0;\ \mu_D \geq 0,\ \mu_D = 0 \text{ if } u_l^d > 0$$

$$(12) \qquad \alpha(1-\theta_l)\left[\frac{1}{2k}(\frac{1}{2}-u_h^d)-\frac{1}{4}\right]+\frac{1-\alpha}{k}u_l^d(\theta_l-q_l^d)+\lambda_D = 0;\ \lambda_D \geq 0,\ \lambda_D = 0 \text{ if } q_l^d > \underline{q}$$

Consider the first order condition (11) that characterizes the symmetric equilibrium. Note that in the LHS of condition (11), $\frac{1}{2k}(\frac{1}{2}-u_h^d)-\frac{1}{4} \leq \frac{1}{4k}-\frac{3}{8}$ since $u_h^d \geq u_h^D = \frac{k}{4}$. Given that $\theta_l^2 < \frac{\alpha}{1-\alpha}(\frac{3}{4}k-\frac{1}{2})$, we have

$$\begin{aligned}
\text{LHS of (11)} &\leq \alpha\left(\frac{1}{4k}-\frac{3}{8}\right)+\frac{1-\alpha}{k}\left(\theta_l q_l-\frac{1}{2}q_l^2-2u_l\right)\\
&\leq \alpha\left(\frac{1}{4k}-\frac{3}{8}\right)+\frac{1-\alpha}{2k}\theta_l^2\\
&< 0.
\end{aligned}$$

This implies that $\mu_D > 0$ and $u_l^d = 0$. Therefore, in equilibrium both firms must offer h contract alone, with $u_h^d = u_h^D = \frac{k}{4}$. Now consider conditions (5) and (6). Condition (5) ensures that the LHS of (12) is negative, thus $q_l^d = \underline{q}$. With $q_l^d = \underline{q}$, condition (6) ensures that the LHS of (11) is negative, thus $u_l^d = 0$. ■

Proof of Lemma 4: First we show that if the DIC does not bind (the full-information solution is feasible) under monopoly, then it does not bind under duopoly either. From the previous analysis, when $k \leq 1/2$, the DIC does not bind under monopoly if and only if

$$(13) \qquad \frac{k}{2}-\frac{\theta_l^2}{4} \geq (1-\theta_l)\theta_l.$$

On the other hand, when $k \leq 1/2$, the DIC does not bind under duopoly if and only if one of the following three conditions hold: (i) $k \in (0, \frac{2}{3}\theta_l^2)$,

$$(14) \qquad \text{(ii) } \frac{1-k}{2}-\frac{k}{4} \geq (1-\theta_l)\theta_l \text{ if } k \in [\frac{2}{3}\theta_l^2, \theta_l^2), \text{ and (iii) } \frac{1-k}{2}-\frac{\theta_l^2}{4} \geq (1-\theta_l)\theta_l \text{ if } k \in [\theta_l^2, 1].$$

Comparing (13) and (14), we see that if (13) is satisfied then (14) must be satisfied. This result implies that whenever the DIC is slack under duopoly, we have $q_l^m \leq q_l^d$.

What remains to be shown is that $q_l^m \leq q_l^d$ when the DIC binds under both monopoly and duopoly. If $q_l^m = \underline{q}$, then $q_l^d \geq q_l^m$ holds trivially. So we focus on the case that $q_l^m > \underline{q}$. Let the optimal solution under monopoly be (q_l^m, u_l^m). Suppose $(q_l^d, u_l^d) = (q_l^m, u_l^m)$. We will Compare the LHS of the first order conditions (9) and (11) with both μ and μ_D being 0. Since $u_h^d \leq 1/2$ (the maximum social surplus of the high type), the first term in (9) is strictly less than that in (11). Given that (9) holds, the LHS of (11) must be strictly positive when $(q_l^d, u_l^d) = (q_l^m, u_l^m)$. By the same procedure, we can show that the LHS of (12) is strictly positive when $(q_l^d, u_l^d) = (q_l^m, u_l^m)$. This means that each firm can increase its profit by offering $(q_l^d, u_l^d) > (q_l^m, u_l^m)$. This proves that $q_l^m \leq q_l^d$

when DIC binds under both monopoly and duopoly. ■

Proof of Proposition 3: We consider two cases in order.

Case (i): only the h contract is offered under monopoly while two contracts are offered under duopoly. As shown earlier, given $k \leq 1/2$, $u_h^d \geq u_h^D > u_h^{fb}$. Thus type-$h$ consumers are strictly better off under duopoly. Given that the low contract is offered under duopoly but is not offered under monopoly, type-l consumers are also strictly better off under duopoly. Since higher consumer utilities lead to more market coverage in the horizontal dimension, the aggregate consumer welfare is also strictly higher under duopoly.

Case (ii): two contracts are offered under both monopoly and duopoly. We consider two subcases.

Subcase a): the DIC is slack under both monopoly and duopoly. It immediately follows that type-h consumers are strictly better off under duopoly, as $u_h^D > u_h^{fb}$. By the previous results, that the DIC is slack under monopoly implies that $k \in (\frac{\theta_l^2}{2}, \frac{1}{2}]$ and $\frac{k}{2} - \frac{1}{4}\theta_l^2 \geq (1-\theta_l)\theta_l$. Since $k \leq 1/2$, the previous inequality implies that $\theta_l \leq 1/3$. Given that $\theta_l \leq 1/3$, the previous inequality further implies that $k \geq 2\theta_l - \frac{3}{2}\theta_l^2 > \theta_l^2$. Therefore, $u_l^{fb} = u_l^D = \frac{1}{4}\theta_l^2$, or type l has the same utility under both monopoly and duopoly.

Subcase b): the DIC binds under both monopoly and duopoly. By Lemma 4 (and its proof), $q_l^d > q_l^m$ and $u_l^d > u_l^m$. By the binding DIC, we have $u_h^d > u_h^m$. Thus both types of consumers are better off under duopoly. ■

Proof of Proposition 4: First consider the case $k \in (0, \frac{2}{3}]$. Note that the full-information u_h^D is increasing in k. Moreover, as k becomes smaller the DIC is less likely to bind. Thus for $k' < k$, if in the duopoly equilibrium two contracts are offered under k, then two contracts must be offered under k'.

To show part (ii), we find two examples in which a decrease in k leads to fighting brands and product line pruning, respectively. Suppose $k' \geq 2/3$. First, we provide an example in which product line pruning occurs. Consider the parameter space such that the following conditions hold: $\frac{k'}{4} \leq (1-\theta_l)\underline{q} < \frac{k}{4}$ and $\theta_l^2 < \frac{\alpha}{1-\alpha}(\frac{3}{4}k' - \frac{1}{2})$. Then by part (iii) of Lemma 3, in the duopoly equilibrium under k both contracts are offered and in the duopoly equilibrium under k' only the h contract is offered. Thus a decrease in k leads to product line pruning. Next, we provide an example in which the number of contracts increases. Consider the parameter space such that the following conditions hold: $(1-\theta_l)\underline{q} \geq \frac{k}{4}$, $\theta_l^2 \leq \frac{\alpha}{1-\alpha}(\frac{3}{4}k - \frac{1}{2})$, and

$$\theta_l \underline{q} - \frac{1}{2}\underline{q}^2 > \frac{\alpha}{1-\alpha}(\frac{k}{8} + \frac{k'}{4} - \frac{1}{4}). \tag{15}$$

By part (iii) of Lemma 3, the first two conditions ensure that in the duopoly equilibrium under k

only the h contract is offered. Now consider the LHS of (11) under k'. Condition (15) implies that when $q_l = \underline{q}$ and $u_h^d = \frac{k}{4} > \frac{k'}{4} = u_h^D(k')$, the LHS is strictly greater than 0. Therefore, under k' the equations (11) and (12) have a solution with $u_l^d > 0$. Hence two contracts must be offered in the duopoly equilibrium. Thus a decrease in k leads to fighting brands. ■

Proof of Lemma 5: First, note that $k < \widehat{k}_2(\overline{q})$ implies that $k < \frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2]$ (there is competition for type h). If $k < \widehat{k}_1(\overline{q})$, we have

$$u_h^I = \frac{1-k}{2} - \frac{1}{6}(1-\overline{q})^2 > \frac{1}{4} > (1-\theta_l)\underline{q}.$$

The above inequality means that the incumbent must offer two contracts. Now suppose $k < \widehat{k}_2(\overline{q})$, then we have

$$u_h^I > u_h^E = \frac{1-k}{2} - \frac{1}{3}(1-\overline{q})^2 > \frac{1}{4} > (1-\theta_l)\underline{q}.$$

The above inequalities imply that two firms must offer two contracts in the duopoly equilibrium. ■

Proof of Proposition 5: From the solutions to the full information benchmark, we observe that $u_h^I > u_h^E$ and $u_l^I = u_l^E$. Therefore, regarding whether the unconstrained solution is feasible, we have three possible scenarios. (1) The unconstrained solutions are feasible both for the incumbent and the entrant. In this case, we have $q_l^i = q_l^e = \theta_l$. (2) The unconstrained solution is feasible for the incumbent, but not feasible for the entrant. In this case, we have $q_l^i = \theta_l > q_l^e$. (3) The unconstrained solution is not feasible for both the incumbent and the entrant. In this case, the DIC must be binding for both firms. We inspect case (3) in more detail.

In case (3), we must have $k > \frac{2}{3}\theta_l^2$. This is because if $k \leq \frac{2}{3}\theta_l^2$, then $k < \frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2]$, and the unconstrained solution is feasible for the incumbent. Given that $k > \frac{2}{3}\theta_l^2$, there is no effective competition for the low type. As a result, the programming problem (P) becomes:

$$\text{I:} \quad \max_{u_l^i, q_l^i} \quad \alpha\left[\frac{1}{4} + \frac{u_l^i - (1-\theta_l)q_l^i - u_h^e}{2k}\right]\left(\frac{1}{2} - u_h^i\right) + (1-\alpha)\frac{u_l^i}{k}\left[\theta_l q_l^i - \frac{1}{2}(q_l^i)^2 - u_l^i\right]$$
$$\text{s.t.} \quad q_l^i \geq \underline{q},\ u_l^i \geq 0$$

and

$$\text{E:} \quad \max_{u_l^e, q_l^e} \quad \alpha\left[\frac{1}{4} + \frac{u_l^e - (1-\theta_l)q_l^e - u_h^i}{2k}\right]\left[\overline{q} - \frac{1}{2}\overline{q}^2 - u_h^e\right] + (1-\alpha)\frac{u_l^e}{k}\left[\theta_l q_l^e - \frac{1}{2}(q_l^e)^2 - u_l^e\right]$$
$$\text{s.t.} \quad \overline{q} > q_l^e \geq \underline{q},\ u_l^e \geq 0$$

The FOC's for I are:

$$\frac{\alpha}{2k}\left(\frac{1}{2} - u_l^i - (1-\theta_l)q_l^i\right) - \alpha\left[\frac{1}{4} + \frac{u_l^i + (1-\theta_l)q_l^i - u_h^e}{2k}\right]$$
$$+\frac{1-\alpha}{k}\left[\theta_l q_l^i - \frac{1}{2}(q_l^i)^2 - 2u_l^i\right] + \mu^i = 0 \tag{16}$$

$$\frac{\alpha(1-\theta_l)}{2k}\left(\frac{1}{2}-u_l^i-(1-\theta_l)q_l^i\right)-\alpha(1-\theta_l)\left[\frac{1}{4}+\frac{u_l^i+(1-\theta_l)q_l^i-u_h^e}{2k}\right]$$
$$+(1-\alpha)\frac{u_l^i}{k}(\theta_l-q_l^i)+\lambda^i=0 \tag{17}$$

and FOC's for E are:

$$\frac{\alpha}{2k}\left[\overline{q}-\frac{1}{2}(\overline{q})^2-u_l^e-(1-\theta_l)q_l^e\right]-\alpha\left[\frac{1}{4}+\frac{u_l^e+(1-\theta_l)q_l^e-u_h^i}{2k}\right]$$

$$+\frac{1-\alpha}{k}\left[\theta_l q_l^e-\frac{1}{2}(q_l^e)^2-u_l^e\right]-\frac{(1-\alpha)}{k}u_l^e+\mu^e=0 \tag{18}$$

$$\frac{\alpha(1-\theta_l)}{2k}\left[\overline{q}-\frac{1}{2}(\overline{q})^2-u_l^e-(1-\theta_l^e)q_l^e\right]-\alpha(1-\theta_l)\left[\frac{1}{4}+\frac{u_l^e+(1-\theta_l)q_l^e-u_h^i}{2k}\right]$$

$$+\frac{(1-\alpha)}{k}u_l^e(\theta_l-q_l^e)+\lambda^e=0 \tag{19}$$

where μ^i and λ^i are Lagrangian multipliers for the constraints in I's problem, and μ^e and λ^e are Lagrangian multipliers for the constraints in E's problem.

Let (u_l^e, q_l^e) be the solution to the above problem for the entrant. Given that the entrant offers two contracts, $u_l^e > 0$ thus $\mu^e = 0$. If $\lambda^e > 0$, then $q_l^e = \underline{q}$. Since the incumbent also offers two contracts, we immediately have $q_l^i \geq q_l^e$. Now suppose that $\lambda^e = 0$. We demonstrate that it must be the case that $(u_l^i, q_l^i) \geq (u_l^e, q_l^e)$. Suppose $(u_l^i, q_l^i) = (u_l^e, q_l^e)$. Then, since $\frac{1}{2} > \overline{q} - \frac{1}{2}\overline{q}^2$, the LHS of (16) is strictly greater than that of (18), which is zero, and the LHS of (17) is strictly greater than that of (19), which is zero. This implies that the incumbent would have incentive to raise either u_l^i, or q^i, or both. Therefore, $(u_l^i, q_l^i) \geq (u_l^e, q_l^e)$. This proves part (i).

Now we show part (ii). If $u_h^I > (1-\theta_l)\underline{q}$, then the incumbent must offer two contracts. So we only need to consider the case $u_h^I \leq (1-\theta_l)\underline{q}$. Since $u_h^I > u_h^E$, we must have $u_h^E < (1-\theta_l)\underline{q}$. Now suppose we have an equilibrium in which the incumbent offers only an h contract and the entrant offers two contracts characterized by u_h^i, $(u_h^e, \overline{q})$ and (u_l^e, q_l^e), where $u_l^e > 0$. Note that for the entrant the DIC must be binding. Moreover, $u_h^e = u_l^e + (1-\theta_l)q_l^e > u_h^i$. This is because if $u_h^i \geq u_l^e + (1-\theta_l)q_l^e$ then the incumbent can profitably introduce a low contract. Let $\Delta u_h \equiv u_h^i - u_h^e < 0$.

For the above contracts to be an equilibrium, the incumbent should have no incentive to raise u_h^i to u_h^e and introduce a low contract (u_l^e, q_l^e). That is,

$$\alpha\left(\frac{1}{4}+\frac{\Delta u_h}{2k}\right)\left(\frac{1}{2}-u_h^i\right) \geq \alpha\frac{1}{4}\left(\frac{1}{2}-u_h^e\right)+(1-\alpha)\frac{u_l^e}{k}\left[\theta_l q_l^1-\frac{1}{2}(q_l^1)^2-u_l^e\right]. \tag{20}$$

Similarly, the entrant should have no incentive to reduce u_h^e to u_h^i and only offer h contract. That is,

$$(21)\quad \alpha\left(\frac{1}{4}-\frac{\Delta u_h}{2k}\right)\left(\overline{q}-\frac{1}{2}\overline{q}^2-u_h^e\right)+(1-\alpha)\frac{u_l^e}{k}\left[\theta_l q_l^1-\frac{1}{2}(q_l^1)^2-u_l^e\right]\geq\alpha\frac{1}{4}\left(\overline{q}-\frac{1}{2}(\overline{q})^2-u_h^i\right).$$

Rearrange the inequalities (20) and (21), we have

(22)

$$\alpha\left[-\frac{1}{4}\Delta u_h+\frac{\Delta u_h}{2k}\left(\frac{1}{2}-u_h^i\right)\right]\geq(1-\alpha)\frac{u_l^e}{k}\left[\theta_l q_l^1-\frac{1}{2}(q_l^1)^2-u_l^e\right]\geq\alpha\left[-\frac{1}{4}\Delta u_h+\frac{\Delta u_h}{2k}\left(\overline{q}-\frac{1}{2}\overline{q}^2-u_h^e\right)\right].$$

Given that $\Delta u_h < 0$, $\overline{q}-\frac{1}{2}(\overline{q})^2 < 1/2$, and both $(\frac{1}{2}-u_h^i) > 0$ and $(\overline{q}-\frac{1}{2}\overline{q}^2-u_h^e) > 0$ (the profit margins for the high type are positive), we have that the first term of (22) is strictly less than the last term of (22), which contradicts the inequality of (22). This proves part (ii). ■

Proof of Proposition 6: From the full information solution when $k \leq \frac{2}{3}[1-\frac{1}{2}(1-\overline{q})^2]$, we see that u_h^I is increasing in $\overline{q}$ and u_l^I is independent of $\overline{q}$. Thus if the unconstrained solution for the incumbent is feasible with $\overline{q}$, it must be feasible with $\overline{q}'$. Therefore, we only need to show that the results hold when the unconstrained solution is not feasible with both $\overline{q}$ and $\overline{q}'$. Note that in this case, the DIC must be binding with both $\overline{q}$ and $\overline{q}'$, and $k > \frac{2}{3}\theta_l^2$.

We first show that the incumbent must offer two contracts in the duopoly equilibrium with $\overline{q}'$. If $u_h^{I\prime} > (1-\theta_l)\underline{q}$, then the incumbent must offer two contracts with $\overline{q}'$. Thus we only need to consider the case that $u_h^{I\prime} \leq (1-\theta_l)\underline{q}$. Since $u_h^I < u_h^{I\prime}$, $u_h^I < (1-\theta_l)\underline{q}$. By part (ii) of Proposition 5, we know that in duopoly equilibrium it cannot be the case that the incumbent offers the h contract only and the entrant offers two contracts. Therefore, it is sufficient to rule out the case that with $\overline{q}'$ both firms offering h contract only cannot be an equilibrium.

First suppose that with $\overline{q}$ the entrant offers two contracts in the duopoly equilibrium. By Proposition 5, the incumbent will also offer two contracts. If $\overline{q}$ is increased to $\overline{q}'$, by inspecting entrant's FOC's (18) and (19) we can see that she will continue to offer both contracts, and thus the incumbent offers two contracts as well.

Now suppose initially the entrant offers one contract only. With $\overline{q}$ in the duopoly equilibrium the incumbent offers two contracts, hence given that the entrant offers h contract only with u_h^E, there is a $u_l^{i*} > 0$, $q_l^{i*} \geq \underline{q}$, and $u_h^{i*} = u_l^{i*} + (1-\theta_l)q_l^{i*} > u_h^I$ such that

(23)

$$\alpha\left(\frac{1}{4}+\frac{u_h^{i*}-u_h^E}{2k}\right)\left(\frac{1}{2}-u_h^{i*}\right)+(1-\alpha)\frac{u_l^{i*}}{k}\left[\theta_l q_l^{i*}-\frac{1}{2}(q_l^{i*})^2-u_l^{i*}\right]>\alpha\left(\frac{1}{4}+\frac{u_h^I-u_h^E}{2k}\right)\left(\frac{1}{2}-u_h^I\right).$$

The above inequality says that the incumbent has an incentive to offer a low contract instead of offering only an h contract. Let $\Delta u_h^I \equiv u_h^{I\prime} - u_h^I > 0$ and $\Delta u_h^E \equiv u_h^{E\prime} - u_h^E > 0$. By earlier results, $\Delta u_h^E = 2\Delta u_h^I$.

Now we show that with $\overline{q}'$, both firms offering only an h contract cannot be an equilibrium. It is sufficient to show that when the entrant offers h contract alone with $u_h^{E\prime}$, the incumbent's best response is to offer two contracts instead of offering h contract alone with $u_h^{I\prime}$. For this purpose, we construct the following two contracts for the incumbent: offering $u_h^{i*} + \Delta u_h^I$ to type h, and offering u_l^{i*} and q_l^{i*} to the low type. Note that these two contracts are not the best response among all the possibilities of offering two contracts (the DIC is not binding). Nevertheless, we show that these two contracts yield a higher profit to the incumbent than the best response of offering h contract alone. That is,

$$
\begin{aligned}
(24) \qquad & \alpha\left(\frac{1}{4}+\frac{u_h^{i*}+\Delta u_h^I-u_h^{E\prime}}{2k}\right)\left(\frac{1}{2}-u_h^{i*}-\Delta u_h^I\right)+(1-\alpha)\frac{u_l^{i*}}{k}\left[\theta_l q_l^{i*}-\frac{1}{2}(q_l^{i*})^2-u_l^{i*}\right] \\
> \ & \alpha\left(\frac{1}{4}+\frac{u_h^{I\prime}-u_h^{E\prime}}{2k}\right)\left(\frac{1}{2}-u_h^{I\prime}\right).
\end{aligned}
$$

To see that (24) holds, it suffices to show that, for (23) and (24), the difference of the first terms, ΔA, equals the difference of the third terms, ΔB. Specifically,

$$
\begin{aligned}
\Delta A &= \frac{1}{4}\Delta u_h^I+\frac{\Delta u_h^E-\Delta u_h^I}{2k}\left(\frac{1}{2}-u_h^{i*}\right)+\frac{u_h^{i*}+\Delta u_h^I-u_h^{E\prime}}{2k}\Delta u_h^I, \\
\Delta B &= \frac{1}{4}\Delta u_h^I+\frac{\Delta u_h^E-\Delta u_h^I}{2k}\left(\frac{1}{2}-u_h^I\right)+\frac{u_h^{I\prime}-u_h^{E\prime}}{2k}\Delta u_h^I, \\
\Delta B-\Delta A &= \frac{\Delta u_h^E-2\Delta u_h^I}{2k}(u_h^{i*}-u_h^I)=0,
\end{aligned}
$$

where the last equality follows since $\Delta u_h^E = 2\Delta u_h^I$. Therefore, with $\overline{q}'$ both firms offering the h contract only cannot be an equilibrium; the incumbent must offer two contracts.

We next show that $q_l^{i\prime} \geq q_l^i$. Following the previous analysis, for the case that we are interested in, the programming problem is the same as (P), and the FOCs for the incumbent are given by (16)-(17). Let (u_l^i, q_l^i) and $(u_l^{i\prime}, q_l^{i\prime})$ be the solutions to (16)-(17) with $\overline{q}$ and $\overline{q}'$, respectively. With $\overline{q}$ whether the entrant offers h contract only or offers two contracts, when $\overline{q}$ increases to $\overline{q}'$, the entrant must respond optimally to (u_l^i, q_l^i) in a way that $u_h^{e\prime} > u_h^e$ because $\overline{q}' - \frac{1}{2}(\overline{q}')^2 > \overline{q} - \frac{1}{2}(\overline{q})^2$. Since with both $\overline{q}$ and $\overline{q}'$ the incumbent offers two contracts, we have $\mu^i = \mu^{i\prime} = 0$. With $u_h^{e\prime} > u_h^e$, from (16)-(17) we see that if $(u_l^{e\prime}, q_l^{e\prime}) = (u_l^e, q_l^e)$, the LHS of (16) and (17) (excluding $\mu^{i\prime}$ and $\lambda^{i\prime}$) are both strictly higher under $\overline{q}'$ than under $\overline{q}$. This implies that $(u_l^{i\prime}, q_l^{i\prime}) \geq (u_l^i, q_l^i)$ and $u_h^{i\prime} > u_h^i$. ■

Proof of Proposition 7: We consider two cases in order:

Case (i): when entry leads the incumbent to introduce a low contract, it is easy to check that $u_h^i \geq u_h^I > u_h^{fb}$. Following the proof in Proposition 3, we show that both type-h and type-l consumers are strictly better off.

Case (ii): when the incumbent offers two contracts under both monopoly and duopoly, there are two subcases to consider.

Subcase a): the incumbent's DIC is slack under both monopoly and duopoly. If the entrant's DIC is also slack, the duopoly solutions are the full information solutions. Clearly, type-h consumers are strictly better off. By the same argument as in the proof of Proposition 3, that DIC is slack under monopoly implies that $k > \theta_l^2$. This means that type-l consumers receive the same utility under both monopoly and duopoly. If the entrant's DIC is binding, it must be the case that $k > \frac{2}{3}\theta_l^2$ (otherwise the unconstrained solutions are feasible for the entrant – see the proof of Proposition 5). This implies that there is no effective competition for type-l consumers. As a result, type-l consumers purchasing from the incumbent receive the same utility as under a monopoly. That DIC is binding means that the entrant has to offer $u_h^e > u_h^E$, which suggests that $u_h^i > u_h^I > u_h^{fb}$. If the entrant offers only the high contract, the duopoly solutions are again the full information solutions and both types of consumers are better off.

Subcase b): the incumbent's DIC is binding under both monopoly and duopoly. The entrant therefore offers only the h contract or two contracts with a binding DIC. The programming problem is therefore the one given in the proof of Proposition 5, as well as the FOC's (16) and (17). Notice that some minor changes to the programming problem and the FOC's are needed if the entrant offers only one contract. Since a firm's horizontal coverage of the h contract must be positive, we have $-\frac{1}{4} < \frac{u_h^i - u_h^e}{2k} < \frac{1}{4}$. Suppose the monopoly solution is (u_l^m, q_l^m) as derived from FOC's (9) and (10). It is easy to check that both the incumbent's FOC's (16) and (17) under duopoly are positive when evaluated at the monopoly solution (u_l^m, q_l^m). Therefore, in duopoly the incumbent offers $u_l^i > u_l^m$ and $q_l^i > q_l^m$ regardless of the number of contracts the entrant offers. As a result, $u_h^i > u_h^m$ and both types of consumers are strictly better off. ■

Proof of Lemma 6: When $k \leq 1/2$, type h is fully covered. When $k \leq \frac{\theta_l^2}{2}$, the DIC's must bind; When $\frac{\theta_l^2}{2} < k \leq \frac{\theta_m^2}{2}$, the DIC$_{hm}$ must bind under full information, and the DIC$_{ml}$ binds if $\frac{k}{2} < \frac{\theta_l^2}{4} + (\theta_m - \theta_l)\theta_l$, which holds if $\theta_m < 3\theta_l$.[46] When $k > \frac{\theta_m^2}{2}$, the DIC$_{hm}$ binds if $\frac{k}{2} < \frac{\theta_m^2}{4} + (1-\theta_m)\theta_m$. Similarly, the DIC$_{ml}$ binds if $\frac{\theta_m^2}{4} < \frac{\theta_l^2}{4} + (\theta_m - \theta_l)\theta_l$, which again holds if $\theta_m < 3\theta_l$.[47]

Therefore, a set of sufficient conditions for both DIC's to bind is that $\theta_m < 3\theta_l$ and $\frac{k}{2} < \frac{\theta_m^2}{4} + (1-\theta_m)\theta_m$. We hence maintain these two assumptions in this subsection.

We first look at the case of fully separating equilibria. The programming problem is as follows:

$$\max_{(u_l, q_l, q_m)} \frac{\alpha_h}{2}\left[\frac{1}{2} - u_l - (\theta_m - \theta_l)q_l - (1-\theta_m)q_m\right] + \alpha_m \frac{u_l + (\theta_m - \theta_l)q_l}{k}\left(\theta_m q_m - \frac{1}{2}q_m^2 - u_m\right)$$

[46] To offer separate contracts, u_m must increase which makes DIC$_{hm}$ even more binding.

[47] When DIC$_{hm}$ binds, to offer separate contracts, u_m needs to be reduced which makes DIC$_{ml}$ more binding.

$$+ \alpha_l \frac{u_l}{k}\left(\theta_l q_l - \frac{1}{2}q_l^2 - u_l\right)$$

subject to: $u_l \geq 0$; $q_m \geq \underline{q}$; $q_l \geq \underline{q}$

Let the Lagrangian multipliers of the three constraints be μ, λ_m, and λ_l respectively. The FOCs are as follows:

$$\begin{aligned} -\frac{\alpha_h}{2} + \frac{\alpha_m}{k}\left(\theta_m q_m - \frac{1}{2}q_m^2 - 2u_m\right) + \frac{\alpha_l}{k}\left(\theta_l q_l - \frac{1}{2}q_l^2 - 2u_l\right) + \mu &= 0, \\ \mu \geq 0,\ \mu = 0 \text{ if } u_l &> 0; \end{aligned} \tag{25}$$

$$\begin{aligned} -\frac{\alpha_h}{2}(1-\theta_m) + \frac{\alpha_m}{k}u_m(\theta_m - q_m) + \lambda_m &= 0, \\ \lambda_m \geq 0,\ \lambda_m = 0 \text{ if } q_l &> \underline{q}. \end{aligned} \tag{26}$$

$$\begin{aligned} -\frac{\alpha_h}{2}(1-\theta_l) + \frac{\alpha_m}{k}\left(\theta_m q_m - \frac{1}{2}q_m^2 - 2u_m\right)(1-\theta_l) + \frac{\alpha_l}{k}u_l(\theta_l - q_l) + \lambda_l &= 0, \\ \lambda_l \geq 0,\ \lambda_l = 0 \text{ if } q_l &> \underline{q}. \end{aligned} \tag{27}$$

From (26), we can see that if $\alpha_h(1-\theta_m) \geq \alpha_m(\theta_m - \underline{q})$, then $\lambda_m > 0$ and $q_m = \underline{q}$ (since $u_m/k \leq 1/2$). Therefore, $\alpha_h(1-\theta_m) \geq \alpha_m(\theta_m - \underline{q})$ implies that fully separating is not optimal. Moreover, if h and m contracts are offered only, $q_m = \underline{q}$.

Now consider the case of partial pooling (types m and l pool at the low contract). The programming problem is as follows:

$$\max_{(u_l,q_l)} \frac{\alpha_h}{2}\left[\frac{1}{2} - u_l - (1-\theta_l)q_l\right] + \left[\alpha_m \frac{u_l + (\theta_m - \theta_l)q_l}{k} + \alpha_l \frac{u_l}{k}\right]\left(\theta_l q_l - \frac{1}{2}q_l^2 - u_l\right)$$

subject to: $u_l \geq 0$; $q_l \geq \underline{q}$

The FOCs are as follows:

$$\begin{aligned} -\frac{\alpha_h}{2} + \frac{\alpha_m}{k}\left[\theta_l q_l - \frac{1}{2}q_l^2 - u_l - u_m\right] + \frac{\alpha_l}{k}\left[\theta_l q_l - \frac{1}{2}q_l^2 - 2u_l\right] + \mu &= 0, \\ \mu \geq 0,\ \mu = 0 \text{ if } u_l &> 0; \end{aligned} \tag{28}$$

$$\begin{aligned} -\frac{\alpha_h}{2}(1-\theta_l) + \frac{\alpha_m}{k}(\theta_m - \theta_l)\left[\theta_l q_l - \frac{1}{2}q_l^2 - u_l\right] + \left[\frac{\alpha_m}{k}u_m + \frac{\alpha_l}{k}u_l\right](\theta_l - q_l) + \lambda &= 0, \\ \lambda \geq 0,\ \lambda = 0 \text{ if } q_l &> \underline{q}. \end{aligned} \tag{29}$$

From (29), we can see that if $\alpha_h(1-\theta_l) \geq \frac{\alpha_m}{k}(\theta_m - \theta_l)\theta_l^2 + (1-\alpha_h)(\theta_l - \underline{q})$, then $\lambda > 0$ and $q_l = \underline{q}$.

To establish that partial pooling is optimal, we must show that partial pooling dominates exclusion, that is, offering an h contract only or only offering h and m contracts. Offering an h contract alone leads to a (half) profit of $\pi_h = \frac{\alpha_h}{4}(1-k)$. When offering h and m contracts only, the optimal

$u_m = \frac{1}{2}(\theta_m \underline{q} - \frac{1}{2}\underline{q}^2 - \frac{\alpha_h}{\alpha_m}\frac{k}{2})$. Let the corresponding profit be π_{hm}. If

$$\theta_m \underline{q} - \frac{1}{2}\underline{q}^2 \leq \frac{\alpha_h}{\alpha_m}\frac{k}{2}, \tag{30}$$

then $u_m \leq 0$, which means that $\pi_{hm} < \pi_h$. In the case of partial pooling, the optimal u_l is given by

$$u_l = \frac{1}{2}\left[\theta_l \underline{q} - \frac{1}{2}\underline{q}^2 - \frac{\alpha_h}{1-\alpha_h}\frac{k}{2} - \frac{\alpha_m}{1-\alpha_h}(\theta_m - \theta_l)\underline{q}\right].$$

Let the corresponding total profit be $\pi_{h(ml)}$. Given u_l, we have

$$\pi_{h(ml)} - \pi_h > -\frac{\alpha_h}{2}\left[u_l + (1-\theta_l)\underline{q} - \frac{k}{2}\right] + (1-\alpha_h)\frac{u_l}{k}\left(\theta_l \underline{q} - \frac{1}{2}\underline{q}^2 - u_l\right) \equiv f(u_l).$$

The maximum $f(u_l)$, $f(u_l^*)$, can be calculated readily. Now if

$$f(u_l^*) = (1-\alpha_h)\frac{(\theta_l \underline{q} - \frac{1}{2}\underline{q}^2)^2 - (\frac{\alpha_h}{1-\alpha_h}\frac{k}{2})^2}{4k} - \frac{\alpha_h}{2}\left[\frac{1}{2}\left(\theta_l \underline{q} - \frac{1}{2}\underline{q}^2 - \frac{\alpha_h}{1-\alpha_h}\frac{k}{2}\right) + (1-\theta_l)\underline{q} - \frac{k}{2}\right] > 0, \tag{31}$$

then $\pi_{h(ml)} > \pi_h$. Overall, if both (30) and (31) hold, then we have $\pi_{h(ml)} > \pi_h > \pi_{hm}$. That is, partial pooling is optimal. ■

Proof of Lemma 7: Given that $k \leq 1/2$, $u_h^D = \frac{1-k}{2}$. Since $(1-\theta_i)\underline{q} < 1/4 \leq \frac{1-k}{2}$, in the duopoly equilibrium at least two contracts are offered. One sufficient condition to guarantee full separation is that $k \leq \frac{2}{3}\theta_l^2$. In this case, competition exists for all three types. The full information solution always satisfies the DICs: for $\theta_i > \theta_j$,

$$u_i^D - u_j^D = \frac{\theta_i^2 - \theta_j^2}{2} > (\theta_i - \theta_j)\theta_j.$$

Therefore, the duopoly equilibrium exhibits full separation and no quality distortion.

We next identify another sufficient condition. Suppose that $k \in [\frac{2}{3}\theta_m^2, \theta_m^2)$, and $2 - 3k \geq 4(1-\theta_m)\theta_m$. By the first condition, $u_m^D = k/4$. By the second condition, the DIC_{hm} is slack in the full-information solution. Therefore, in the duopoly equilibrium we must have $q_m^d = \theta_m$ (no quality distortion for type m). We further assume that $(\theta_m - \theta_l)\underline{q} < \frac{k}{4}$. This condition implies that if q_l is low enough, offering a contract to type l will not affect the DIC_{ml}. Therefore, in the duopoly equilibrium, each firm must offer three contracts (fully separating). ■

Proof of Proposition 9: Now suppose both firms offer three separate contracts. The problem becomes:

$$
\begin{aligned}
\max \quad & \alpha_h \left[\tfrac{1}{4} + \tfrac{u_l+(\theta_m-\theta_l)q_l+(1-\theta_m)q_m-u_h^{\mathrm{II}}}{2k}\right]\left(\tfrac{1}{2} - u_l - (\theta_m-\theta_l)q_l - (1-\theta_m)q_m\right) \\
& +\alpha_m \tfrac{u_l+(\theta_m-\theta_l)q_l}{k}\left[\theta_m q_m - \tfrac{1}{2}q_m^2 - u_l - (\theta_m-\theta_l)q_l\right] + \alpha_l \tfrac{u_l}{k}\left[\theta_l q_l - \tfrac{1}{2}q_l^2 - u_l\right] \\
\text{s.t.} \quad & u_l \geq 0,\ q_m \geq \underline{q},\ q_l \geq \underline{q}
\end{aligned}
$$

The LHS (excluding λ_m) of the FOC for q_m is:

$$(u_l + (\theta_m - \theta_l)q_l)\left[\frac{\alpha_m}{k}(\theta_m - q_m) - \alpha_h \frac{1-\theta_m}{2k}\right] + \alpha_h \frac{1-\theta_m}{2k}\left[\frac{1-k}{2} - (1-\theta_m)q_m\right].$$

If

$$\frac{\alpha_m}{k}(\theta_m - \underline{q}) - \alpha_h \frac{1-\theta_m}{2k} < 0, \tag{32}$$

and

$$\frac{1-k}{2} - (1-\theta_m)\underline{q} < 0, \tag{33}$$

then $q_m = \underline{q}$, which means that fully separating is not optimal.

Next consider the case where partial pooling occurs. The problem now becomes:

$$
\begin{aligned}
\max_{(u_l,q_l)} \quad & \alpha_h \left[\frac{1}{4} + \frac{u_l + (1-\theta_l)q_l - u_h^{\mathrm{II}}}{2k}\right]\left(\frac{1}{2} - u_l - (1-\theta_l)q_l\right) \\
& + \left[\alpha_m \frac{u_l + (\theta_m - \theta_l)q_l}{k} + \alpha_l \frac{u_l}{k}\right]\left(\theta_l q_l - \frac{1}{2}q_l^2 - u_l\right) \\
\text{s.t.} \quad & u_l \geq 0, q_m \geq \underline{q}, q_l \geq q
\end{aligned}
$$

The FOC for u_l is as follows:

$$\frac{\alpha_h}{2k}\left[\frac{1-k}{2} - (1-\theta_l)q_l\right] + \frac{1-\alpha_h}{k}\left(\theta_l q_l - \frac{1}{2}q_l^2\right) - \frac{\alpha_m}{k}(\theta_m - \theta_l)q_l - \left[\frac{\alpha_h}{2k} + 2\frac{1-\alpha_h}{k}\right]u_l + \mu = 0$$

And the LHS (excluding λ) of the FOC for q_l is:

$$\alpha_h \frac{1-\theta_l}{2k}\left[\frac{1-k}{2} - (1-\theta_l)q_l\right] + \alpha_m \frac{\theta_m - \theta_l}{k}\left(2\theta_l q_l - \frac{3}{2}q_l^2\right) + \left[\frac{1-\alpha_h}{k}(\theta_l - q_l) - \alpha_h \frac{1-\theta_l}{2k} - \alpha_m \frac{\theta_m - \theta_l}{k}\right]u_l$$

If

$$(1-\alpha_h)(\theta_l - \underline{q}) - \frac{\alpha_h}{2}(1-\theta_l) - \alpha_m(\theta_m - \theta_l) < 0,$$

which is equivalent to:

$$(\theta_l - \underline{q}) - \alpha_h\left(\frac{\theta_l + 1}{2} - \underline{q}\right) < \alpha_m(\theta_m - \theta_l), \tag{34}$$

then the LHS of the FOC for q_l is less than or equal to

$$\alpha_h \frac{1-\theta_l}{2k}\left[\frac{1-k}{2}-(1-\theta_l)q_l\right]+\alpha_m\frac{\theta_m-\theta_l}{k}\left(2\theta_l q_l-\frac{3}{2}q_l^2\right).$$

Define $A=\alpha_h\frac{1-\theta_l}{2}$ and $B=\alpha_m(\theta_m-\theta_l)$. Then the above expression is proportional to

$$-\frac{3B}{2}q_l^2+(2\theta_l B-(1-\theta_l)A)q_l+\frac{1-k}{2}A.$$

The above expression is decreasing in q if $2\theta_l B-(1-\theta_l)A<0$, or more explicitly,

$$4\theta_l(\theta_m-\theta_l)\alpha_m<(1-\theta_l)^2\alpha_h. \tag{35}$$

Therefore, if (35) and the following condition hold,

$$\alpha_h\frac{1-\theta_l}{2}\left[\frac{1-k}{2}-(1-\theta_l)\underline{q}\right]+\alpha_m(\theta_m-\theta_l)\left(2\theta_l\underline{q}-\frac{3}{2}\underline{q}^2\right)<0, \tag{36}$$

then we have $q_l=\underline{q}$.

Next we will compare the expected profit from partial pooling with those from offering high contract only and offering both high and middle contracts.

If only the high contract is offered, the expected profit would be $\pi_h=\frac{\alpha_h k}{8}$. If both high and middle contracts are offered, the LHS (excluding multiplier) of the FOC for q_m is

$$u_m\left[\frac{\alpha_m}{k}(\theta_m-q_m)-\alpha_h\frac{1-\theta_m}{2k}\right]+\alpha_h\frac{1-\theta_m}{2k}\left[\frac{1-k}{2}-(1-\theta_m)q_m\right].$$

From condition (32) and (33), we have $q_m=\underline{q}$.

Given $q_m=\underline{q}$, the LHS (excluding multiplier) of the FOC for u_m is

$$\alpha_h\frac{1}{2k}\left[\frac{1-k}{2}-u_m-(1-\theta_m)\underline{q}\right]+\frac{\alpha_m}{k}\left[\theta_m\underline{q}-\frac{1}{2}\underline{q}^2-2u_m\right].$$

If

$$\frac{\alpha_h}{2}\left(\frac{1-k}{2}-(1-\theta_m)\underline{q}\right)+\alpha_m\left(\theta_m\underline{q}-\frac{1}{2}\underline{q}^2\right)<0, \tag{37}$$

then the optimal $u_m=0$ which means that it is not profitable to offer a middle contract along with high contract.

With partial pooling, we denote expected profit as $\pi_{h(ml)}$. From the previous discussion, we know that

$$\begin{aligned}\pi_{h(ml)} - \pi_h &= \frac{\alpha_h}{4}\left[\frac{1-k}{2} - u_l - (1-\theta_l)\underline{q}\right] + \left[(1-\alpha_h)\frac{u_l}{k} + \frac{\alpha_m}{k}(\theta_m - \theta_l)\underline{q}\right]\left(\theta_l\underline{q} - \frac{\underline{q}^2}{2} - u_l\right) \\ &\equiv g(u_l);\end{aligned}$$

When

$$\begin{aligned}\max g(u_l) &= \frac{\left[(1-\alpha_h)(\theta_l\underline{q} - \frac{\underline{q}^2}{2}) - \frac{\alpha_h}{4}k - \alpha_m(\theta_m - \theta_l)\underline{q}\right]^2}{4(1-\alpha_h)} + \frac{\alpha_h}{4}\left(\frac{1-k}{2} - (1-\theta_l)\underline{q}\right) \\ &\quad + \frac{\alpha_m}{k}(\theta_m - \theta_l)\underline{q}(\theta_l\underline{q} - \frac{\underline{q}^2}{2}) \\ &\geq 0, \end{aligned} \tag{38}$$

partial pooling is optimal in duopoly.

We can simplify the conditions a little bit. First, condition (33) implies that $(\theta_m - \frac{2}{3})^2 < \frac{6k-2}{9}$, which is one sufficient condition for binding DICs. Second, condition (32), (35), (36), and (37) are all about the proportions of high type and middle type, and they can be summarized by the following condition:

$$\alpha_h > \delta\alpha_m, \tag{39}$$

where

$$\delta = \max\left\{\frac{2(\theta_m - \underline{q})}{1-\theta_m}, \frac{4\theta_l(\theta_m - \theta_l)}{(1-\theta_l)^2}, \frac{2(\theta_m - \theta_l)(2\theta_l\underline{q} - \frac{3}{2}\underline{q}^2)}{(1-\theta_l)((1-\theta_l)\underline{q} - \frac{1-k}{2})}, \frac{2(\theta_m\underline{q} - \frac{1}{2}\underline{q}^2)}{(1-\theta_m)\underline{q} - \frac{1-k}{2}}\right\}.$$

■

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