Abstract

We study equilibria of first- and second-price auctions with resale in a model with independent private values. With asymmetric bidders, the resulting inefficiencies create a motive for post-auction trade. In our basic model, resale takes place via monopoly pricing—the winner of the auction makes a take-it-or-leave-it offer to the loser after updating his prior beliefs based on his winning. We show that a first-price auction with resale has a unique monotonic equilibrium. Our main result is that with resale, the expected revenue from a first-price auction exceeds that from a second-price auction. The results extend to other resale mechanisms: monopsony and, more generally, probabilistic $k$-double auctions. The inclusion of resale possibilities thus permits a general revenue ranking of the two auctions that is not available when these are excluded.

1 Introduction

In a first-price auction, asymmetries among bidders typically result in inefficient allocations—that is, the winner of the auction may not be the person who values the object the most. This inefficiency creates a motive for post-auction resale and when bidders take resale possibilities into account, their bidding behavior is affected. Standard models of such auctions, by and large, implicitly assume either that resale possibilities do not exist or that bidders do not take these into account when formulating bids.

There are at least two reasons why resale possibilities should be considered explicitly. The first one is positive. If, after the auction is over, bidders see that there are potential gains from trade, then they will naturally engage in such trade. And it seems unlikely that the seller can prevent bidders from engaging in post-auction...
trade, even if, for some reason, resale was deemed disadvantageous. In the auction of 3G spectrum licenses in the UK, post-auction trade was restricted by the government. The bidders, however, were easily able to circumvent these restrictions. TIW, a Canadian firm, bid successfully for the most valuable license “A” with a winning bid in excess of £4 bn. Another telecommunications company, Hutchison, then acquired the license by buying TIW itself. British Telecom created a wholly-owned subsidiary that bid in the auction and also purchased a license. After the auction, this subsidiary was floated on the stock market and sold. Thus, restrictions on the buying and selling of licenses were circumvented by the buying and selling of companies that owned the licenses. The actions of British Telecom—it created the subsidiary before the auction—suggest that bidders fully anticipated post-auction resale possibilities.

The second reason to consider auctions with resale is normative. There has been recent interest in the design of efficient auctions, especially in the context of privatization. If post-auction resale results in efficiency, however, then from the planner’s perspective, an inefficient auction is just as good. Are the allocations from an inefficient auction followed by post-auction resale indeed efficient?

In this paper, we study the effects of post-auction resale in a simple model with two bidders whose private values are independently, but perhaps asymmetrically, distributed. Equilibrium allocations in first-price auctions are then inefficient and bidders have the incentive to engage in post-auction trade. In our basic model, resale takes place via monopoly pricing—the winner of the auction makes a take-it-or-leave-it offer to the loser.

We show that a first-price auction with resale has a unique monotonic equilibrium (Theorems 1 and 2). We do this by first showing that every equilibrium has the feature that, despite the asymmetries, the distributions of bids of the two bidders are identical. Symmetry would not be surprising if resale took place under complete information and so was always efficient. In that case, each bidder would bid as if his value were the maximum of the two values. In our model, the symmetry of bid distributions is striking because post-auction resale also takes place under incomplete information and so is not always efficient. Here it occurs as a result of some cost-benefit calculations at the margin. The symmetry of the bid distributions is key—it is used both to construct an equilibrium and to show that it is unique.

The possibility of resale also affects incentives in second-price auctions. It is no longer a dominant strategy to bid one’s value. It is, nevertheless, a robust equilibrium—the strategies do not depend on the value distributions—and uniquely so. In this equilibrium, of course, the auction allocates efficiently and so there is no resale.

Our main result (Theorem 3) is that once resale possibilities are admitted, the expected revenue from a first-price auction exceeds that from a second-price auction. We thus obtain a general revenue ranking of the two auction formats. We require only that the value distributions be regular in the sense of Myerson (1981), ensuring that the monopoly pricing problem at the resale stage is well behaved. In particular,
we do not assume that one of the bidders is “stronger” than the other in the sense of stochastic dominance. We remind the reader that in the absence of resale, the two auctions cannot be unambiguously ranked, even if the bidders can be classified as being “weak” and “strong” (Maskin and Riley (2000)).

The proof of Theorem 3 uses a variational technique borrowed from the calculus of variations. To the best of our knowledge, the use of this technique is new to auction theory and it will, perhaps, prove useful in other applications as well.

The results reported above concern a particular resale institution—monopoly pricing—in which the winner of the auction has all the bargaining power. In Section 6, we show, however, that this is inessential by first considering the monopsony mechanism in which the loser has all the bargaining power and then more generally, mechanisms in which bargaining power is shared, perhaps unequally. All of the results reported above extend to these alternative, and more general, resale institutions.

**Related Literature**

Equilibrium analysis of asymmetric first-price auctions has posed many challenges. Some of the difficulties were already pointed out by Vickrey (1961) in his pioneering paper. He constructed an example in which bidder 1’s value, say $a$, was commonly known while the other’s was uniformly distributed. In that case, there is an equilibrium of the first-price auction in which bidder 1 randomizes. Vickrey (1961) showed that for some values of $a$, the revenue from a first-price auction exceeded that from a second-price auction; for other values of $a$, the revenue ranking was reversed.

Since then, progress in the area has been sporadic at best. In asymmetric first-price auctions without resale, pure strategy equilibria exist under quite permissive conditions, as a consequence of general existence results (see, for instance, Reny (1999), Athey (2001) or Jackson and Swinkels (2005)). There is, again under weak conditions, a unique equilibrium (see, for instance, Maskin and Riley (2003) orLebrun (2006)). But characterization results and revenue comparisons are few and far between. Griesmer, et al (1967) derive closed-form equilibrium bidding strategies in a first-price auction in which bidders draw values from uniform distributions, but over different supports. Plum (1992) extends this to situations in which the two value distributions are of the form $x^n$, again over different supports. Cheng (2005) identifies a class of distribution pairs which lead to linear bidding strategies. For this class, he shows that the first-price auction is revenue superior to the second-price auction. For specific examples of distribution pairs, Cantillon (2005) shows how asymmetry affects revenue in first-price auctions. In the absence of general analytic results, some researchers have resorted to numerical methods (Marshall, et al (1994)).

Maskin and Riley (2000) derive the most comprehensive characterization and revenue ranking results concerning first- and second-price auctions in the presence of asymmetries. They consider problems in which one of the bidders is unambiguously stronger than the other. Specifically, the distribution of one bidder (conditionally) stochastically dominates that of the other. Maskin and Riley (2000) are able to
identify circumstances in which one or the other auction is revenue superior. For instance, the second-price auction is revenue superior if the distribution of the weak bidder is obtained from that of the strong bidder by reassigning probability mass toward lower values. Fibich, Gavious and Sela (2004) have shown that when the bidders are “nearly symmetric,” the difference in revenues is of a smaller magnitude than the difference in the underlying distributions. Thus, for small asymmetries, the auctions are nearly revenue equivalent.

Gupta and Lebrun (1999) consider resale possibilities in a manner not unlike this paper. They assume, however, that at the end of the auction both values are announced. This means, of course, that resale is always efficient. But it is not clear how the auctioneer would come to know the values themselves. In contrast, in our model, the auctioneer knows only the bids and not the values. Haile (2003) considers resale possibilities in a symmetric model. At the time of bidding, however, buyers have only noisy information regarding their true values, which are revealed to them only after the auction. There is a motive for resale because although the winner of the auction may have received the highest signal, he may not have the highest true value. No general revenue ranking obtains. Zheng (2002) identifies conditions under which the outcomes of Myerson’s (1981) optimal auction can be achieved when resale is permitted.

The model of Garratt and Tröger (2006) is closest to ours in spirit. The crucial difference is that they assume, as in Vickrey’s (1961) example, that the value of one of the bidders is commonly known, and moreover, is equal to zero. This bidder, thus, participates in the auction for purely “speculative” reasons—he has no use value for the object. He only benefits if he can resell the object to the other bidder. In the efficient equilibrium of the second-price auction, the revenue is obviously zero. Garratt and Tröger (2006) show that there is a unique mixed strategy equilibrium in the first-price auction in which the revenue is positive. We allow for general continuous distributions and so their model may be viewed as a limiting case of ours.

2 Preliminaries

A single indivisible object is for sale. There are two risk-neutral buyers, labelled 1 and 2, with independently distributed private values, $X_1$ and $X_2$. Buyer $i$’s value for the object, $X_i$, is distributed according to the cumulative distribution function $F_i$ with support $[0, \omega_i]$. It is assumed that each $F_i$ admits a continuous density $f_i \equiv F_i'$ and that this density is positive on $(0, \omega_i)$. We suppose, without loss of generality, that $\omega_1 \geq \omega_2$.

We assume that both $F_i$ are regular in the sense of Myerson (1981) so that for $i = 1, 2$, the virtual value, defined as

$$x = \frac{1 - F_i(x)}{f_i(x)}$$
is a strictly increasing function of the actual value $x$. This ensures that the price at the resale stage is uniquely determined and is characterized by the first-order conditions for a maximum.\footnote{As shown by Bulow and Roberts (1989), the virtual value can be interpreted as the “marginal revenue” of a monopolist who faces a demand curve $1 - F_i(p)$.}

In later sections we will need to consider conditional distributions of the form $F_i(x \mid X_i \leq a) = F_i(x) / F_i(a)$ with support $[0, a]$. The associated conditional virtual value is

$$x - \frac{1 - F_i(x \mid X_i \leq a)}{f_i(x \mid X_i \leq a)} = x - \frac{F_i(a) - F_i(x)}{f_i(x)}$$

The monotonicity of the virtual values implies the monotonicity of the conditional virtual values. To see this, note that if we write $q = F_i(a) \leq 1$, then the latter is equivalent to the statement: if $x < x' \leq a$ then

$$f_i(x') f_i(x) (x' - x) > f_i(x) [q - F_i(x')] - f_i(x') [q - F_i(x)]$$

If $f_i(x) < f_i(x')$, then the right-hand side is nonpositive for all $q$ while the left-hand side is positive. If $f_i(x) \geq f_i(x')$, then the right-hand side is a nondecreasing function of $q$ and since regularity implies that the inequality holds for $q = 1$, it holds for all $q < 1$. Thus, we have that for all $a < \omega_i$, the conditional virtual values are strictly increasing. In other words, if $F_i$ is regular then the conditional distribution $F_i(\cdot \mid X_i \leq a)$ is also regular.

## 3 First-Price Auction with Resale

Our model of the first-price auction with resale (FPAR) is the following. The buyers first participate in a standard sealed-bid first-price auction. The winning bid is publicly announced. We assume—as is common in real-world auctions—that the losing bid is not announced.\footnote{This assumption is discussed in more detail below in Remark 1.}

In the second stage, the winner of the auction—say $j$—may, if he wishes to, offer to sell the object to the other bidder $i \neq j$ at some price $p$. If the offer is accepted by $i$, a sale ensues. If the offer is rejected, the original owner $j$ retains the object. Thus resale takes place via a take-it-or-leave-it offer by the winner of the auction.\footnote{All bargaining power thus lies with the seller and from his perspective, this is, of course, the optimal resale mechanism. In Section 6 below, we show that our analysis extends to resale mechanisms in which all bargaining power lies with the buyer and then, more generally, to mechanisms in which it is shared.}

### Strategies and Beliefs

A strategy for bidder $i$ in a first-price auction with resale has two components: (i) a bidding strategy $\beta_i : [0, \omega_i] \to \mathbb{R}$ that specifies the
bid \( \beta_i(x_i) \) that \( i \) will submit in the auction if his value is \( x_i \); (ii) a pricing strategy \( p_i(b_i, x_i) \) which specifies the asking price \( i \) will set if he wins the auction with a bid of \( b_i \) and his value is \( x_i \). We allow \( i \) to set \( p_i = \infty \), which is interpreted to be the same as a decision not to offer the object for sale.

In addition, \( i \) must specify the set of price offers \( A_i(b_j, x_i) \) he will accept if \( j \) wins with a bid of \( b_j \) and his own value is \( x_i \).

A belief function \( \mu_i \) specifies a probability distribution \( \mu_i(b_i, \cdot) \) over \([0, \omega_j] \) that represents the beliefs that \( i \) holds regarding \( j \)'s values if he wins the auction with a bid of \( b_i \).

Equilibrium A perfect Bayesian equilibrium (henceforth, equilibrium) consists of a pair of bidding-pricing strategies \( (\beta_i, p_i) \) and belief functions \( \mu_i \) for \( i = 1, 2 \) with the property that: (i) if \( i \) loses the auction, then the set of offers he will accept is \( A_i(b_j, x_i) = \{ p_j : p_j \leq x_i \} \); (ii) if \( i \) wins the auction with a bid of \( b_i \) when his value is \( x_i \), then \( p_i(b_i, x_i) \) is optimal given \( \mu_i(b_i, \cdot) \) and the acceptance strategy above; (iii) for each \( x_i, \beta_i(x_i) \) is optimal given \( \beta_j, p_i \) and \( p_j \); and (iv) the beliefs \( \mu_i \) are generated from \( F_i \) and \( \beta_i \) using Bayes rule whenever possible.

Note that if \( i \) loses the auction, then the announcement of the winning bid \( b_j \) carries no useful information—that is, the set of price offers \( i \) will accept is independent of \( b_j \). Thus the equilibrium would be unaffected if neither bid were announced.

As usual, we work backwards and first outline behavior in the resale stage.

### 3.1 Resale Stage

Suppose that the two bidders follow continuous and strictly increasing bidding strategies \( \beta_1 \) and \( \beta_2 \) with inverses \( \phi_1 \) and \( \phi_2 \), respectively.\(^4\)

Suppose that bidder \( j \) with value \( x_j \) won the auction with a bid of \( b \). As a result, he would infer that bidder \( i \)'s value \( X_i \leq \phi_i(b) \). His beliefs, therefore, are \( \mu_j(b, x_i) = \mu_i(x_i | X_i \leq \phi_i(b)) \). If \( x_j < \phi_i(b) \), then there are potential gains from trade and so bidder \( j \) will set a ("monopoly") price \( p \) that maximizes

\[
[F_i(\phi_i(b)) - F_i(p)]p + F_i(p)x_j
\]

The first term in the maximand is \( j \)'s expected payoff from the event \( X_i \geq p \) in which bidder \( i \) accepts his offer. The second term is his payoff from the event \( X_i < p \), in which case bidder \( i \) rejects it.

The first-order condition for \( j \)'s maximization problem can be rewritten as

\[
p = \frac{F_i(\phi_i(b)) - F_i(p)}{f_i(p)} = x_j
\]

\(^4\)We will show later that all equilibria must have these properties.
Since $F_i$ is regular, the left-hand side is increasing and so (1) has a unique solution. Moreover, (1) is also sufficient for $j$’s maximization problem. Thus there is a unique price

$$p_j(b, x_j) = \arg \max_p [F_i(\phi_i(b)) - F_i(p)]p + F_i(p)x_j$$

(2)

that maximizes $j$’s payoff from resale and clearly, $x_j < p_j(b, x_j) < \phi_i(b)$. It follows immediately from (1) that the optimal price $p_j(b, x_j)$ is an increasing function of both $b$ and $x_j$.

Let

$$R_j(b, x_j) = \max_p [F_i(\phi_i(b)) - F_i(p)]p + F_i(p)x_j$$

(3)

denote bidder $j$’s optimal expected revenue from resale. For future reference, note that as a result of the envelope theorem,

$$\frac{\partial}{\partial b} R_j(b, x_j) = f_i(\phi_i(b)) \phi_i'(b) p_j(b, x_j)$$

(4)

If bidder $j$ won the auction with a bid of $b$ and $x_j \geq \phi_i(b)$, then there are no potential gains from trade and so bidder $j$ does not offer the object for sale.

### 3.2 Bidding Stage

We begin by deriving some necessary conditions that equilibrium bidding strategies must satisfy. At the time of the auction, both bidders anticipate that behavior in the resale stage will be as specified above.

#### 3.2.1 Necessary Conditions

Suppose that, in equilibrium, each bidder $i$ follows a continuous and strictly increasing bidding strategy $\beta_i : [0, \omega_i] \to \mathbb{R}$, so that $\beta_i(x_i)$ is the bid submitted by $i$ when his value is $x_i$.

First, notice that we must have $\beta_1(0) = \beta_2(0) = 0$. Suppose that $\beta_i(0) > 0$ and without loss of generality, suppose that there is a sequence $x^n \downarrow 0$ such that $\beta_i(x^n) \geq \beta_j(x^n)$. For $n$ large enough, $x^n$ is less than $\beta_i(0)$ and if bidder $i$ with value $x^n$ wins and does not resell then his net gain is less than zero. If bidder $i$ with value $x^n$ wins and resells to bidder $j$, the price he will receive is also less than $\beta_i(0)$. This is because if he wins with bid of $\beta_i(x^n)$ then $j$’s value $X_j \leq \beta_j^{-1} \beta_i(x^n)$ and by continuity, this is close to zero, when $x^n$ is close to zero. Thus his gain from winning is less than $\beta_i(0)$ whether or not the object is resold. Moreover, if bidder $i$ with value $x^n$ loses, then $x^n \leq X_j$ and there will be no resale. Thus his payoff from losing is 0. As a result, his overall payoff is negative and it is better for bidder $i$ with value $x^n$ to bid 0. Hence, $\beta_i(0) > 0$ is not possible.

It is also easy to verify that $\beta_1(\omega_1) = \beta_2(\omega_2) \equiv b$. 


As above, let \( \phi_i : [0, b_i] \to [0, \omega_i] \) denote \( i \)'s inverse bidding strategy in equilibrium, that is, \( \phi_i = \beta_i^{-1} \). Fix a bid \( b \) and suppose that \( \phi_j (b) < \phi_i (b) \). This means that if \( j \) wins with a bid of \( b \), then there are potential gains from trade and so \( j \) will make an offer to \( i \). If, on the other hand, \( i \) wins with bid of \( b \), then there are no potential gains from trade and so \( i \) will not make an offer to \( j \). Thus the bid \( b \) itself determines the direction the resale transaction, that is, the identities of the seller and the buyer.

Suppose bidder \( i \) follows \( \phi_i \). Bidder \( j \)'s expected payoff when his value is \( x_j \equiv \phi_j (b) \) and he deviates by bidding a \( c \) close to \( b \) is

\[
\Pi_j (c, x_j) = R_j (c, x_j) - F_i (\phi_i (c)) c
\]

where \( R_j (c, x_j) \), defined in (3), is his expected payoff from resale if he wins the auction. If \( j \) loses the auction, then \( \phi_j (c) < \phi_i (c) \) implies that bidder \( i \) will not offer to resell to him and so his payoff is 0. Since it is optimal for \( j \) to bid \( b \), the first-order condition for maximizing \( \Pi_j \), together with (4), results in

\[
f_i (\phi_i (b)) \phi'_i (b) p_j (b, x_j) - f_i (\phi_i (b)) \phi'_i (b) b - F_i (\phi_i (b)) = 0
\]

where \( p_j (b, x_j) \) is defined in (2). Since \( x_j = \phi_j (b) \), writing \( p (b) \equiv p_j (b, \phi_j (b)) \), the first-order condition results in the differential equation

\[
\frac{d}{db} \ln F_i (\phi_i (b)) = \frac{1}{p (b) - b} \tag{5}
\]

Note that \( p \) depends on both \( \phi_1 \) and \( \phi_2 \).

Now suppose bidder \( j \) follows an equilibrium strategy \( \phi_j \). Bidder \( i \)'s expected payoff when his value is \( x_i \equiv \phi_i (b) \) and he deviates by bidding a \( c \) close to \( b \) is

\[
\Pi_i (c, x_i) = (x_i - c) F_j (\phi_j (c)) + \int_{\phi_j (c)}^{\omega_j} \max \{ [x_i - p_j (\beta_j (x_j), x_j)] , 0 \} f_j (x_j) dx_j
\]

This is because if \( i \) wins the auction, he never resells to \( j \) and so his profit is simply \( x_i - c \). The second term is \( i \)'s expected payoff from buying the object from \( j \). Since it is optimal for \( i \) to bid \( b \), the first-order condition for maximizing \( \Pi_i \) is

\[
[x_i - b] f_j (\phi_j (b)) \phi'_j (b) - F_j (\phi_j (b)) - [x_i - p_j (b, \phi_j (b))] f_j (\phi_j (b)) \phi'_j (b) = 0
\]

Again writing \( p_j (b, \phi_j (b)) = p (b) \), the first-order condition becomes

\[
\frac{d}{db} \ln F_j (\phi_j (b)) = \frac{1}{p (b) - b}
\]

which is the same as (5).

We have argued that if \( \phi_1, \phi_2 \) are the equilibrium inverse bid functions in a first-price auction with resale, then they must both satisfy (5). This was derived using the first-order necessary conditions for local deviations to be unprofitable.\(^5\)

\(^5\)We have argued that the differential equations hold at any \( b \) such that \( \phi_j (b) < \phi_i (b) \). If \( b \) is such that \( \phi_j (b) = \phi_i (b) \), then whoever wins at that bid will set a price \( p (b) = \phi_i (b) = \phi_j (b) \) and the same arguments as given above show that the differential equations still hold.
3.2.2 Sufficiency

We now show that solutions to the differential equations (5) for \( j = 1, 2 \) are indeed equilibrium bidding strategies—that is, no deviations are profitable.

**Proposition 1** The strictly increasing and onto functions \( \phi_1 : [0, \overline{b}] \to [0, \omega_1] \) and \( \phi_2 : [0, \overline{b}] \to [0, \omega_2] \) are equilibrium inverse bidding strategies for the first-price auction with resale if and only if for all \( b \in [0, \overline{b}] \),

\[
\frac{d}{db} \ln F_1(\phi_1(b)) = \frac{1}{p(b) - b} \tag{6}
\]

\[
\frac{d}{db} \ln F_2(\phi_2(b)) = \frac{1}{p(b) - b} \tag{7}
\]

where, if \( \phi_j(b) \leq \phi_i(b) \),

\[
p(b) = \arg \max_p [F_i(\phi_i(b)) - F_i(p)]p + F_i(p)\phi_j(b) \tag{8}
\]

**Proof.** See Appendix A. \( \blacksquare \)

Note that the boundary conditions are determined by the condition that the \( \phi_i \) be strictly increasing and onto.

A word regarding the *equilibrium pricing function* \( p(\cdot) \) and its relationship to the equilibrium inverse bidding strategies, \( \phi_1 \) and \( \phi_2 \), is in order. Suppose, as depicted in Figure 1, that \( \phi_1(b) < \phi_2(b) \). Then if bidder 1 were to win with a bid of \( b \), he...
would infer that \( X_2 \leq \phi_2(b) \) and since there are potential gains from trade, bidder 1 would make an offer \( p(b) \) to bidder 2 \((j = 1 \text{ and } i = 2 \text{ in formula (8)})\). On the other hand, if bidder 2 were to win with a bid of \( b \), he would infer that 1’s value \( X_1 \leq \phi_1(b) < \phi_2(b) = x_2 \) and so would not make 1 an offer. But for bid \( b' \), \( \phi_2(b') < \phi_1(b') \), again as in the figure, the opposite holds. Upon winning with a bid of \( b' \), bidder 2 would make an offer \( p(b') \) to bidder 1 \((j = 2 \text{ and } i = 1 \text{ in formula (8)})\) but not the other way around. The value of \( b \) thus uniquely determines the identity of the seller \( j \) who sets the price \( p(b) \).

Recall that in any increasing equilibrium, the highest bids must be the same, say \( \overline{b} \). Thus \( F_1(\phi_1(\overline{b})) = 1 = F_2(\phi_2(\overline{b})) \). Since the boundary conditions for the two differential equations are the same, it now follows immediately from Proposition 1 that

**Corollary 1** If \( \phi_1 : [0, \overline{b}] \to [0, \omega_1] \) and \( \phi_2 : [0, \overline{b}] \to [0, \omega_2] \) are strictly increasing equilibrium inverse bidding strategies, then for all \( b \in [0, \overline{b}] \),

\[
F_1(\phi_1(b)) = F_2(\phi_2(b))
\]

that is, the bid distributions of the two bidders are identical.

The equality of the bid distributions implies that for all \( x \in [0, \omega_2] \), \( F_i(x) \leq F_j(x) \) if and only if \( \beta_i(x) \leq \beta_j(x) \). In particular, the bidding functions intersect at \( x \) if and only if the distributions intersect at \( x \).

### 3.2.3 Symmetrization

Corollary 1 identifies a remarkable property of first-price auctions with resale—even though the bidders are asymmetric, in equilibrium they bid in a way that the resulting bid distributions \( F_i(\phi_i(\cdot)) \) are the same. In this sense, resale *symmetrizes* the auction. Since this property plays an important role in what follows, it is worth exploring the underlying reasons.\(^6\)

As a first step, consider a standard first-price auction *without resale* (FPA) and let \( \varphi_1 \) and \( \varphi_2 \) be the equilibrium inverse bidding strategies. Suppose bidder \( i \) with value \( x_i = \varphi_i(b) \) raises his bid slightly from \( b \) to \( b + \varepsilon \). This make a difference only against the types of bidder \( j \) to whom bidder \( i \) loses the auction by bidding \( b \) but wins by bidding \( b + \varepsilon \). By doing this bidder \( i \) gains approximately \( x_i - b = \varphi_i(b) - b \) whenever \( \varphi_j(b) < x_j < \varphi_j(b + \varepsilon) \). Writing the first-order condition for optimality yields the pair of differential equations: for \( j = 1, 2 \) and \( i \neq j \),

\[
\frac{d}{db} \ln F_j(\varphi_j(b)) = \frac{1}{\varphi_i(b) - b} \tag{9}
\]

\(^6\)Gupta and Lebrun (1998) allude to this kind of symmetry in passing although the main thrust of their analysis is in the context of a different model—one in which values are announced at the end of the auction.
Notice that the right-hand side is the inverse of the marginal gain accruing to \( i \) from increasing his bid.

Now consider a first-price auction with resale (FPAR) with equilibrium inverse bidding strategies \( \phi_1 \) and \( \phi_2 \). Let \( b \) be such that \( \phi_j(b) < \phi_i(b) \). This means that in equilibrium if \( j \) wins with a bid of \( b \), so that his value \( x_j = \phi_j(b) \), then he will try to resell the object to bidder \( i \) since there are potential gains from trade. On the other hand, if \( i \) wins with a bid of \( b \), he will not resell the object to bidder \( j \) since there are no gains from trade.

Suppose bidder \( j \) with value \( x_j = \phi_j(b) \) raises his bid slightly from \( b \) to \( b + \varepsilon \). As before, we look at how much \( j \) gains against bidder \( i \) types such that \( \phi_j(b) < x_i < \phi_i(b + \varepsilon) \). When he bids \( b \), bidder \( j \) loses against these types of bidder \( i \) and since there is no resale, bidder \( j \)'s payoff is 0. When he bids \( b + \varepsilon \), however, he wins against these types of bidder \( i \) and is able to resell to them at a price of \( p(b) \) for a gain of \( p(b) - b \).

What about bidder \( i \)? Suppose bidder \( i \) with value \( x_i = \phi_i(b) \) raises his bid slightly from \( b \) to \( b + \varepsilon \) and again consider the benefit to \( i \) against those bidder \( j \) types such that \( \phi_j(b) < x_j < \phi_j(b + \varepsilon) \). When he bids \( b \), bidder \( i \) loses against these types of bidder \( j \) but is able to buy the object from them at a price of approximately \( p(b) \). His payoff thus approximately equals \( x_i - p(b) \). When he bids \( b + \varepsilon \), he wins against these types of bidder \( j \) and so his payoff is \( x_i - b \). The gain in payoff for \( i \) from increasing his bid from \( b \) to \( b + \varepsilon \) is thus approximately equal to \( (x_i - b) - (x_i - p(b)) = p(b) - b \), the same as \( i \)'s gain!

In contrast to (9), the right-hand sides of (6) and (7) are identical.

The symmetrization effects of resale come from the fact that the marginal gain to both bidders from a higher bid is the same: \( p(b) - b \). For bidder \( j \) (the “seller”), the marginal gain is just the profit from resale, that is, \( p(b) - b \). For bidder \( i \) (the “buyer”), the marginal gain is the difference in the “retail price” \( p(b) \) he pays when he loses the auction but buys from bidder \( j \) and the “wholesale price” \( b \) that he pays when he wins the auction and buys directly from the auctioneer.

The distributions of equilibrium bids in an asymmetric first-price auction with resale are thus observationally equivalent to the distribution of equilibrium bids in a symmetric first-price auction. In other words, given \( F_1 \) and \( F_2 \), there exists a distribution \( F \) such that a first-price auction (FPA) in which both bidders draw values from \( F \) is equivalent, in terms of equilibrium bid distributions, to a first-price auction with resale (FPAR) in which bidders draw values from \( F_1 \) and \( F_2 \), respectively. This also means that the two auctions are revenue equivalent.

We now show how \( F \) may be obtained from \( F_1 \) and \( F_2 \).

**Lemma 1** Given distributions \( F_1 \) and \( F_2 \), define \( F \) as follows: if \( F_i(p) \leq F_j(p) \), then

\[
F(p) = F_j \left(p - \frac{F_j(p) - F_i(p)}{f_i(p)}\right)
\]

(10)
Then $F$ is a uniquely determined distribution function such that $F_i(p) \leq F(p) \leq F_j(p)$. Moreover, if $F_i(p) < F_j(p)$, then $F_i(p) < F(p) < F_j(p)$.

**Proof.** See Appendix A. □

The construction of $F$ has a simple geometric interpretation, as depicted in Figure 2. For the point $p$, $F_2(p) < F_1(p)$ and so $i = 2$ and $j = 1$ in the formula (10). The distribution $F$ is such that it passes through the point $b$, which bisects the line segment $ac$. The length of the line segment $ab$ is just $p - F_1^{-1}(F(p))$. And since $bd/bc$ is $f_2(p)$, the slope of $F_2$ at $p$, the length of $bc$ is $[F(p) - F_2(p)]/f_2(p)$. Equation (10) requires that these be equal.

### 3.2.4 Equivalent Symmetric Auction

Now consider a symmetric first-price auction without resale in which there are two bidders and both draw values independently from the distribution function $F$ on $[0,\bar{p}]$ as defined above in (10).

The equilibrium strategies in a symmetric auction can, of course, be derived explicitly and are given by

$$
\beta(x) = \frac{1}{F(x)} \int_0^x yf(y) \, dy
$$
Let $\bar{b} = \beta(\pi)$. Define the equilibrium inverse bid function for the symmetric auction as

$$
\phi(b) \equiv \beta^{-1}(b)
$$

so that the distribution of bids for each bidder is $F(\phi(b))$. A necessary condition for $\phi$ to be the equilibrium inverse bidding strategy in the symmetric auction is that

$$
\frac{d}{db} \ln F(\phi(b)) = \frac{1}{\phi(b) - b}
$$

### 3.3 Existence and Uniqueness of Equilibrium

In this section, we establish that the first-price auction with resale has a pure strategy equilibrium in which each bidder follows a strictly increasing bidding strategy. The equilibrium is unique in the class of pure strategy equilibria with nondecreasing bidding strategies.

The proof that there is a strictly increasing equilibrium is constructive. Given regular distribution functions $F_1$ and $F_2$, construct $F$ as in Lemma 1. Consider a symmetric first-price auction in which each bidder draws values independently from $F$. In symmetric auctions, it is known that a symmetric equilibrium $\beta$ exists and is strictly increasing. We will use the equilibrium $\beta$ to construct equilibrium bidding strategies $\beta_1$ and $\beta_2$ for the asymmetric first-price auction with resale.

#### 3.3.1 Existence of Equilibrium

**Theorem 1** Suppose $F_1$ and $F_2$ are regular. Then there exists an equilibrium in the first-price auction with resale in which the bidding strategies are strictly increasing.

**Proof.** The proof is by construction.

Given $F_1$ and $F_2$, let $F$ be determined as in the statement of Lemma 1. Let $\phi$, as defined above in (11), be the equilibrium inverse bidding strategy in the symmetric auction in which bidders draw values from $F$. Let $\bar{b}$ be the maximum bid in the symmetric auction and define inverse bidding strategies $\phi_1 : [0, \bar{b}] \to [0, \omega_1]$ and $\phi_2 : [0, \bar{b}] \to [0, \omega_2]$ in the asymmetric first-price with resale as follows:

$$
F_1(\phi_1(b)) = F(\phi(b)) \quad \text{(13)} \\
F_2(\phi_2(b)) = F(\phi(b)) \quad \text{(14)}
$$

Then using (12), we have that for $i = 1, 2$

$$
\frac{d}{db} \ln F_i(\phi_i(b)) = \frac{1}{\phi(b) - b}
$$

We claim that $\phi_1$ and $\phi_2$ are equilibrium inverse bidding strategies in the first-price auction with resale.
The definition of $F$ in (10) implies that if $F_i(\phi(b)) < F_j(\phi(b))$, then
\[
F_j^{-1}(F(\phi(b))) = \phi(b) - \frac{F(\phi(b)) - F_i(\phi(b))}{f_i(\phi(b))}
\]
and since $F(\phi(b)) = F_j(\phi_j(b)) = F_i(\phi_i(b))$,
\[
\phi_j(b) = \phi(b) - \frac{F_i(\phi_i(b)) - F_i(\phi(b))}{f_i(\phi(b))}
\]
which is precisely the first-order condition for
\[
p(b) = \arg \max_p [F_i(\phi_i(b)) - F_i(p)]p + F_i(p)\phi_j(b)
\]
Regularity implies that the first-order condition is both necessary and sufficient for a maximum. Thus we have that for all $b$
\[
p(b) = \phi(b)
\]
Finally, note that $F_i(p(b)) < F_j(p(b))$ is equivalent to $\phi_j(b) < \phi_i(b)$. This is because (13) and (14) imply that $F_i(p(b)) < F(p(b))$, which is equivalent to $F_i(p(b)) < F_i(\phi_i(b))$ and so also to $p(b) < \phi_i(b)$. Similarly, $F(p(b)) < F_j(p(b))$ is equivalent to $F_j(\phi_j(b)) < F_j(p(b))$ and so also to $\phi_j(b) < p(b)$. Thus, $F_i(p(b)) < F_j(p(b))$ if and only if $\phi_j(b) < \phi_i(b)$.

We have thus argued that if $\phi_1$ and $\phi_2$ are determined by (13) and (14), then they satisfy the differential equations (6) and (7) where $p(b)$ is determined by (8).

Thus as constructed, the functions $\phi_1$ and $\phi_2$ satisfy the conditions of Proposition 1 and so constitute equilibrium inverse bidding strategies. 

**Remark 1** Theorem 1 relies on the assumption that at the end of the auction, the losing bid is not announced. If the losing bid is announced, the value of the losing bidder would be revealed in any strictly increasing equilibrium. This creates an incentive for a bidder to bid lower, so that if he were to lose, then the other bidder would think that his value is smaller than it actually is. This effect overwhelms the loss from not winning with a lower bid and it is known that no strictly increasing equilibrium exists (Krishna, 2002, Chapter 4). In fact, a stronger result holds: if the losing bid is announced, there is no nondecreasing equilibrium with (partial) pooling either.

### 3.3.2 Uniqueness of Equilibrium

In this section we show that the equilibrium constructed in Theorem 1 is, in fact, the only equilibrium in which bidders follow nondecreasing bidding strategies. We first show that it is unique in the class of equilibria with strictly increasing strategies.
The proof is completed by showing that if both equilibrium bidding strategies are nondecreasing, then they must be strictly increasing and continuous.

The following proposition is a key step. It shows that the distribution of resale prices in any equilibrium is given by $F$. Since $F$ is determined without reference to the equilibrium, this shows that the distribution of resale prices is uniquely determined.

**Proposition 2** Suppose $\phi_1$ and $\phi_2$ are strictly increasing equilibrium inverse bidding strategies. Then $F$, defined in (10), is the distribution of equilibrium resale prices and for all $b$, $F(p(b)) = F_j(\phi_j(b))$, $j = 1, 2$.

**Proof.** Let the random variable $P$ denote the resale price resulting from $\phi_1$ and $\phi_2$. Fix a $b$ such that $\phi_j(b) < \phi_i(b)$ and suppose bidder $j$ wins the auction with a bid of $b$. Then we have

$$\Pr[P \leq p(b)] = \Pr[X_j \leq \phi_j(b)] = F_j(\phi_j(b))$$

Because in equilibrium, for all $b$, $F_1(\phi_1(b)) = F_2(\phi_2(b))$, it is also the case that $\Pr[P \leq p(b)] = F_i(\phi_i(b))$. And since $p(b)$ is the monopoly resale price when the winning bid is $b$, it must satisfy the first-order condition

$$\phi_j(b) = p(b) - \frac{F_i(\phi_i(b)) - F_i(p(b))}{f_i(p(b))}$$

So

$$\Pr[P \leq p(b)] = F_j(\phi_j(b)) = F_j\left(p(b) - \frac{\Pr[P \leq p(b)] - F_i(p(b))}{f_i(p(b))}\right)$$

where we have used the fact that $\Pr[P \leq p(b)] = F_i(\phi_i(b))$ also. Now (10) implies that

$$\Pr[P \leq p(b)] = F(p(b))$$

and this completes the proof. ■

Since $F(p(b)) = F_j(\phi_j(b))$, Proposition 1 implies that

$$\frac{d}{db}(F(p(b))) = \frac{1}{p(b) - b}$$

This means that $p(b)$ satisfies (12), the differential equation characterizing the equilibrium inverse bidding strategy in the symmetric first-price auction in which both bidders draw values independently from the same distribution $F$. But given $F$, the
equilibrium inverse bidding strategy in a symmetric first-price auction is uniquely
determined.

We have thus argued that given \( F_1 \) and \( F_2 \), the equilibrium distribution of re-
sale prices, \( F \), and the function \( p(b) \) are uniquely determined. Now it follows from
\( F_i(\phi_i(b)) = F(p(b)) \) that \( \phi_1 \) and \( \phi_2 \) are uniquely determined. Thus there is only one
equilibrium in which bidders follow \textit{strictly increasing} bidding strategies.

In Appendix A it is shown that there are no equilibria with nondecreasing strate-
gies that are not strictly increasing. Thus, we obtain

\textbf{Theorem 2} The first-price auction with resale has a unique equilibrium in the class
of equilibria with nondecreasing bidding strategies.

\subsection{An Example}

It is useful to consider an explicit example to illustrate the various constructs.

\textbf{Example 1} Suppose that \( F_1(x) = x/\omega_1 \) over \([0, \omega_1]\) and \( F_2(x) = x/\omega_2 \) over \([0, \omega_2]\)
where \( \omega_1 \geq \omega_2 \); that is, the value distributions are both uniform, but over different
supports.

Since \( F_1 \leq F_2 \), it can only be that (the “weak”) bidder 2 resells to (the “strong”)
bidder 1. It may be verified that, the distribution of resale prices, \( F \), is also uniform. Specifi-
cally, \( F(p) = 2p/ (\omega_1 + \omega_2) \) over support \([0, 1/2 (\omega_1 + \omega_2)]\). The associated pric-
ing function \( p(b) = 2b \). The equilibrium inverse bidding strategies are: \( \phi_1(b) = 4\omega_1 b/ (\omega_1 + \omega_2) \) and \( \phi_2(b) = 4\omega_2 b/ (\omega_1 + \omega_2) \). The highest bid \( \bar{b} = 1/4 (\omega_1 + \omega_2) \).

Notice that if \( 3\omega_2 < \omega_1 \), then \( \phi_2(b) < b \), or equivalently, \( \beta_2(x) > x \); that is, bid-
er 2 bids more than his value in a first-price auction with resale. The reason, of
course, is that he anticipates being able to resell the object to bidder 1 for a profit.
Thus, bidder 2’s motives have a substantial “speculative” component. The model of
Garratt and Tröger (2006), in which one of the bidders is known to have a value of
0, is an extreme instance of this. There speculation is the only motive.

\subsection{Revenue}

The auxiliary symmetric first-price auction in which both bidders draw values from \( F \)
has the same distribution of bids as the first-price auction with resale in which bidders
draw values from \( F_1 \) and \( F_2 \), respectively. This is because \( p(\cdot) \) is the equilibrium
inverse bidding strategy in the auxiliary auction and from Lemma 2, for all \( b \in [0, \bar{b}] \),
\( F(p(b)) = F_j(\phi_j(b)) \).
Hence, in equilibrium, the expected revenue accruing to the auctioneer from a first-price auction with resale (FPAR) is

\[ R_{FPAR} (F_1, F_2) = R_{FPA} (F, F) = R_{SPA} (F, F) = \int_0^1 (1 - F (p))^2 \, dp \quad (15) \]

where \( F \) is defined in (10) and \( R_{SPA} (F, F) \) denotes the revenue from a symmetric second-price auction (SPA). The second equality is a consequence of the revenue equivalence principle. The third equality is a well-known formula for the expectation of the minimum of two independent random variables, both of which are distributed according to \( F \).

Note that the expected revenue from a first-price auction with resale can thus be calculated without direct reference to the equilibrium bidding strategies.

For the asymmetric uniform distributions in Example 1, we have

\[ R_{FPAR} (F_1, F_2) = \frac{1}{6} (\omega_1 + \omega_2) \]

4 Second-Price Auction with Resale

We now study properties of the second-price auction with resale. Our model is the same as that in previous sections except for the change in the auction format—that is, there is a second-price auction and then the winner, if he so wishes, can resell the object to the other bidder via a take-it-or-leave-it offer. There is one important difference, however. Under second-price rules, the winner of the auction inevitably knows the losing bid—after all this is the price he pays in the auction. Thus, unlike in a first-price auction, the winner can condition the price offered in the resale stage on the losing bid.\(^7\) This, of course, considerably simplifies the inference problem faced by a winning bidder and puts the losing bidder in a weak position during resale.

Resale Stage Suppose bidder \( i \) follows a nondecreasing bidding strategy \( \beta_i \) in the auction. Suppose also that bidder \( j \) wins the auction and pays a price of \( b_i \) which is in the range of \( \beta_i \); that is, \( i \)'s bid. He then infers that bidder \( i \)'s value is in the set \( \beta_i^{-1} (b_i) = \{ x : \beta_i (x) = b_i \} \).

If \( \beta_i^{-1} (b_i) \) is a singleton, say \( \beta_i^{-1} (b_i) = \{ x_i \} \), then it is optimal for \( j \) to offer the object to \( i \) only if \( x_j < x_i \) and in that case, set a price \( p = x_i \). If \( \beta_i^{-1} (b_i) \) is an interval, then it is optimal for \( j \) to offer the object to \( i \) only if \( x_j < \sup \beta_i^{-1} (b_i) \) and in that case, set a price \( p_j (b_i, x_j) \) that maximizes

\[ [F_i (\sup \beta_i^{-1} (b_i)) - F_i (p)] \, p + [F_i (p) - F_i (\inf \beta_i^{-1} (b_i))] \, x_j \]

\(^7\)Recall that a first-price auction with resale does not have a monotonic equilibrium if the losing bid is known to the winner. See Remark 1.
Bidding Stage  With private values, a standard second-price auction—without the possibility of resale—has some important and well-known features. First, it is a weakly dominant strategy for each bidder to bid his true value. Second, the resulting equilibrium is, of course, efficient, even in an asymmetric environment. Third, there is a continuum of other (inefficient) equilibria (see Blume and Heidhues (2004) for a complete classification).

Our first observation is that once there is the possibility of resale, it is not a weakly dominant strategy to bid one’s value in a second-price auction. As the example below shows, if one of the bidders, say 1, bids more than his value, the other bidder may gain by bidding less than his value. This is because a lower bid in the auction may lead to a lower resale price.

Example 2  The values $X_1, X_2 \in [0, 1]$. Suppose that bidder 1 bids according to a continuous and strictly increasing strategy $\beta_1(x)$ such that $\beta_1(x) > x$, for all $x \in (0, 1)$, and, if he wins, has beliefs $\mu_1(b_2, x) = 1$ if $x = b_2$ and 0 otherwise; that is, 1 believes that 2 is following the strategy $\beta_2(x_2) = x_2$.

Suppose that bidder 1 has value $x_1 \in (0, 1)$ and bidder 2’s value $x_2$ is such that $x_1 < x_2 < \beta_1(x_1)$. If bidder 2 bids $x_2$, then 1 will win the auction and will offer the object to 2 at price $p = x_2$. So bidder 2’s payoff from bidding his value is 0. If bidder 2 reduces his bid to a $b_2$ such that $x_1 < b_2 < x_2$, then again bidder 1 will win the auction but now offer to sell the object to 2 at price $p = b_2$. By accepting this offer, bidder 2 can make a profit of $x_2 - b_2$. Thus in this situation it is strictly better for bidder 2 to bid $b_2 < x_2$ than to bid $x_2$.

Robust Equilibrium  While not weakly dominant, if both bidders bid their values and the winner prices optimally, then this nevertheless results in an equilibrium of the second-price auction with resale. In fact, it constitutes a robust equilibrium—that is, the proposed strategies constitute a perfect Bayesian for all distributions $F_1, F_2$ of values that are strictly increasing and continuous.\(^8\)

The property that such equilibria are “distribution independent” makes them particularly attractive. Of course, every weakly dominant equilibrium is a robust equilibrium.

Proposition 3  There is a robust equilibrium of the second-price auction with resale in which both bidders bid their values.

\(^8\)In normal form games, a robust equilibrium also has the ex post property—players do not suffer any regret if after the game is played, all private information is made public. Börgers and McQuade (2006) have pointed out that this equivalence does not hold in extensive form games because of the possibility of ex post regret after a deviation. They have also shown that a SPAR does not have an ex post equilibrium. Their comments helped us correct a claim we made in an earlier version of this paper.
**Proof.** Consider the following strategies. In the auction, each bidder bids his value; that is, $\beta_i(x_i) = x_i$. After the auction, the winner $i$ believes that $j$’s value $X_j = b_j$ and offers to sell at a price $p_i = b_j$ if and only if $b_j > x_i$; the loser responds optimally to the price offer, if any.

Suppose bidder $i$ follows the strategy outlined above.

Suppose bidder $j$ deviates and bids $b < x_j$. If $x_i < b < x_j$, bidder $j$ wins for a price of $x_i$ and so there is no resale. So his payoff is $x_j - x_i$ which is the same as if he bids $x_j$. If $b < x_i < x_j$, then $i$ wins and $j$’s payoff is zero since again there is no offer of resale. So if $b < x_i < x_j$, $j$’s payoff is zero if he bids $x$ and $x_j - x_i$ if he bids $x_j$. Finally, if $b < x_j < x_i$, bidder $j$ loses the auction and his payoff is 0 whether he bids $b$ or $x_j$. Thus underbidding is not profitable.

Now suppose bidder $j$ bids $b > x_j$. If $x_i > b$, then $j$’s payoff is 0 since he loses and $i$ will not resell to him. If $b > x_i > x_j$, then again his payoff is zero, because he will pay $x_i$ for the object and then resell to $i$ for $X_i$. If $b > x_j > x_i$, then it makes no difference whether he bids $b$ or $x_j$. Thus overbidding is not profitable either.

We have thus argued it is a best response for bidder $j$ to follow the strategy $\beta_j(x) = x$, also. The optimality of the proposed strategies in the resale stage is clear.

None of the arguments above use any properties of the particular value distributions. Thus the proposed strategies constitute a robust equilibrium. ■

**Uniqueness** In a second-price auction (SPA) without resale, it is a dominant strategy for each bidder to bid his value. This, of course, is also a robust equilibrium. But in a SPA there are also other robust equilibria.

**Example 3** The values $X_1, X_2 \in [0,1]$. Bidder 1 bids according to the strategy $\beta_1(x) = 1$ and bidder 2 according to $\beta_2(x) = 0$.

These strategies, while weakly dominated, nevertheless constitute a robust equilibrium. Thus in the second-price auction without resale, there is a multiplicity of robust equilibria.

When there is resale, however, there is (essentially) a unique robust equilibrium—both bidders bid their values and since the resulting allocation is efficient, there is no resale.

**Proposition 4** Suppose $\beta_1$ and $\beta_2$ are bidding strategies in a robust equilibrium of the second-price auction with resale. Then for all $x \in [0,\omega_2]$, $\beta_1(x) = \beta_2(x) = x$.

**Proof.** See Appendix B. ■

If $\omega_1 > \omega_2$, then there are robust equilibria in which for $x > \omega_2$, bidder 1 bids more than $x$. Since bidder 1 wins for sure in these cases, all such equilibria are outcome equivalent to the one in which for all $x$, bidder 1 bids his value.
In what follows, we restrict attention to the unique robust equilibrium outcome of the second-price auction with resale in which bidders bid their values and the outcome is efficient. Henceforth, we will refer to this as the efficient equilibrium.

**Revenue** The expected revenue from the efficient equilibrium of a second-price auction with resale (SPAR) is

\[ R_{SPAR} (F_1, F_2) = \int_0^{\omega_2} (1 - F_1 (x)) (1 - F_2 (x)) \, dx \] (16)

The right-hand side of the formula above is simply \( E[\min \{X_1, X_2\}] \).

It is easily verified that for the asymmetric uniform distributions in Example 1, we have

\[ R_{SPAR} (F_1, F_2) = \frac{\omega_2 (3\omega_1 - \omega_2)}{6\omega_1} \]

and thus,

\[ R_{FPAR} (F_1, F_2) - R_{SPAR} (F_1, F_2) = \frac{(\omega_1 - \omega_2)^2}{6\omega_1} \]

and this is positive as long as \( \omega_1 > \omega_2 \).

In the next section, we show that the revenue superiority of the first-price auction over its second-price counterpart is general.

## 5 Revenue Comparison

In this section we establish our main result.

**Theorem 3** The seller’s revenue from a first-price auction with resale is at least as great as that from a second-price auction with resale.

Before proving Theorem 3, some preliminaries are in order.

**Calculus of Variations** In what follows, we will make use of a simple technique from the calculus of variations that is used to derive the *Euler equation*. (See, for instance, Section 3 in Kamien and Schwartz, 1981).

Consider the integral

\[ \Delta = \int_0^p \Phi (p, M (p), m (p)) \, dp \]

where \( M : [0, \bar{p}] \to \mathbb{R} \) and \( m (p) = M' (p) \). Suppose \( Z (p) : [0, \bar{p}] \to \mathbb{R} \) is a variation satisfying \( Z (0) = Z (\bar{p}) = 0 \) and let \( z (\bar{p}) = Z' (\bar{p}) \). Define

\[ \Delta (\varepsilon) = \int_0^p \Phi (p, M + \varepsilon Z, m + \varepsilon z) \, dp \]
to be the value of the integral when $M$ is perturbed by $\varepsilon Z$. Differentiating with respect to $\varepsilon$,
\[
\Delta' (\varepsilon) = \int_0^P [\Phi_M Z + \Phi_m z] \, dp
\]
where $\Phi_M \equiv \partial \Phi / \partial M$ and $\Phi_m \equiv \partial \Phi / \partial m$. Integrating by parts,
\[
\int_0^P \Phi_m z \, dp = \Phi_m Z \bigg|_0^P - \int_0^P \frac{d}{dp} (\Phi_m) \, Z \, dp = - \int_0^P \frac{d}{dp} (\Phi_m) \, Z \, dp
\]
since $Z(0) = Z(\bar{p}) = 0$.
Thus we obtain,
\[
\Delta' (0) = \int_0^P \left[ \Phi_M - \frac{d}{dp} (\Phi_m) \right] Z \, dp
\]  
(17)

**Notation** In the proof below it is convenient to reformulate the problem in terms of the *decumulative* distribution functions. Let
\[
H_1 (p) \equiv 1 - F_1 (p), \quad H_2 (p) \equiv 1 - F_2 (p) \quad \text{and} \quad H (p) \equiv 1 - F (p)
\]
and
\[
h_1 (p) = H'_1 (p), \quad h_2 (p) = H'_2 (p) \quad \text{and} \quad h (p) = H' (p)
\]
Notice that in terms of decumulative functions, (10) can be rewritten as: if $H_i (p) \geq H_j (p)$, then
\[
H (p) = H_j \left( p - \frac{H (p) - H_i (p)}{h_i (p)} \right)
\]
or equivalently,
\[
H (p) = H_i (p) + \left[ p - H_j^{-1} (H (p)) \right] h_i (p)
\]  
(18)
The regularity assumption; that is, the monotonicity of virtual values $p + (H_i (p) / h_i (p))$, is equivalent to
\[
2 h_i (p)^2 - H_i (p) h'_i (p) > 0
\]  
(19)

**Proof of Theorem 3.** In terms of the decumulative functions, the difference between the revenue from a FPAR (15) and the revenue from a SPAR (16) is
\[
\Delta = \int_0^P H (p) \, dp - \int_0^{\omega_2} H_i (x) H_j (x) \, dx
\]
Define the function $M : [0, \bar{p}] \rightarrow [0, \omega_2]$ as follows: if $H_i (p) \geq H_j (p)$, then
\[
M (p) = H_j^{-1} (H (p))
\]
$M$ is an increasing function satisfying $M (0) = 0$, $M (\bar{p}) = \omega_2$ and since for all $p$, $H_i (p) \geq H (p) \geq H_j (p)$ it is the case that $M (p) \leq p$. (See Figure 3.) Define
Figure 3: The Function $M$

$m(p) = M'(p)$ wherever this is well defined. By changing the variable of integration from $x \in [0, \omega_2]$ to $p = M^{-1}(x) \in [0, \bar{p}]$ in the second integral in the expression for $\Delta$, we obtain

$$\Delta = \int_0^\bar{p} (H(p))^2 \, dp - \int_0^\bar{p} H_i(M(p)) H_j(M(p)) m(p) \, dp$$

which, in turn, is directly obtained from (18).

Lemma 1 implies that any $H_i$ and $H_j$ uniquely determine an $H$ satisfying (18) and this in turn uniquely determines an $M$. Conversely, given an $H_i$ and an $M$, (20) determines a unique $H$ and this in turn determines a unique $H_j$ since $H_j(x) = H(M^{-1}(x))$.

Note that $H_i = H_j$ if and only if $M(p) = p$ for all $p$, and in that case, $\Delta = 0$. We will argue that, in general; that is, for all $H_i$ and $H_j$, $\Delta \geq 0$. 

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Consider the integrand in the expression for $\Delta$, that is, the function\(^9\)
\[
\Phi(p, M, m) = H^2 - H_i(M) H m
\]
and the perturbation $Z(p) \equiv p - M(p) \geq 0$. Then define
\[
\Delta(\varepsilon) = \int_0^p \Phi(p, M + \varepsilon Z, m + \varepsilon z) \, dp
\]
to be the revenue difference when $M$ is perturbed slightly in the direction of $p$. (Figure 4 shows how a variation of $M$ in the direction of $p$ moves both $H_j$ and $H$ closer to $H_i$; that is, in the direction of increased symmetry.)

We now use (17) to evaluate $\Delta'(0)$. For this we need
\[
\Phi_M = 2H \frac{\partial H}{\partial M} - h_i(M) H m - H_i(M) \frac{\partial H}{\partial M} m
\]
\[
= -2H h_i - h_i(M) H m + H_i(M) h_i m
\]
using $\partial H / \partial M = -h_i$, derived from (20).

\(^9\)To ease the notational burden, we suppress the argument $p$ for the remainder of the proof.
And since,
\[ m = H_i(M) \]
we have
\[ \frac{d}{dp} (\Phi_m) = -h_i(M) H m - H_i(M) h \]  
\[ (22) \]

Now (21) and (22) result in
\[ \Phi_M - \frac{d}{dp} (\Phi_m) = -2 H h_i + H_i(M) h_i m + H_i(M) h \]
\[ = -2 H h_i + H_i(M) h_i m + H_i(M) [2 h_i + p - M] h_i' - h_i m \]
\[ = 2 [H_i(M) - H] h_i + H_i(M) [p - M] h_i' \]
\[ = H_i(M) \left( 2 \left[ 1 - \frac{H}{H_i(M)} \right] h_i - \left[ \frac{H - H_i}{h_i} \right] h_i' \right) \]

where the second equality is obtained by substituting \( h = 2 h_i + [p - M] h_i' - h_i m \), which is derived from (20). The fourth equality is obtained by substituting \( [p - M] = - (H_i - H) / h_i \), again from (20).

But since \( H_i \) is decreasing and \( M(p) \leq p \), we have \( H_i(M(p)) \geq H_i(p) \) and using the fact that \( h_i < 0 \),
\[ \Phi_M - \frac{d}{dp} (\Phi_m) \leq H_i(M) \left( 2 \left[ 1 - \frac{H}{H_i(M)} \right] h_i - \left[ \frac{H - H_i}{h_i} \right] h_i' \right) \]
\[ = H_i(M) \left[ \frac{H_i - H}{H_i h_i} \right] (2 h_i^2 - H_i h_i') \]

Since \( F_i \) is regular, \( 2 h_i^2 - H_i h_i' > 0 \), as in (19). Together with \( H_i > H \) and \( h_i < 0 \), this implies
\[ \Phi_M - \frac{d}{dp} (\Phi_m) < 0 \]

whenever \( M(p) < p \).

If \( M(p) < p \) for all \( p \), then \( Z(p) = p - M(p) > 0 \), and so (17) now implies
\[ \Delta'(0) = \int_0^p \left[ \Phi_M - \frac{d}{dp} (\Phi_m) \right] [p - M] dp < 0 \]

We have shown that given any \( M \) such that \( M(p) < p \) for all \( p \), a small perturbation \((1 - \varepsilon) M(p) + \varepsilon p\) in the direction of \( p \) always decreases \( \Delta \). Thus, \( \Delta \) is minimized at \( M = p \). But when \( M(p) = p \) for all \( p \), \( H_i = H_j \) and this means that \( F_1 = F_2 \), and in that case, the revenue equivalence principle implies that \( \Delta = 0 \). We have thus shown that for all \( F_1 \) and \( F_2 \) that are regular, \( \Delta \geq 0 \).

This completes the proof of Theorem 3.
Remark 2 The proof of Theorem 3 actually shows not only that the difference in revenues between the two auctions is nonnegative but moreover that it is, in fact, increasing in the degree of asymmetry. Suppose that \( F_1 \leq F_2 \) so that \( i = 1 \) and \( j = 2 \) holds everywhere. For fixed \( F_1 \), if \( F_2 \) is perturbed in the direction of \( F_1 \), thereby bringing the situation closer to symmetry, then the difference in revenues decreases.

Remark 3 Can Theorem 3 be strengthened so that its conclusion holds for all equilibria (not necessarily robust) of the second-price auction? The answer is no. With resale, there may exist inefficient equilibria of the second-price auction which are not robust, but result in a higher expected revenue than the first-price auction. An example is available from the authors.

6 Other Resale Mechanisms

In our analysis of asymmetric auctions with resale, we assumed that post-auction trade took place via a take-it-or-leave-it offer from the winner of the auction. Since all bargaining power resides in the hands of the seller, it is natural to refer to this as the monopoly mechanism.

In this section we show that our results are quite robust. We first show that they continue to hold if post-auction trade takes place via a take-it-or-leave-it offer from the loser of the auction. Now all bargaining power lies with the buyer and so by analogy, we will refer to this as the monopsony mechanism.

We then extend our results to a class of mechanisms in which the bargaining power is shared, perhaps unequally. Specifically, we consider a mechanism in which with probability \( k \), the seller makes a take-it-or-leave-it offer and with probability \( 1 - k \), the buyer makes a take-it-or-leave-it offer. We refer to this as the (probabilistic) \( k \)-double auction.\(^{10}\)

Recall that in our basic model, at the end of the auction, only the winning bid was announced. When resale takes place via the monopoly mechanism, this is the same as if no bid were announced. This is because information about the winner’s bid is irrelevant to the other bidder—he faces a take-it-or-leave-it offer from the winner. In other resale mechanisms, say when the buyer makes a take-it-or-leave-it offer, information regarding the winner’s value is no longer irrelevant—for instance, if the winning bid revealed the winner’s value, then the loser could extract all surplus from the winner during resale. It can be shown that if the winning bid is announced when resale takes place via monopsony, then there is no monotonic equilibrium.

In the extensions of the basic model that follow, we assume that no bids are announced at the end of the auction. Thus only the identity of the winner is commonly known.

\(^{10}\)The term \( k \)-double auction usually refers to a situation in which the price is a weighted average of the price demanded by the seller and that offered by the buyer. This mechanism is known to have a continuum of equilibria and so is not useful for our analysis.
6.1 Monopsony Resale

In this case, the loser of the auction can make an offer to buy the object from the winner.

Suppose that the two bidders follow continuous and strictly increasing inverse bidding strategies \( \phi_1 \) and \( \phi_2 \), respectively.

Suppose bidder \( i \) with value \( x_i \) bids \( b \) and loses the auction. He infers that bidder \( j \)'s value \( X_j \geq \phi_j (b) \). If \( x_i > \phi_j (b) \), there are potential gains from trade. Let \( r_i (b, x_i) \) be the optimal (monopsony) price set by bidder \( i \) with value \( x_i \) when he bids \( b \) and loses. This results in a resale profits of

\[
S_i (b, x_i) = \max_r \left[ F_j (r) - F_j (\phi_j (b)) \right] (x_i - r) \tag{23}
\]

and the optimal price \( r_i (b, x_i) \) must satisfy the first-order condition

\[
r - \frac{F_j (\phi_j (b)) - F_j (r)}{f_j (r)} = x_i \tag{24}
\]

Regularity again implies that the left-hand side is increasing and so there is a unique solution.

Notice that the monopsony pricing formula in (24) is the same as the monopoly pricing formula in (1). The only difference is that now the optimal price \( r_i (b, x_i) < x_i \) Notice also that the envelope theorem implies

\[
\frac{\partial}{\partial b} S_i (b, x_i) = -f_j (\phi_j (b)) \phi_j' (b) (x_i - r_i (b, x_i))
\]

Now fix a \( b \) satisfying \( \phi_j (b) < \phi_i (b) \). Suppose bidder \( j \) follows \( \phi_j \). Bidder \( i \)'s expected payoff when his value is \( x_i = \phi_i (b) \) and he deviates by bidding a \( c \) close to \( b \) is

\[
\Pi_i (c, x_i) = F_j (\phi_j (c)) (x_i - c) + S_i (c, x_i)
\]

where \( S_i (c, x_i) \) as defined above, is his expected payoff from resale if he loses the auction. Maximizing \( \Pi_i \), using the envelope theorem to evaluate the derivative of \( S_i (c, x_i) \) and writing \( r (b) \equiv r_i (b, \phi_i (b)) \) leads to

\[
\frac{d}{db} \ln F_j (\phi_j (b)) = \frac{1}{r (b) - b}
\]

Now consider bidder \( j \). A bid of \( c \) satisfying \( \phi_j (c) < \phi_i (c) \) when his value is \( x_j = \phi_j (b) \) results in an expected payoff of

\[
\Pi_j (c, x_j) = F_i (z_i) x_j + \int_{z_i}^{\phi_i (c)} r (\beta_i (x_i)) f_i (x_i) dx_i - F_i (\phi_i (c)) c
\]
where $z_i$ is highest type of bidder $i$ whose offer is accepted, that is, $r(\beta_i(z_i)) = x_j$. Since it is optimal for $j$ to bid $b$, the first-order condition for maximizing $\Pi_j$ results in the same differential equation as above.

Thus, just as (6), (7) and (8) characterize the equilibrium when resale is via the monopoly mechanism, the following characterize the equilibrium bidding when resale is via the monopsony mechanism.

\[
\frac{d}{db} \ln F_1(\phi_1(b)) = \frac{1}{r(b) - b} \tag{25}
\]

\[
\frac{d}{db} \ln F_2(\phi_2(b)) = \frac{1}{r(b) - b} \tag{26}
\]

where if $\phi_j(b) \leq \phi_i(b)$,

\[
r(b) = \arg \max_r \left[ F_j(r) - F_j(\phi_j(b)) \right] (\phi_i(b) - r) \tag{27}
\]

It should be noted that although the formulae defining the two are similar, in general, $r(b) \neq p(b)$. This implies of course that the equilibrium bidding strategies when resale is via the monopsony mechanism—that is, the solutions to (25) and (26)—are different from the equilibrium bidding strategies when resale is via the monopoly mechanism. So that there is no ambiguity, let us denote by $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ the equilibrium inverse bidding strategies in a first-price auction in which resale is via monopsony.

The remainder of the analysis parallels that in the case of monopoly exactly once the monopoly pricing function $p(b)$ is replaced by the monopsony pricing function $r(b)$. Specifically,

1. As in Theorem 1, the differential equations (25) and (26), together with (27), are both necessary and sufficient for $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ to be an equilibrium.

2. As in Lemma 1, $F_1$ and $F_2$ uniquely determine a distribution $G$ of monopsony resale prices where if $F_i(r) \leq F_j(r)$, then

\[
G(r) = F_i \left( r - \frac{G(r) - F_j(r)}{f_j(r)} \right) \tag{28}
\]

The distribution $G$ has a geometric interpretation similar to that of $F$ in Figure 2. In general, $G \neq F$.

3. As in Theorem 1, there exists an equilibrium with strictly increasing bidding strategies $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ and it is unique in the class of all nondecreasing equilibria.

4. As in Theorem 3, the revenue from a first-price auction with monopsony resale is at least as large as that from a second-price auction with resale.
6.2 Probabilistic $k$-Double Auctions

In this mechanism, resale takes place as follows. With probability $k$, the winner of the auction makes a take-it-or-leave-it offer to the loser and with probability $1-k$ the loser makes a take-it-or-leave-it offer to the winner. Either side may decide not to participate, in which case no transaction takes place.

When $k = 1$, this reduces to the monopoly resale mechanism considered earlier. When $k = 0$, it reduces to the monopsony mechanism of the previous subsection.

Again, fix a $b$ satisfying $\phi_j(b) < \phi_i(b)$. Suppose bidder $j$ follows $\phi_j$. Bidder $i$’s expected payoff when his value is $x_i = \phi_i(b)$ and he deviates by bidding a $c$ close to $b$ is

$$\Pi_i(c, x_i) = (x_i - c) F_j(\phi_j(c)) + k \int_{\phi_j(c)}^{\omega_j} \max\{x_i - p(\beta_j(x_j)), 0\} f_j(x_j) \, dx_j + (1 - k) S_i(c, x_i)$$

This is because if $i$ loses, with probability $k$ bidder $j$ will offer to sell him the object at price $p(\beta_j(x_j))$. With probability $1-k$, he will offer to buy from the other bidder and obtain a profit of $S_i(c, x_i)$, as defined in (23). Differentiating with respect to $c$ and using the equilibrium condition $x_i = \phi_i(b)$ results in

$$(kp(b) + (1-k) r(b) - b) f_j(\phi_j(b)) \phi'_j(b) - F_j(\phi_j(b)) = 0$$

If we write $s(b) = kp(b) + (1-k) r(b)$, then

$$\frac{d}{db} \ln F_j(\phi_j(b)) = \frac{1}{s(b) - b}$$

Now consider bidder $j$. A bid of $c$ satisfying $\phi_j(c) < \phi_i(c)$ when his value is $x_j = \phi_j(b)$ results in an expected payoff of

$$\Pi_j(c, x_j) = k R_j(c, x_j) + (1 - k) \int_{z_i}^{\phi_i(c)} r(\beta_i(x_i)) f_i(x_i) \, dx_i - F_i(\phi_i(c)) c$$

where $z_i$ is lowest type of bidder $i$ whose offer is accepted, that is, $r(\beta_i(z_i)) = x_j$. Differentiating, and using the equilibrium condition $x_j = \phi_j(b)$, leads to

$$[kp(b) + (1-k) r(b)] f_i(\phi_i(b)) \phi'_i(b) - b f_i(\phi_i(b)) \phi'_i(b) - F_i(\phi_i(b)) = 0$$

which again results in

$$\frac{d}{db} \ln F_i(\phi_i(b)) = \frac{1}{s(b) - b}$$

Thus the equilibrium when resale is via a $k$-double auction is characterized by

$$\frac{d}{db} \ln F_1(\phi_1(b)) = \frac{1}{s(b) - b}$$

$$\frac{d}{db} \ln F_2(\phi_2(b)) = \frac{1}{s(b) - b}$$

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where \( s(b) = kp(b) + (1-k)r(b) \) and \( p(b) \) and \( r(b) \) are given in (8) and (27), respectively.

Once again, in general, \( p(b) \neq r(b) \) and so for \( k \in (0,1) \), \( s(b) \) is distinct from both \( p(b) \) and \( r(b) \). Thus the equilibrium bidding strategies are now different from both those in the case of monopoly resale and those in the case of monopsony resale. This in turn implies that the pricing functions \( p(b) \) and \( r(b) \) in the case of a \( k \)-double auction are also different from those resulting in the case of a pure monopoly or a pure monopsony.

Define \( \hat{\varphi}_1, \hat{\varphi}_2 \) to be the equilibrium inverse bidding strategies in a first-price auction in which resale is via the \( k \)-double auction. Similarly, define \( \hat{p}(b) \) and \( \hat{r}(b) \) to be the monopoly and monopsony pricing functions in the \( k \)-double auction. Thus \( \hat{\varphi}_1, \hat{\varphi}_2, \hat{p}, \hat{r} \) are the simultaneous solutions to the two differential equations above together with (8) and (27). For notational consistency, let \( \hat{s}(b) = k\hat{p}(b) + (1-k)\hat{r}(b) \).

### 6.2.1 Distribution of \( \hat{s}(b) \)

We wish to determine the distribution of the random variable \( \hat{s}(b) \). As before, the differential equations imply once again that the distributions of bids of the two bidders are identical; that is, \( F_i(\hat{\varphi}_1(b)) = F_j(\hat{\varphi}_2(b)) \). Let \( A(b) \equiv F_j(\hat{\varphi}_j(b)) \) be the common distribution of bids.

Assume, without loss of generality, that \( \hat{\varphi}_j(b) < \hat{\varphi}_i(b) \) and notice that \( \hat{p}(b), \hat{r}(b) \) and \( \hat{s}(b) \) satisfy:

\[
\begin{align*}
\hat{\varphi}_j(b) &= \hat{p}(b) - \frac{F_i(\hat{\varphi}_i(b)) - F_i(\hat{p}(b))}{f_i(\hat{p}(b))} \\
\hat{\varphi}_i(b) &= \hat{r}(b) - \frac{F_j(\hat{\varphi}_j(b)) - F_j(\hat{r}(b))}{f_j(\hat{r}(b))} \\
\hat{s}(b) &= k\hat{p}(b) + (1-k)\hat{r}(b)
\end{align*}
\]

Define distributions \( \hat{F}, \hat{G} \) and \( L \) as follows

\[
\hat{F}(\hat{p}(b)) = \hat{G}(\hat{r}(b)) = L(\hat{s}(b)) = A(b)
\]

\( F, G \) and \( L \) are well defined since \( \hat{p}(\cdot), \hat{r}(\cdot) \) and \( \hat{s}(\cdot) \) are increasing functions.

Note that (29) and (30) then can be rewritten as follows:

\[
\begin{align*}
\hat{F}(p) &= F_j\left(p - \frac{\hat{F}(p) - F_i(p)}{f_i(p)}\right) \\
\hat{G}(r) &= F_i\left(r - \frac{\hat{G}(r) - F_j(r)}{f_j(r)}\right)
\end{align*}
\]

But now notice that since (33) is the same as (10) and \( F \) was uniquely determined there, \( \hat{F} = F \), the distribution of prices when resale is via monopoly. Similarly,
is the same as (28) and so \( \hat{G} = G \), the distribution of prices when resale is via monopsony. We thus obtain the conclusion that even though, in general, \( \hat{p} \neq p \) and \( \hat{p}_i \neq \hat{\phi}_i \), we have

\[
F(\hat{p}(b)) = F_{i}(\hat{\phi}_i(b)) \text{ and } F(p(b)) = F_{i}(\phi_i(b))
\]

Similarly, even though, in general, \( \hat{r} \neq r \) and \( \hat{p}_i \neq \hat{\phi}_i \),

\[
G(\hat{r}(b)) = F_{i}(\hat{\phi}_i(b)) \text{ and } G(r(b)) = F_{i}(\phi_i(b))
\]

Finally, since \( F = \hat{F} \) and \( G = \hat{G} \), (32) implies that for all \( b \),

\[
L^{-1}(A(b)) = kF^{-1}(A(b)) + (1 - k)G^{-1}(A(b))
\]

Since \( A(b) \) varies from 0 to 1, we obtain

\[
L^{-1}(q) = kF^{-1}(q) + (1 - k)G^{-1}(q)
\]

for all \( q \in [0, 1] \), where \( F \) and \( G \) satisfy (33) and (34), respectively.

### 6.2.2 Revenue from FPA with Resale via \( k \)-Double Auction

**Lemma 2**

\[
\int_{0}^{\bar{s}} (1 - L(s))^2 \, ds = k \int_{0}^{\bar{p}} (1 - F(p))^2 \, dp + (1 - k) \int_{0}^{\bar{r}} (1 - G(r))^2 \, dr
\]

**Proof.** Let \( L(s) = q \), then from (35) we obtain

\[
s = L^{-1}(q) = kF^{-1}(q) + (1 - k)G^{-1}(q)
\]

so that

\[
ds = \left( \frac{k}{f(F^{-1}(q))} + \frac{1 - k}{g(G^{-1}(q))} \right) dq
\]

By changing the variable of integration from \( s \in [0, \bar{s}] \) to \( q = L(s) \in [0, 1] \) we obtain

\[
\int_{0}^{\bar{s}} (1 - L(s))^2 \, ds = \int_{0}^{1} (1 - q)^2 \left( \frac{k}{f(F^{-1}(q))} + \frac{1 - k}{g(G^{-1}(q))} \right) \, dq
\]

\[
= k \int_{0}^{1} \frac{(1 - q)^2}{f(F^{-1}(q))} \, dq + (1 - k) \int_{0}^{1} \frac{(1 - q)^2}{g(G^{-1}(q))} \, dq
\]

Changing the variables again from \( q \in [0, 1] \) to \( p = F^{-1}(q) \in [0, \bar{p}] \) in the first integral and \( r = G^{-1}(q) \in [0, \bar{r}] \) in the second, we obtain the required equality.

The expected revenue of the original seller when resale is via the \( k \)-double auction is given by

\[
\int_{0}^{\bar{s}} (1 - L(s))^2 \, ds
\]
where $L$ satisfies (35).

We have already shown in Theorem 3 that

$$\int_{p}^{p} (1 - F(p))^2 dp \geq \int_{0}^{\omega^2} (1 - F_1(x))(1 - F_2(x)) dx$$

and it is similarly the case that

$$\int_{r}^{r} (1 - G(r))^2 dr \geq \int_{0}^{\omega^2} (1 - F_1(x))(1 - F_2(x)) dx$$

Lemma 2 now implies that the revenue from a first-price auction with resale via the $k$-double auction is also greater than or equal to the revenue from a second-price auction.

The following theorem extends our main result.

**Theorem 4** The seller’s revenue from a first-price auction with resale via a probabilistic $k$-double auction is at least as great as that from a second-price auction.

Theorem 4 subsumes the case of monopoly (when $k = 1$) and the case of monopsony (when $k = 0$). The reader may thus wonder why we did not establish only the more general result. The proof of Theorem 4, however, makes use of both Theorem 3 and its counterpart for the monopsony mechanism.

Finally, one may well ask whether the revenue ranking is valid for arbitrary (balanced-budget) resale mechanisms. In other words, suppose that the resale mechanism is of the direct form $(Q, T)$, where $Q(z_b, z_s)$ is the probability that the object is transferred from the seller to the buyer when $(z_b, z_s)$ are the values reported by the two parties and $T(z_b, z_s)$ is the payment from the buyer to the seller. As usual, we ask that the mechanism be incentive compatible and individually rational. Does Theorem 4 extend to this general specification? Clearly, the answer is no. For instance, the result does not hold for the “no-trade” mechanism ($Q = 0$ and $T = 0$) that, in effect, bans resale—as Vickrey’s (1961) example shows. This suggests that the revenue ranking to hold, the probability of a resale transaction should not be “too low.”

**7 Conclusion**

We have shown that a consideration of resale possibilities allows for a simpler characterization of equilibrium strategies in first-price auctions than available when resale is not admitted. In our model, equilibrium strategies can be explicitly computed in a relatively simple manner as in the proof of Theorem 1. Moreover, we obtain a general revenue ranking result between first- and second-price auctions that is not available in the standard model. Thus this appears to be one of those happy circumstances
where complicating the model with a real-world feature—resale—actually simplifies the analysis.

In this paper, we have restricted attention to the case of two bidders. Considering resale when there are three or more bidders poses some conceptual difficulties.\footnote{Zheng (2002) also finds that when there are three or more bidders, Myerson’s optimal auction is robust to resale only under stringent conditions (see Mylovanov and Tröger (2005)).} There are simply too many modelling choices and the challenge is to model resale in a way that is both realistic and analytically tractable. To fix ideas, consider a first-price auction with three bidders and suppose that bidder 3 wins the auction. At the resale stage, among the many available options are:

1. Bidder 3 invites both bidders 1 and 2 to participate in an auction.

2. Bidder 3 approaches one of the bidders, say bidder 1, and makes a take-it-or-leave-it offer.

The results of this paper have a direct bearing on any analysis of the first option. This is because the resale auction involves only two bidders. A model with these features could, however, be legitimately criticized on the grounds that most resale transactions are bilateral—we simply do not observe multiple rounds of auctions with the winner in each round holding an auction for the remaining bidders. But it is not clear why the bilateral transaction should consist of only one take-it-or-leave-it offer. Bidder 3 could approach bidder 1 and if his offer is refused, then approach bidder 2. And this raises the question of the order in which bidders 1 and 2 should be approached. We hope to consider some of these issues in future work.

A Appendix: Proofs from Section 3

This appendix contains proofs of results on first-price auctions with resale.

A.1 Proof of Proposition 1

\textbf{Proof.} The necessity of (6) and (7) for all \(b \in [0, b^\hat{b}]\) has already been shown. It remains to show that these are sufficient.

Suppose bidder \(j\) follows the equilibrium inverse bidding strategy \(\phi_j\). We will argue that when bidder \(i\) has a value of \(x_i\), he cannot do better than to bid \(b\ such that \(\phi_i(b) = x_i\). We do this by showing that neither underbidding nor overbidding can be profitable.

Notice that the differential equations can be rewritten as: for \(j = 1, 2\)

\[
(p(b) - b)f_j(\phi_j(b))\phi_j'(b) - F_j(\phi_j(b)) = 0 \tag{36}
\]

\textbf{Case A (Underbidding):} Suppose bidder \(i\) bids \(b\ such that \(\phi_i(b) < x_i\).
Case A1: $\phi_j(b) < \phi_i(b) < x_i$. If $i$ wins the auction with a bid of $b$, then his payoff is simply $(x_i - b)$ since there are no benefits to reselling. If $i$ loses, however, $j$ will offer to sell the object to him for a price of $p(\beta_j(x_j))$ and so $i$’s payoff is $\max \{x_i - p(\beta_j(x_j)), 0 \}$. Thus $i$’s expected payoff is

$$
\Pi_i (b, x_i) = (x_i - b) F_j \left( \phi_j (b) \right) + \int_{\phi_j(b)}^{x_j} \max \{x_i - p \left( \beta_j (x_j) \right), 0 \} f_j (x_j) \, dx_j
$$

Differentiating with respect to $b$ and using (36), results in

$$
\frac{\partial \Pi_i}{\partial b} = (p(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) = 0
$$

Case A2: $\phi_i(b) \leq \phi_j(b) < x_i$. If $i$ wins the auction with a bid of $b$, then his payoff is simply $(x_i - b)$ since again there are no benefits to reselling. Similarly, if $i$ loses, bidder $j$ will not offer to sell the object to him since from $j$’s perspective, there appear to be no benefits to selling to $i$. Thus $i$’s expected payoff is simply

$$
\Pi_i (b, x_i) = (x_i - b) F_j \left( \phi_j (b) \right)
$$

and so again by using (36),

$$
\frac{\partial \Pi_i}{\partial b} = (x_i - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b))
\geq (p(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b))
\geq (p(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b))
= 0
$$

Case A3: $\phi_i(b) < x_i \leq \phi_j(b)$. If $i$ wins the auction with a bid of $b$, then he may resell it to bidder $j$ since again there are potential gains from trade. His expected payoff from winning is

$$
R_i(b, x_i) = \max[F_j(\phi_j(b)) - F_j(p)]p + F_j(p)x_i
$$

If $i$ loses, bidder $j$ will not offer to sell the object to him since from $j$’s perspective, there appear to be no gains from trade. Thus $i$’s expected payoff from bidding $b$ is

$$
\Pi_i(b, x_i) = R_i(b, x_i) - F_j \left( \phi_j (b) \right) b
$$

and using the envelope theorem as in (4) and the fact that $p_i(b, x_i) \geq p_i(b, \phi_i(b)) \equiv p(b),

$$
\frac{\partial \Pi_i}{\partial b} = (p_i(b, x_i) - b) f_j \left( \phi_j (b) \right) \phi_j'(b) - F_j \left( \phi_j (b) \right)
\geq (p(b) - b) f_j \left( \phi_j (b) \right) \phi_j'(b) - F_j \left( \phi_j (b) \right)
= 0
$$
CASE B (OVERBIDDING): Suppose bidder $i$ bids $b$ such that $x_i < \phi_i(b)$.

CASE B1: $\phi_j(b) < x_i < \phi_i(b)$. If $i$ wins the auction with a bid of $b$, then his payoff is simply $(x_i - b)$ since there is no benefit from reselling to $j$. On the other hand, if $i$ loses, $j$ will offer to sell the object to him for a price of $p(\beta_j(x_j))$ and so $i$’s payoff if he loses is $\max\{x_i - p(\beta_j(x_j)), 0\}$. Thus $i$’s expected payoff from bidding $b$ is

$$\Pi_i(b, x_i) = (x_i - b) F_j(\phi_j(b)) + \int_{\phi_j(b)}^{\omega_j} \max\{x_i - p(\beta_j(x_j)), 0\} f_j(x_j) \, dx_j$$

Differentiating with respect to $b$,

$$\frac{\partial \Pi_i}{\partial b} = (x_i - b) f_j(\phi_j(b)) \phi'_j(b) - F_j(\phi_j(b)) - \max\{x_i - p(b), 0\} f_j(\phi_j(b)) \phi'_j(b)$$

$$\leq (x_i - b) f_j(\phi_j(b)) \phi'_j(b) - F_j(\phi_j(b)) - (x_i - p(b)) f_j(\phi_j(b)) \phi'_j(b)$$

$$= (p(b) - b) f_j(\phi_j(b)) \phi'_j(b) - F_j(\phi_j(b))$$

$$= 0$$

CASE B2: $x_i \leq \phi_j(b) < \phi_i(b)$. If $i$ wins the auction with a bid of $b$, then he may resell it to bidder $j$ since again there are potential gains from trade. If he loses, bidder $j$ will offer to sell the object to him for a price of $p(\beta_j(x_j))$ but this price will always exceed $x_i$ and so $i$ will refuse the offer. Thus $i$’s expected payoff from bidding $b$ is just

$$\Pi_i(b, x_i) = R_i(b, x_i) - F_j(\phi_j(b)) b$$

and again using (4) and the fact that $p_i(b, x_i) \leq \phi_j(b) \leq p_j(b, \phi_j(b)) \equiv p(b)$,

$$\frac{\partial \Pi_i}{\partial b} = (p_i(b, x_i) - b) f_j(\phi_j(b)) \phi'_j(b) - F_j(\phi_j(b))$$

$$\leq (p(b) - b) f_j(\phi_j(b)) \phi'_j(b) - F_j(\phi_j(b))$$

$$= 0$$

CASE B3: $x_i < \phi_i(b) \leq \phi_j(b)$. If $i$ wins the auction with a bid of $b$, then he may resell it to bidder $j$ since again there are potential gains from trade. His expected payoff from winning is the monopoly profit $R_i(b, x_i)$. If he loses, bidder $j$ will not offer to sell the object to him since from $j$’s perspective, there appear to be no gains from trade. Thus $i$’s expected payoff from bidding $b$ is again

$$\Pi_i(b, x_i) = R_i(b, x_i) - F_j(\phi_j(b)) b$$

and the argument is the same as in Case B2, except that now $p_i(b, x_i) \leq p_i(b, \phi_i(b)) \equiv p(b)$.

We have thus argued that for all $b$ such that $\phi_i(b) < x_i$, $\frac{\partial \Pi_i}{\partial b} \geq 0$ and for all $b$ such that $\phi_i(b) > x_i$, $\frac{\partial \Pi_i}{\partial b} \leq 0$. Thus bidding a $b$ such that $\phi_i(b) = x_i$ is a best response to $\phi_j$. ■
A.2 Proof of Lemma 1

Proof. Fix a \( p \) and note that

\[
\Psi(p, q) = F_j \left( p - \frac{q - F_i(p)}{f_i(p)} \right)
\]

is a strictly decreasing function of \( q \), where \( F_i(p) \leq q \leq 1 \).

If \( F_i(p) = F_j(p) \), then \( F(p) = F_i(p) \) also. If \( F_i(p) < F_j(p) \), then \( \Psi(p, F_i(p)) > F_i(p) \) and \( \Psi(p, F_j(p)) < F_j(p) \). Thus for every \( p \), there exists a unique fixed-point \( q \in (F_i(p), F_j(p)) \) such that \( \Psi(p, q) = q \) and by (10) \( F(p) \equiv q \). Thus, \( F_i(p) \leq F(p) \leq F_j(p) \) and the inequalities are strict if \( F_i(p) < F_j(p) \).

Clearly, \( F(0) = 0 \). We now argue that \( F \) is strictly increasing. It suffices to show that \( F(p') < F(p'') \) for \( p' < p'' \) such that for all \( p \in [p', p''] \), \( F_i(p) \leq F_j(p) \). The regularity of \( F_i \) (see Section 2) implies that for all \( q \),

\[
p'' - \frac{q - F_i(p'')}{f_i(p'')} > p' - \frac{q - F_i(p')}{f_i(p')}
\]

Thus, if \( p' < p'' \), then for all \( q \), \( \Psi(p'', q) > \Psi(p', q) \). This implies that if \( \Psi(p', q') = q' \) and \( \Psi(p'', q'') = q'' \), then \( q'' > q' \). So \( F(p'') > F(p') \). Finally, note that since \( \omega_1 \geq \omega_2 \), \( F \) has support \([0, \overline{p}]\), where \( \overline{p} \) satisfies

\[
\omega_2 = \overline{p} - \frac{1 - F_1(p)}{f_1(p)}
\]

We have thus shown that \( F \) is a well-defined distribution function over \([0, \overline{p}]\). ■

A.3 Proof of Theorem 2

It has already been established that the equilibrium constructed in Theorem 1 is unique in the class of equilibria with strictly increasing strategies. Here we complete the proof Theorem 2 by showing that every equilibrium in nondecreasing strategies must, in fact, have strictly increasing bidding strategies.

Lemma 3 If \( \beta_1 \) and \( \beta_2 \) are nondecreasing equilibrium bidding strategies in the first-price auction with resale, then \( \beta_1 \) and \( \beta_2 \) are continuous.

Proof. Suppose that there exists an \( x_i > 0 \) such that \( \lim_{x \to x_i} \beta_j(x) = b' < b'' = \lim_{x \to x_i} \beta_i(x) \). First, note that in that case bidder \( j \) also does not bid between \( b' \) and \( b'' \); that is, there does not exist an \( x_j \) such that \( b' < \beta_j(x_j) < b'' \). Otherwise, bidder \( j \) with value \( x_j \) could increase his payoff by decreasing his bid to \( \beta_j(x_j) - \varepsilon > b' \). This change does not affect his payoff if he were to lose but increases it by \( \varepsilon \) if he were to win (which happens with positive probability since \( x_i > 0 \)). Second, note that bidder
\(j\) bids \(b'\) or lower with positive probability. Otherwise, \(\lim_{x \to 0} \beta_j (x) \geq b'\) and bidder \(j\) with value \(x_j\) close to zero can improve his payoff by reducing his bid to \(b'\). This is because bidder \(j\) would gain at least \(b'' - b'\) whenever bidder \(i\)'s value was between 0 and \(x_i\) and suffer only a small loss in the cases when bidder \(i\)'s value is just slightly above \(x_i\). So there does not exist an \(x_j\) such that \(b' < \beta_j (x_j) < b''\).

Now consider bidder \(i\) with a value slightly greater than \(x_i\), say \(x_i + \delta\). By reducing his bid from \(\beta_i (x_i + \delta) \geq b''\) to \(b'\), bidder \(i\) could increase his payoff. Once again, this change does not affect his payoff if he were to lose and increases it by at least \(b'' - b'\) if he were to win (which happens with positive probability). Thus for \(\delta\) small enough, bidder \(i\) with value \(x_i + \delta\) has a profitable deviation.

We have argued that the bidding strategies \(\beta_i\) are continuous at any \(x_i > 0\). It remains to argue that they are also continuous at 0.

Suppose that \(\lim_{x \to 0} \beta_i (x) = b_0 > 0\) and without loss of generality, suppose that for some small \(\delta\), it is the case that for all \(x \in (0, \delta)\), \(\beta_i (x) \geq \beta_j (x)\). Then we must have that \(\lim_{x \to 0} \beta_j (x) = b_0\) also. Otherwise, bidder \(i\) with a value close to zero could reduce his bid an improve his payoff. If \(\beta_i\) is increasing in \((0, \delta)\), then the same argument that shows that \(\beta_i (0) = 0\) when both strategies are increasing shows that this is impossible. If both \(\beta_i\) and \(\beta_j\) are constant over \((0, \delta)\), then bidder \(i\) with value close to zero can improve his payoff by increasing his bid slightly. Thus \(\lim_{x \to 0} \beta_i (x) = 0\).

**Lemma 4** If \(\beta_1\) and \(\beta_2\) are nondecreasing equilibrium bidding strategies in the first-price auction with resale, then \(\beta_1\) and \(\beta_2\) are strictly increasing.

**Proof.** Suppose that there is an interval \([x', x'']\) such that for all \(x_i \in (x', x'')\), \(\beta_i (x_i) = b > 0\); that is, bidder \(i\)’s strategy is constant. Consider bidder \(j\) with value \(x_j\) such that \(\lim_{x \to x_j} \beta_j (x) = b\). For \(x\) close to \(x_j\), bidder \(j\) can improve his payoff by bidding higher than \(b\). This is because he then wins whenever \(X_i \in (x', x'')\) and the loss is arbitrarily small. If \(b = 0\), then bidder \(j\) with a small value \(x_j\) can improve his payoff by reducing his bid.

### B Appendix: Proofs from Section 4

#### B.1 Proof of Proposition 4

The proof that there is a unique robust equilibrium in the second-price auction with resale follows from the lemmas below.

We suppose that equilibrium bidding strategies are right continuous. Since the expected payoff functions of the bidders are continuous in equilibrium, if there is a discontinuity in the bidding strategies, bidders must be indifferent between bidding the right and left limits. Therefore, focusing on equilibria with right-continuous bidding strategies is without loss of generality.
Consider a robust equilibrium of the SPAR: \((\beta_1(\cdot), p_1(\cdot, \cdot), \beta_2(\cdot), p_2(\cdot, \cdot))\).

**Lemma 5** For all \(x_i\) and for all \(b \leq \beta_i(x_i)\), the set \(\{x_j : \beta_j(x_j) = b \text{ and } x_j > x_i\}\) is either a singleton or empty.

**Proof.** Suppose there are two points, say \(x_j'\) and \(x_j''\) are in the set \(\{x_j : \beta_j(x_j) = b \text{ and } x_j > x_i\}\). Upon winning, the optimal price \(p_i(b, x_i)\) that \(i\) will set must then depend on the distribution \(F_j\), contradicting the definition of a robust equilibrium.

A simple but important consequence of Lemma 5 is that if \(j\) loses in the auction, his payoff must be zero. Either bidder \(i\) makes no offer to him or makes an offer equal to \(x_j\).

**Lemma 6** \(\inf_{x_1} \beta_1(x_1) = \inf_{x_2} \beta_2(x_2)\)

**Proof.** Suppose \(\inf_{x_1} \beta_1(x_1) < \inf_{x_2} \beta_2(x_2)\). By right continuity, there is an open interval \(I\) of values \(X_1\) such that for all \(x_1 \in I\), \(\beta_1(x_1) < \inf_{x_2} \beta_2(x_2)\). Consider \(x_1', x_1'' \in I\) such that \(0 < x_1' < x_1''\). For all \(x_2 < x_1\), bidder 2 with value \(x_2\) wins against both \(x_1'\) and \(x_1''\) and from Lemma 5, offers to sell to both at prices equal to their values. But this means that bidder 1 with value \(x_1''\) is better off by bidding \(\beta_1(x_1') \neq \beta_1(x_1'')\).

**Lemma 7** For \(i = 1, 2\) there exists an \(x_i^0\) such that bidder \(i\) with value \(x_i^0\) makes an overall expected payoff of 0.

**Proof.** From Lemma 6, \(\inf_{x_1} \beta_1(x_1) = \inf_{x_2} \beta_2(x_2) = m\), say. Let \(x_i^0 = \inf\{x_i : \beta_i(x_i) = m\}\). Since \(\beta_i(0) \geq m\), from Lemma 5 there is at most one \(x_j\) such that \(\beta_j(x_j) = m\). This means that bidder \(i\) with value \(x_i^0\) wins the object with probability 0. Since the payoff from losing is also 0, his overall payoff is 0.

**Lemma 8** \(\beta_1(x) = \beta_2(x) = x, \text{ for all } x \in [0, \omega_2]\).

**Proof.** Suppose that \(\beta_j(x_j') < x_j'\) for some \(x_j'\). By right continuity, there exists a \(\delta\) such that for all \(x_j \in I = [x_j', x_j' + \delta]\), \(\beta_j(x_j) < x_j'\). Thus there exists an \(\varepsilon > 0\) such that \(x_j' - \beta_j(x_j) \geq \varepsilon\) for all \(x_j \in I\). Suppose now that bidder \(i\) with value \(x_i^0\) (whose overall profit is zero), bids \(x_j'\). Whenever bidder \(j\)'s value \(x_j \in I\), \(i\) will win and make a profit of at least \(\varepsilon\). Consider a distribution of values \(F_j\) such that \(\Pr[X_j \in I]\) is close enough to 1 so that the overall expected profit of \(x_i^0\) is also positive, which is a contradiction. So \(\beta_j(x_j') < x_j'\) is impossible.

Similarly, suppose \(\beta_j(x_j') > x_j'\) for some \(x_j'\). Again, by right continuity, there exists a \(\delta\) such that for all \(x_j \in I = [x_j', x_j' + \delta]\), \(\beta_j(x_j) > x_j' + \delta\). For all \(x_j \in I\), if \(\beta_j(x_j) > x_j' + \delta\), then bidder \(i\) with value \(x_i\) makes a negative profit when he faces bidder \(j\) with value \(X_j \in I\). Consider a distribution of values \(F_j\) such that \(\Pr[X_j \in I]\) close enough to 1 so that his overall expected profit is also negative,
which is a contradiction. This shows that for all $x_i \in I$, $\beta_i(x_i) \leq x_i' + \delta$. Now consider $x_i'$ and $x_i''$ such that $x_j < x_i' < x_i'' < x_i' + \delta$. Bidder $j$ with value $x_j$ such that $x_j < x_j < x_i' < x_i''$ wins against both $x_i'$ and $x_i''$ and from Lemma 5, offers to sell to both at prices equal to their values. But this means if the distribution of values $F_j$ is such that $Pr[x_j < x_j < x_i']$ is close to 1, then bidder $i$ with value $x_i''$ is better off by bidding $\beta_i(x_i') \neq \beta_i(x_i'')$.

References


