A Solution for Games with Inadequate Information

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Abstract

We propose a framework to analyze a certain family of games called games with inadequate information, where an outside observer of a game intends to predict the payoff outcome without knowing the players' interaction process. We define a parametric solution called the valuation, where the parameters in the solution characterize the belief the outside observer holds about how the game is played. We discuss the properties of this solution, especially in some special forms of games. We show that the valuation is usually a more appropriate solution to a game with inadequate information than some traditional solutions such as the Nash equilibrium.

1 Introduction

Suppose a group of people are playing a game. There is an outside observer who cares about the outcome of the game. The outside observer knows the basic physical structure of the game, including the set of players, whether the game is cooperative or non-cooperative, the actions each player can take, and the outcome that associates with each combination of actions taken by all players. However, since the outside observer does not participate in the game himself, he is ignorant of one important information that can affect the outcome of the game: the process of interactions. In other words, the outside observer does not know whether the

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game is static or sequential, and in the latter case, played in what sequence. We say that this is a game with inadequate information (GII for short).[1]

There exist lots of works in the literature that investigate games from the view of outside observers, rather than players. For example, models of implementation in complete information environments usually assume that the outside observer does not know all players' preference.[2] However, to the best of our knowledge, little has been done to systematically analyze a game whose process or procedure under which the players interact is unknown to the outside observer.

In reality, the situations of inadequate information are common and worth studying. For instance, suppose the government is considering whether or not to set up a new policy to regulate a certain industry. However, as an outside observer, the government does not know how the firms in this industry interact. Hence, an appropriate suggestion regarding this policy depends on the government's estimation of the impact of the policy on each firm under the circumstance of inadequate information.

The main purpose of this paper is to provide a framework to analyze GII, and to suggest a solution based on this framework so that the outside observer can use this solution to predict the expected payoff of each player. At first glance, it seems impossible to predict the outcome without knowing the process of a game. However, the outside observer can build some "beliefs" of the missing information about how the game is played by observing some performance of players, e.g. their relative "bargaining power" that reflects their distinctions in experience or wisdom. It turns out that these beliefs can help to solve GII, and an appropriate solution of GII should be consistent with the outside observer's observation.

Our basic idea is to solve GII using the bargaining approach, which can be illustrated in Figure 1. Roughly speaking, in this approach we introduce an underlying bargaining game for each original game with inadequate information. Although the outside observer does not know how the original game G is played, he believes

[1] We emphasize on the difference between inadequate information and some well known information issues, especially incomplete (asymmetric) information and imperfect information. "Incomplete information" describes a situation when one player does not know the type of another player, while "imperfect information" is about one player not knowing which action has been taken by another player at an earlier stage. In contrast, "inadequate information" involves a totally different kind of missing information.

[2] See, for instance, Mas-colell et al. (1995, Chapter 23, Appedix B.).

that the players are interacting through the underlying bargaining game $G'(\mu)$, which can be solved by some tradition equilibrium solutions, e.g. stationary subgame perfect equilibrium. The inadequate information is characterized by some parameters μ that appear in the underlying game and its equilibrium solution $x(\mu)$. The outside observer regards this parametric solution $x(\mu)$ as a solution to the original game.

$$\begin{tikzcd} G \arrow[r, "\mu"] \arrow[dr, dashed] & G'(\mu) \arrow[d] \\ & x(\mu) \end{tikzcd}$$

Figure 1: The bargaining approach for GII.

In fact, the outside observer does not know whether $G'(\mu)$ is the actual interaction procedure followed by the players. Nevertheless, $x(\mu)$ is still an acceptable solution as long as it can explain what is observed under some proper belief μ. In this paper (e.g. section 2), we shall present some examples to show that this is indeed the case, at least in some particular situations.

There is a large literature on the bargaining approach in game theory. Here we only mention a few that are most closely related to this paper. The reader may refer to Osborne and Rubinstein (1990), Ray (2007) and Serrano (2008) for some surveys on this topic.

Nash (1953) first suggests to explain a solution of a cooperative game (original game) as an equilibrium of a noncooperative game (underlying bargaining game). This research agenda is later known as the Nash program. Rubinstein (1982) considers a two-player bargaining game with alternating offers procedure and fixed discount factors. He proves that this game has a unique subgame perfect equilibrium outcome. A lot of works aims to extend Rubinstein's model to more general settings. For example, Gul (1989) discusses a bargaining among many players where in each stage two players first match by random and then bargain bilaterally. Chatterjee et al. (1993) proposes a multi-player bargaining game with common discount factor and a fixed order of proposers. Okada (1996) and Yan (2002) propose multi-player bargaining models that the proposer in the bargaining is randomly chosen. Bloch (1996), Ray and Vohra (1999) extend the model to partition function form games.

Alternately, Binmore et al. (1986) proposes a two-player bargaining model so that after an offer is rejected, instead of going to the next stage with payoffs discounted, there is a risk that the bargaining breakdowns and some player becomes inactive. It is usually more convenient to analyze games where payoffs are not necessarily positive with this model. Hart and Mas-colell (1996, 2010) extend this model to multi-player games under a variety of situations.

In this paper, we extend the bargaining approach, especially the models in Hart and Mas-colell (1996, 2010), to analyze games with inadequate information. Specifically, the outside observer believes that the players are bargaining over which actions to take through a bargaining procedure. This procedure is random in two aspects. In each round of the bargaining, one active player will be chosen to be the proposer of this round according to an exogenous probability. If the chosen player refuses to be proposer, or some player rejects the offer proposed by the chosen player, then there is a given chance that the chosen player becomes inactive in the remaining part of the bargaining. These two types of uncertainty is characterized by some parameters. The outside observer thinks that the values of these parameters are the only information he may not know about the game.

Given a game with inadequate information and some combination of parameters, we define a valuation of this game relative to these parameters to be a payoff vector that can be supported by an stationary subgame perfect equilibrium of the underlying bargaining game. The dependence of the valuation on the parameters can be used to investigate how this solution will be affected by outside observer's belief about the inadequate information.

Our model is general enough to explore GII in a variety of situations. However, due to the complexity of the model in the general setting, it is not easy to obtain important properties of the valuation, including its existence, uniqueness, efficiency, and how the solution depends on the parameters. For this reason in this paper we mainly discuss three special forms of games: two-player zero-sum game, two-player pure bargaining game, and three-player TU coalitional game.

In addition to exploring the above mentioned properties of the valuation in these games, we are especially interested in one question: whether the valuation is essentially different from, and even more suitable than, some traditional solutions such as the Nash equilibrium? This question is important because if the solutions are substantially different, then a situation with inadequate information should not be modeled as a traditional game (without inadequate information), since otherwise systematic errors may occur. In general, our analysis provides a positive answer to this question[3]. For example, the von Neumann (1928)'s value becomes not fully satisfactory when solving two-player zero-sum games from the viewpoint of outside observers, but this problem is fixed by the valuation. This justifies the method and the solution introduced in this paper.

This paper is organized as follows. Section 2 introduces our motivation by some simple examples. Section 3 and 4 develop a general framework to analyze GII and define a particular solution (the valuation). We briefly discuss the properties of this solution in section 5 and apply it to three special types of games in section 6. Section 7 concludes.

2 Introductory examples

In this section, we illustrate the basic idea and motivation through some simple examples.

The first example in Figure 2 is known as the Battle-of-sex (BOS) game played by two players in a family. The wife and the husband negotiate to determine whether to watch a dancing show (D) or a boxing match (B). The wife prefers D and the husband prefers B, while they both agree that the worst outcome is for them to take different actions. This game has three Nash equilibria: two are pure strategies, (D, D) and (B, B), and one mixed strategies.

		husband D	husband B
wife	D	$3, 1$	$0, 0$
wife	B	$0, 0$	$1, 3$

Figure 2: *BOS*

Suppose an outside observer of this game (for example, a neighbor of the family)

[3]There are exceptions, though. For instance, in section 6.2 we show that for a two-player pure bargaining game, the limit of valuation as $p \to 0$ is the weighted Nash bargaining solution.

has observed that the family would take (D, D) in about seven times out of every ten times, and (B, B) in the remaining three times. The neighbor believes that the wife has a larger "power" in this family, but he does not know the detailed process in which the family plays the BOS game. Given the limited information, how much payoff does the neighbor expect each player would get? Intuitively, the expected payoff vector should be $0.7 \times (3,1) + 0.3 \times (1,3) = (2.4, 1.6)$. Note that none of the three Nash equilibrium outcomes coincides with this outcome.

More generally, imagine a family in which the wife has probability α to first choose an action, which leads to payoff profile $H_1 = (3,1)$; while the husband has probability $1-\alpha$ to act first, leading to $H_2 = (1,3)$. Then the expected payoff profile for such a family should be $(u_1, u_2) = \alpha H_1 + (1-\alpha) H_2 = (1+2\alpha, 3-2\alpha)$. The parameter α actually characterizes the power of the wife relative to the husband, since this game clearly exhibits first mover advantage.

		player 2		
		R	P	S
	R	$0,0$	$-1,1$	$1,-1$
player 1	P	$1,-1$	$0,0$	$-1,1$
	S	$-1,1$	$1,-1$	$0,0$

Figure 3: *Rock, paper and scissors*

The second example in Figure 3 is usually referred to as Rock, paper and scissors (RPS). According to the Minmax theorem (von Neumann, 1928), each player would receive an expected payoff of zero. However, an outside observer of this game often observes that an experienced player may have a positive expected payoff when playing with a rookie, since the former has a larger probability of correctly guessing which action the latter takes.[4] Suppose from the outside observer's view, with probability α player 1 loses while with probability $1-\alpha$ player 1 wins[5], then the expected payoff profile is $\alpha \times (-1,1) + (1-\alpha) \times (1,-1) = (1-2\alpha, 2\alpha-1)$. Although the players are supposed to act simultaneously according to the rule of RPS, the outsider observer may analyze the game as if the players move in turn,

[4]For example, see Zhijian et al. (2014).

[5]We have already taken into consideration the case of tie when defining α. For each player, a draw can be treated as a combination of a half winning and a half losing.

and the player who moves first loses the game. Hence, we may regard RPS as a GII that has first mover disadvantage.

Both of the above examples are noncooperative games, but the idea used to analyze them can also be applied to solve cooperative games. The next example we consider is a pure bargaining game where two players (1 and 2) negotiate over how to divide a pie. An outside observer assumes that the game proceeds in rounds. In each round with probability α (or $1-\alpha$) player 1 (or 2, respectively) is chosen as the proposer and will suggest a partition of the pie. If the other player accepts this proposal, then the game ends with this partition; otherwise with probability p the bargaining breaks down and both player receive a payoff of zero, while with probability $1-p$ the game proceeds to the next round. Player i's payoff u_i is his expected share of the pie. Let (u_1, u_2) be the expected payoff profile. If player 1 is chosen in a round, then he will propose to player 2 an offer which player 2 is indifferent between accepting and rejecting, and thus lead to the payoff vector $H_1 = \big(1-(1-p)u_2, (1-p)u_2\big)$. Similarly, the expected payoffs are $H_2 = \big((1-p)u_1, 1-(1-p)u_1\big)$ if player 2 is chosen in a round. Therefore, we have $(u_1, u_2) = \alpha H_1 + (1-\alpha)H_2$, and consequently $(u_1, u_2) = (\alpha, 1-\alpha)$. Note that this game also has first mover advantage, and (u_1, u_2) is independent of p.

In each example above, the outside observer can find an expected payoff profile which may dependend on (i) his knowledge of the structure of the game, and (ii) some parameters (α and p) reflecting his belief on the missing information with respect to how the game is played. These two aspects of games are formally described in the next two sections.

3 The Game

To unify and simplify analysis in a variety of situations, we consider a general game setting where some players interact to determine their actions that will yield certain payoffs. The physical structure of the situation, which is commonly known to all players and the outside observer, can be characterized by the set of players, the feasible partitions of the player set, the possible actions players may take, and the payoffs associated with their actions.

Formally, let $G = (N, \mathscr{M}, \mathscr{A}, \pi)$ denote a game. Roughly speaking, it tells us that players in N may form certain coalitions according to $\mathscr{M}$, take actions according to $\mathscr{A}$, and receive corresponding payoffs according to π.

Player set

Let $N = \{1, 2, \ldots, n\}$ be the finite set of players, and let 2^N denote the set of all subsets of N. Each nonempty subset M is called a coalition. The cardinality of a coalition M is denoted by $m = |M|$.

Coalition and partition

A coalition M is said to be feasible if the players in M can sign binding agreements to coordinate their actions. Let $\mathscr{M}$ denote the set of all feasible coalitions. We assume that $\{i\} \in \mathscr{M}$ for each $i \in N$. Let $\mathscr{M}^i = \{M \in \mathscr{M} \,|\, i \in M\}$ denote the set of all feasible coalitions containing player i.

A game is said to be noncooperative if $\mathscr{M} = \big\{\{1\}, \{2\}, \ldots, \{n\}\big\} \equiv \mathscr{M}_0$, while it is said to be cooperative if $\mathscr{M} = 2^N \backslash \emptyset \equiv \mathscr{M}_N$. In addition, a game is said to be partially cooperative if $\mathscr{M}_0 \subsetneq \mathscr{M} \subsetneq \mathscr{M}_N$.

A partition (or coalition structure) of a coalition S is a set of disjoint coalitions whose union is S. If β is a partition of N and $\beta \subseteq \mathscr{M}$, that is, every coalition $M \in \beta$ is feasible, then we say that β is a feasible partition of N. There exists at least one feasible partition $\beta_0 = \mathscr{M}_0$ for each game. Let $\mathscr{B}(\mathscr{M})$ be the set of all feasible partitions of N. In particular, $\mathscr{B}(\mathscr{M}_0) = \{\beta_0\}$, while $\mathscr{B}(\mathscr{M}_N)$ consists of all partitions of N. If $M \in \mathscr{M}$, let $\mathscr{B}^M(\mathscr{M}) = \{\beta \in \mathscr{B}(\mathscr{M}) \,|\, M \in \beta\}$ be the set of feasible partitions of N that contain coalition M.

Moreover, given $\mathscr{M}$ and coalition S, let $\mathscr{M}(S) = \{M \in \mathscr{M} \,|\, M \subseteq S\}$, and let $\mathscr{M}^i(S) = \{M \in \mathscr{M}(S) \,|\, i \in M\}$. Let $\mathscr{B}(\mathscr{M}, S)$ denote the set of all feasible partitions of S. Note that $\mathscr{M}_0(S) \equiv \big\{\{i\} \,|\, i \in S\big\} \in \mathscr{B}(\mathscr{M}, S)$. If $M \in \mathscr{M}(S)$, then let $\mathscr{B}^M(\mathscr{M}, S) = \{\beta \in \mathscr{B}(\mathscr{M}, S) \,|\, M \in \beta\}$.

Action and choice

If $M \in \mathscr{M}$, let A^M denote the set of all possible actions taken by coalition M. A vector $a^M = (a_i^M)_{i \in M} \in A^M$ is called an action of M, where a_i^M is player i's action.

For simplicity, we may write A^i and a_i instead of $A^{\{i\}}$ and $a_i^{\{i\}}$ when $M = \{i\}$, write A^{ij} instead of $A^{\{i,j\}}$ when $M = \{i, j\}$, and so on.

A game is said to be finite if, for any $M \in \mathscr{M}$, A^M only contains finitely many actions.

For any feasible partition of N, $\beta = \{M_1, \dots, M_k\}$, a combination of actions $a = (a^{M_1}, \dots, a^{M_k})$ is called a list of actions under β, where $a^{M_j} \in A^{M_j}$, $\forall M^j \in \beta$.

Let $\mathscr{A}^\beta$ denote the set of all lists of actions under β. That is, $\mathscr{A}^\beta = \times_{M \in \beta} A^M$. Let $\mathscr{A} = \cup_{\beta \in \mathscr{B}(\mathscr{M})} \mathscr{A}^\beta$ denote the set of all lists of actions.

Example 1. Suppose $N = \{1, 2, 3\}$, $\mathscr{M} = \big\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 2, 3\}\big\}$, then we have $\mathscr{B}(\mathscr{M}) = \big\{\beta_0, \big\{\{1, 2\}, \{3\}\big\}, \{N\}\big\}$.

$\mathscr{A}^{\beta_0} = \big\{\big(a_1, a_2, a_3\big) \,|\, a_1 \in A^1,\, a_2 \in A^2,\, a_3 \in A^3\big\} = A^1 \times A^2 \times A^3$. If $\beta_1 = \big\{\{1, 2\}, \{3\}\big\}$, then $\mathscr{A}^{\beta_1} = \big\{\big(a_1^{12}, a_2^{12}, a_3\big) \,|\, \big(a_1^{12}, a_2^{12}\big) \in A^{12},\, a_3 \in A^3\big\} = A^{12} \times A^3$. If $\beta_2 = \big\{N\big\}$, then $\mathscr{A}^{\beta_2} = A^N$. Finally, $\mathscr{A} = \mathscr{A}^{\beta_0} \cup \mathscr{A}^{\beta_1} \cup \mathscr{A}^{\beta_2}$. □

A pair (β, a) is called a choice of N, where $\beta \in \mathscr{B}(\mathscr{M})$, $a \in \mathscr{A}^\beta$. The physical outcome of the game can be fully characterized by a choice of N.

Likewise, given any coalition S and any $\beta \in \mathscr{B}(\mathscr{M}, S)$, we use $\mathscr{A}^\beta(S)$ to denote the set of all lists of actions under β. Also, we use $\mathscr{A}(S)$ to denote the set of all lists actions of S. A pair (β, a) is called a choice of S, if $\beta \in \mathscr{B}(\mathscr{M}, S)$, $a \in \mathscr{A}^\beta(S)$.

Payoff function

A payoff function of player i, denoted by π_i, assigns a real value $\pi_i(a)$ to each $a \in \mathscr{A}$. If q is a probability distribution over $\mathscr{A}$ where $q(a)$ is the probability that a is realized, then the expected payoff of player i under q is

$$\pi_i(q) = \sum_{a \in \mathscr{A}} q(a)\pi_i(a). \tag{1}$$

Let $\pi(q) = \big(\pi_1(q), \dots, \pi_n(q)\big)$ denote the payoff vector of N under q.

Some special cases

Some traditional forms of game can be reformulated as special cases of the form defined above. We list some examples below. The player sets in all these examples are $N = \{1, 2, \ldots, n\}$.

- Let $(S_i, u_i)_{i \in N}$ be a noncooperative strategic form game, where S_i is the set of player i's pure strategies, and u_i is i's payoff function defined on $S = \times_{i \in N} S_i$. Then $\mathscr{M} = \mathscr{M}_0$, $\mathscr{A} = S$, $\pi(a) = u(a) = \big(u_1(a), \ldots, u_n(a)\big)$, $\forall a \in \mathscr{A}$.

- Let (N, X, u^0) denote a pure bargaining game, where $X \subset \mathbb{R}^n_+$ is the feasible utility space collecting all payoff profiles that can be achieved by coordinating the actions of all players in N, and $u^0 \in X$ is the reservation payoff profile describing the payoff outcome when the bargaining breaks down. Then $\mathscr{M} = \mathscr{M}_0 \cup \{N\}$; $\mathscr{A}^\beta = X$ if $\beta = \{N\}$, and $\mathscr{A}^\beta = \{u_0\}$ if $\beta = \beta_0$; $\pi(a) = a$, $\forall a \in \mathscr{A}$.

- Suppose (N, V) is a NTU cooperative coalitional form game. For each $M \in 2^N$, $V(M) \subset \mathbb{R}^m_+$ is a set of possible payoff profiles of coalition M. Then $\mathscr{M} = \mathscr{M}_N$; $\mathscr{A}^\beta = \times_{M \in \beta} V(M)$, $\forall \beta$; $\pi(a) = a$, $\forall a \in \mathscr{A}$. Furthermore, consider a TU coalitional form game (N, v), where v is the characteristic function that for any $M \in 2^N$, $v(M) \geq 0$ is the total payoff of coalition M, satisfying $v(\emptyset) = 0$. Then $\mathscr{M} = \mathscr{M}_N$; for any partition β and $M \in \beta$, $A^M = \big\{(a_i)_{i \in M} \mid \sum_{i \in M} a_i = v(M), a_i \geq 0, \forall i\big\}$, $\mathscr{A}^\beta = \times_{M \in \beta} A^M$; $\pi(a) = a$, $\forall a \in \mathscr{A}$.

- Consider a partition function form game (N, P), where P is the partition function. For any β and any $M \in \beta$, $P(M; \beta) \geq 0$ is the total payoff of coalition S under the partition β. Then $\mathscr{M} = \mathscr{M}_N$; for any partition β and $M \in \beta$, $A^M(\beta) = \big\{(a_i)_{i \in M} \mid \sum_{i \in M} a_i = P(M; \beta), a_i \geq 0, \forall i\big\}$, $\mathscr{A}^\beta = \times_{M \in \beta} A^M(\beta)$; $\pi(a) = a$, $\forall a \in \mathscr{A}$.

It is worth noting that although these traditional forms can be used to formulate the physical structure of a game with inadequate information, they do not require the usual assumptions with regard to the process of iterations. For instance, a game with inadequate information formulated in strategic form (e.g. the BOS game in section 2) should not be understood as a situation where players act simultaneously.

4 The solution

To solve a game with inadequate information is to predict the expected payoff outcome of the game from the viewpoint of the outside observer, who does not know the detailed interactive process of the players. Therefore, a solution of a game $G = (N, \mathscr{M}, \mathscr{A}, \pi)$ should depend on the belief the outside observer holds about how the game is played. We assume that this belief can be summarized by a bargaining among the players with a parameter $p \in (0, 1]$ characterizing the risk of breakdown of the bargaining when a player chooses not to finish the game right away, and a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ describing players' personal possibilities to move first in the bargaining, where $\alpha_1, \ldots, \alpha_n > 0$, $\sum_{i=1}^n \alpha_i = 1$. Given α, coalition M and $i \in M$, let $\alpha_i(M) = \alpha_i / \sum_{j \in M} \alpha_j$ denote the possibilities that i moves first in M.[6]

Formally, given a game $G = (N, \mathscr{M}, \mathscr{A}, \pi)$ and parameters (α, p), we introduce an underlying bargaining game $G'(\alpha, p)$ described by (g1)–(g5) below. In this bargaining game, players may take actions in discrete time periods $t = 0, 1, 2, \ldots$

(g1) At the beginning of $t = 0$, the set of active players is $N_0 = N$.

(g2) Suppose by induction that the game has come to period $t = k$ with the set of active players $N_k \neq \emptyset$. All actions taken in previous periods are commonly known. With probability $\alpha_i(N_k)$, player $i \in N_k$ will be chosen, and can either refuse or agree to be the proposer in this period.

(g3) If i refuses to be the proposer, then with probability $1 - p$ all players in N_k remain active; the game restarts from the beginning of $t = k$. With probability p player i chooses an action $a_i \in A^i$, becomes inactive alone and forms coalition $\{i\}$; period $t = k$ ends.

(g4) If i agrees to be the proposer, then he makes an offer (M_i, a^{M_i}), where $M_i \in \mathscr{M}^i(N_k)$, $a^{M_i} \in A^{M_i}$. Player i himself trivially accepts this offer. If $M_i \backslash \{i\} \neq \emptyset$, then all players in $M_i \backslash \{i\}$ sequentially determine whether to accept or reject it in a predetermined order[7].

[6] See Hart and Mas-colell (1996, section 6) for a more general setting of $\alpha_i(M)$.

[7] The specification of the order does not affect the equilibrium outcome of the bargaining.

(g4-1) If all players in M_i accept the offer, then coalition M_i is formed and a^{M_i} is enforced. All players in M_i become inactive. Period $t = k$ ends.

(g4-2) If at least one player rejects the offer, then with probability $1 - p$ all players in N_k remain active; the game restarts from the beginning of $t = k$. With probability p the proposer i chooses an action $a_i \in A^i$, becomes inactive alone, and forms coalition $\{i\}$; period $t = k$ ends.

(g5) By the end of period $t = k$, if there still remains at least one active player, then the game proceeds to $t = k + 1$ with N_{k+1} consisting of all remaining active players. If all players in N has become inactive with the resulting choice (β, a) of N, then $G'(\alpha, p)$ ends with the payoff outcome $\pi(a)$.

Unlike some bargaining procedures used in the literature (e.g., Okada (1996) and Hart and Mas-colell (1996, 2010)), the bargaining game $G'(\alpha, p)$ has several different features. First, in our model each player i is chosen according to a given probability $\alpha_i(N_k)$. Sometimes, $\alpha_i(N_k)$ characterizes the relative bargaining power of player i in the coalition N_k from the outside observer's view.[8] Second, we assume that the chosen player in each round has an option to agree or refuse to be the proposer, mainly to emphasize the distinction between games with early-mover advantage and games with early-mover disadvantage. Third, we do not allow the players to use mixed strategies. Note that this restriction is innocuous as players are assumed to move in turns in $G'(\alpha, p)$ – even a player is allowed to use mixed strategies at some stage, the chosen action would have already been realized before other players move.

A strategy of player i in game $G'(\alpha, p)$, denoted by f_i, specifies the action he will take at each time when it's his turn to move, given the history by that time. Let $f = (f_1, \ldots, f_n)$ denote a strategy profile of all players in N, and let f_{-i} denote a strategy profile of all players other than i.

Due to (1), we can define a payoff function of $G'(\alpha, p)$ to be a mapping $\overline{\pi}(\cdot)$ that assigns a vector

$$\overline{\pi}(f) = \sum_{a \in \mathscr{A}} P(a; f)\pi(a) \tag{2}$$

[8]A larger $\alpha_i(N_k)$ may imply a stronger or weaker bargaining power of player i, depending on whether i has early-mover advantage or early-mover disadvantage.

to each strategy profile f, where $P(a;f)$ is the probability that $a \in \mathscr{A}$ will be realized under f. Since $p > 0$, with probability one $G'(\alpha, p)$ will end up with some $a \in \mathscr{A}$ under any f, implying $\sum_{a\in\mathscr{A}} P(a;f) = 1$. Therefore, $\overline{\pi}(f)$ is the expected payoff vector of N under f. We illustrate how to derive $\overline{\pi}(f)$ by an example.

Example 2. Suppose $N = \{1, 2\}$, $\mathscr{M} = \mathscr{M}_N$, $A^1 = A^2 = \{1\}$, $A^{12} = \big\{(3, 0), (1, 2)\big\}$, $\pi(a_1, a_2) = (a_1, a_2)$, $\forall(a_1, a_2) \in \mathscr{A}$, $\alpha = (\frac{2}{3}, \frac{1}{3})$, $p = \frac{1}{4}$.

Let $f^0 = (f^0_1, f^0_2)$ be the strategy pair such that: (i) if player 1 is chosen at $t = 0$, then 1 will propose $\big(N, (3, 0)\big)$, and player 2 will accept all offers proposed to him; (ii) if player 2 is chosen at $t = 0$, then 2 will refuse to be the proposer; (iii) at $t = 1$, any active player i will choose $a_i = 1$.

We first calculate $P(a; f^0)$ for all $a \in \mathscr{A}$ so that $P(a; f^0) > 0$:

$$\begin{aligned} P\big((3,0); f^0\big) =& \tfrac{2}{3} + \big(\tfrac{1}{3} \times \tfrac{3}{4}\big) \times \tfrac{2}{3} + \big(\tfrac{1}{3} \times \tfrac{3}{4}\big)^2 \times \tfrac{2}{3} + \cdots = \tfrac{8}{9}, \\ P\big((1,1); f^0\big) =& \tfrac{1}{3} \times \tfrac{1}{4} + \big(\tfrac{1}{3} \times \tfrac{3}{4}\big) \times \tfrac{1}{3} \times \tfrac{1}{4} + \big(\tfrac{1}{3} \times \tfrac{3}{4}\big)^2 \times \tfrac{1}{3} \times \tfrac{1}{4} + \cdots = \tfrac{1}{9}. \end{aligned}$$

Due to (2), $\overline{\pi}(f^0) = \frac{8}{9} \times (3, 0) + \frac{1}{9} \times (1, 1) = (\frac{25}{9}, \frac{1}{9})$. □

The following lemma shows that for a chosen player in some period of $G'(\alpha, p)$, making an offer that will be rejected by at least one player can always be replaced by refusing to be the proposer. Thus we shall assume from now on that no chosen player will ever make an offer that will be rejected by any other player.

Lemma 1. *Suppose $\hat{f}$ is a strategy profile such that there exists a player $i \in N$ who, when chosen in some period $t = k \geqslant 0$, will agree to be the proposer and propose an offer according to $\hat{f}_i$ so that the offer will be rejected by at least one player according to $\hat{f}_{-i}$. Then there exists f'_i such that according to f'_i, player i will refuse to be the proposer when chosen in $t = k$, and $\overline{\pi}(f'_i, \hat{f}_{-i}) = \overline{\pi}(\hat{f})$.*

Proof. Let f'_i be the strategy of i such that i will refuse to be the proposer if he is chosen in $t = k$; this is the only difference between $\hat{f}_i$ and f'_i. If i is chosen in $t = k$, then both $\hat{f}_i$ and f'_i lead to the same result: restarting $t = k$ with probability $1-p$, and going to $t = k+1$ with $N_{k+1} = N_k \backslash \{i\}$ with probability p. The expected payoffs are the same under these two strategies. Therefore $\overline{\pi}(f'_i, \hat{f}_{-i}) = \overline{\pi}(\hat{f})$. □

We refer to a combination $\gamma = (M^\gamma, \beta^\gamma, a^\gamma)$ as a state of $G'(\alpha, p)$, which indicates that at some point of $G'(\alpha, p)$ the active player set is $M^\gamma \subseteq N$, while the choice of inactive players $N\backslash M^\gamma$ is (β^γ, a^γ). In particular, if $M^\gamma = N$, then the state can be simply denoted by $\gamma = N$; if $M^\gamma = \emptyset$, then we write $\gamma = (\beta^N, a^N)$. Let Γ denote the set of all states of G, and let $\Gamma^0 = \{\gamma \in \Gamma \mid M^\gamma \neq \emptyset\}$.

A strategy profile f is said to be stationary, if in any period of $G'(\alpha, p)$ when the state is $\gamma \in \Gamma^0$, the actions specified by f do not depend on any information before this period other than γ.

When f is stationary, we have another (maybe easier) method to calculate $\overline{\pi}(f)$, other than directly applying (2). For instance, we consider Example 2 again. It is easy to see that f^0 is stationary. Since $\overline{\pi}(\cdot)$ is well defined for any f, we let $\overline{\pi}(f^0) = (w_1, w_2)$. Due to the stationarity of f^0, when player 2 refuses to be the proposer and the game restarts from the beginning of $t = 0$, the expected payoff outcome under f^0 is also (w_1, w_2). Thus, one has

$$(w_1, w_2) = \tfrac{2}{3} \times (3, 0) + \tfrac{1}{12} \times (1, 1) + \tfrac{1}{4} \times (w_1, w_2),$$

implying $\overline{\pi}(f^0) = (w_1, w_2) = (\frac{25}{9}, \frac{1}{9})$.

A strategy profile f is said to be a stationary subgame perfect equilibrium (SSPE for short) of $G'(\alpha, p)$, if it is stationary and is subgame perfect — that is, it induces a Nash equilibrium in each subgame of $G'(\alpha, p)$. Following the literature[9] we use SSPE to predict the outcome of $G'(\alpha, p)$.

Definition 1. Given parameters (α, p), a payoff vector $\psi = (\psi_1, \ldots, \psi_n)$ is called a valuation of G relative to (α, p), if there exists a strategy profile f such that f is an SSPE of $G'(\alpha, p)$, and $\psi = \overline{\pi}(f)$.

A valuation of G is associated with an SSPE of the underlying bargaining game $G'(\alpha, p)$, and thus is a bargaining solution of G. The outside observer perceives a valuation as a possible expected payoff outcome of G, given his knowledge of the situation and his belief on the inadequate information.

[9]See, among others, Okada (1996), Hart and Mas-colell (1996, 2010).

5 Discussions of the solution

In this section, we provide some preliminary analysis on the properties of the valuation.

An important problem is that the valuation of a game may not exist or be unique, and thus may not always provide an effective prediction for the game. In fact, recall that we do not allow mixed strategies in $G'(\alpha, p)$, therefore the Nash equilibrium (and hence SSPE) of $G'(\alpha, p)$ may not exist in general.

Example 3. Suppose $N = \{1, 2, 3\}$, $\mathscr{M} = \big\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\big\}$, $A^1 = A^2 = A^3 = \{0\}$, $A^{12} = \big\{(a_1, a_2) = (2, 1)\big\}$, $A^{23} = \big\{(a_2, a_3) = (5, 1)\big\}$, $A^{13} = \{(a_1, a_3) = (1, 2)\}$, $\pi(a_1, a_2, a_3) = (a_1, a_2, a_3)$, $\forall (a_1, a_2, a_3) \in \mathscr{A}$, $\alpha = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $p = \frac{1}{2}$. For notational simplicity, in this example we write a stationary strategy profile as $f = (s_1, s_2, s_3)$, where $s_i = 0$ implies i will refuse to be the proposer, $s_i = i$ implies i will choose $(\{i\}, 0)$, and $s_i = ij$ means i will propose $(\{i, j\}, a^{ij})$ where $a^{ij} \in A^{ij}$, when i is chosen at $t = 0$; other players will not reject i's offer. Table 1 collects the payoff outcomes $\overline{\pi}(f)$ for all such strategy profile f, each one of which can be derived by the method described in the previous section. Of course there are other strategy profiles, involving some player's offer being rejected by other players. However, due to Lemma 1, if no strategy profile listed in Table 1 is SSPE, then neither is any other strategy profile.

Table 1

f	$\overline{\pi}(f)$	f	$\overline{\pi}(f)$	f	$\overline{\pi}(f)$	f	$\overline{\pi}(f)$
0,0,0	1., 2., 1.	0,2,0	1., 1.5, 1.25	0,12,0	1.5, 2., 0.25	0,23,0	0.5, 4., 0.75
0,0,3	1.25, 1.75, 0.75	0,2,3	1.2, 1.4, 1.	0,12,3	1.6, 1.8, 0.2	0,23,3	0.8, 3.4, 0.6
0,0,13	0.75, 1.25, 1.75	0,2,13	0.8, 1., 1.8	0,12,13	1.2, 1.4, 1.	0,23,13	0.4, 3., 1.4
0,0,23	0.25, 3.75, 1.25	0,2,23	0.4, 3., 1.4	0,12,23	0.8, 3.4, 0.6	0,23,23	0., 5., 1.
1,0,0	0.75, 2.75, 1.	1,2,0	0.8, 2.2, 1.2	1,12,0	1.2, 2.6, 0.4	1,23,0	0.4, 4.2, 0.8
1,0,3	1., 2.4, 0.8	1,2,3	1., 2., 1.	1,12,3	1.33, 2.33, 0.333	1,23,3	0.667, 3.67, 0.667
1,0,13	0.6, 2., 1.6	1,2,13	0.667, 1.67, 1.67	1,12,13	1., 2., 1.	1,23,13	0.333, 3.33, 1.33
1,0,23	0.2, 4., 1.2	1,2,23	0.333, 3.33, 1.33	1,12,23	0.667, 3.67, 0.667	1,23,23	0., 5., 1.
12,0,0	1.75, 0.75, 0.5	12,2,0	1.6, 0.6, 0.8	12,12,0	2., 1., 0.	12,23,0	1.2, 2.6, 0.4
12,0,3	1.8, 0.8, 0.4	12,2,3	1.67, 0.667, 0.667	12,12,3	2., 1., 0.	12,23,3	1.33, 2.33, 0.333
12,0,13	1.4, 0.4, 1.2	12,2,13	1.33, 0.333, 1.33	12,12,13	1.67, 0.667, 0.667	12,23,13	1., 2., 1.
12,0,23	1., 2.4, 0.8	12,2,23	1., 2., 1.	12,12,23	1.33, 2.33, 0.333	12,23,23	0.667, 3.67, 0.667
13,0,0	1.25, 0.25, 1.5	13,2,0	1.2, 0.2, 1.6	13,12,0	1.6, 0.6, 0.8	13,23,0	0.8, 2.2, 1.2
13,0,3	1.4, 0.4, 1.2	13,2,3	1.33, 0.333, 1.33	13,12,3	1.67, 0.667, 0.667	13,23,3	1., 2., 1.
13,0,13	1., 0., 2.	13,2,13	1., 0., 2.	13,12,13	1.33, 0.333, 1.33	13,23,13	0.667, 1.67, 1.67
13,0,23	0.6, 2., 1.6	13,2,23	0.667, 1.67, 1.67	13,12,23	1., 2., 1.	13,23,23	0.333, 3.33, 1.33

It can be verified that no strategy profile in Table 1 is SSPE, since each f, except $(12, 23, 13)$, will be upset by some other f'. For example, $f = (0, 0, 0)$ will be upset by $f' = (0, 0, 23)$, since if 3 unilaterally deviates from f to propose

$(\{2,3\},(5,1))$ and 2 accepts this new offer, both 2 and 3 will be better off. Also note that although $f=(12,23,13)$ cannot be upset by any other f' in the table, player 2 has an incentive to reject 1's offer leading to outcome $(0.4,3,1.4)$, which in turn will be upset by $f''=(13,23,13)$, and so on. Therefore, there exists no SSPE of $G'(\alpha,p)$, and G does not have a valuation relative to (α,p). □

There may be some special situations under which the valuation exists. For example, the following propositions show that valuation exists when G is finite and $p=1$, or when G is a finite noncooperative two-player game, or when each player has a weakly dominant action.

Proposition 1. *For any finite game G and any α, there exists a valuation of G relative to $(\alpha,1)$.*

Proof. In each period of $G'(\alpha,1)$, the chosen player either refuses to be the proposer, or proposes an offer that will be accepted by others. In both cases, this period ends. Therefore, $G'(\alpha,1)$ will end in finite periods of time, and thus an SSPE of $G'(\alpha,1)$ can be easily derived by backward induction. □

Proposition 2. *If G is a two-player noncooperative finite game, then G has a valuation relative to any (α,p).*

Proof. When player i is chosen in period $t=0$, he can either refuse to be the proposer, or choose $a_i\in A^i$. Let $(a_1^i,a_2^i)\in A^1\times A^2$ denote a pair of actions that can be derived by backward induction when i is chosen in period $t=0$ and agrees to be the proposer. Let $s_i\in\{0,i\}$ be a strategy of player i such that: (a) $s_i=0$ represents refusing to be the proposer in period $t=0$; (b) $s_i=i$ represents choosing a_i^i in period $t=0$; (c) no matter what s_i is, the action chosen by player i in period $t=1$ is a best response to the action chosen by player j in period $t=0$ (in particular, i will choose a_i^j in period $t=1$ if j chooses a_j^j in period $t=0$).

Let $\pi^i=\pi(a_1^i,a_2^i)$, and let $\widetilde{\pi}(s_1,s_2)$ denote the expected payoff outcome of the game when players strategy pair is (s_1,s_2). It is easy to show that $\widetilde{\pi}(1,2)=\widetilde{\pi}(0,0)=\alpha_1\pi^1+\alpha_2\pi^2$, $\widetilde{\pi}(0,2)=\frac{\alpha_1 p}{\alpha_2+\alpha_1 p}\pi^1+\frac{\alpha_2}{\alpha_2+\alpha_1 p}\pi^2$, $\widetilde{\pi}(1,0)=\frac{\alpha_1}{\alpha_1+\alpha_2 p}\pi^1+\frac{\alpha_2 p}{\alpha_1+\alpha_2 p}\pi^2$.

Note that $\frac{\alpha_1 p}{\alpha_2+\alpha_1 p}\le\alpha_1$, $\frac{\alpha_2}{\alpha_2+\alpha_1 p}\ge\alpha_2$, $\frac{\alpha_1}{\alpha_1+\alpha_2 p}\ge\alpha_1$, $\frac{\alpha_2 p}{\alpha_1+\alpha_2 p}\le\alpha_2$. Thus, if $\pi_1^1\ge\pi_1^2$, $\pi_2^2\ge\pi_2^1$, then $\widetilde{\pi}_1(1,2)\ge\widetilde{\pi}_1(0,2)$, $\widetilde{\pi}_2(1,2)\ge\widetilde{\pi}_2(1,0)$. Hence, $(s_1,s_2)=(1,2)$

is an SSPE of $G'(\alpha, p)$, while $\widetilde{\pi}(1,2) = \alpha_1\pi^1 + \alpha_2\pi^2$ is a valuation of G relative to (α, p). Similarly, if $\pi_1^1 < \pi_1^2$, $\pi_2^2 < \pi_2^1$, then $\widetilde{\pi}_1(0,0) \geq \widetilde{\pi}_1(1,0)$, $\widetilde{\pi}_2(0,0) \geq \widetilde{\pi}_2(0,2)$, and $\widetilde{\pi}(0,0) = \alpha_1\pi^1 + \alpha_2\pi^2$ is a valuation of G relative to (α, p); if $\pi_1^1 \geq \pi_1^2$, $\pi_2^2 < \pi_2^1$, then $\widetilde{\pi}_1(1,0) \geq \widetilde{\pi}_1(0,0)$, $\widetilde{\pi}_2(1,0) \geq \widetilde{\pi}_2(1,2)$, and $\widetilde{\pi}(1,0)$ is a valuation of G relative to (α, p); if $\pi_1^1 < \pi_1^2$, $\pi_2^2 \geq \pi_2^1$, then $\widetilde{\pi}_1(0,2) \geq \widetilde{\pi}_1(1,2)$, $\widetilde{\pi}_2(0,2) \geq \widetilde{\pi}_2(0,0)$, and $\widetilde{\pi}(0,2)$ is a valuation of G relative to (α, p). In sum, G always has a valuation relative to any (α, p). □

According to the proof of Proposition 2, in the SSPE we constructed, the chosen player will agree to be the proposer if and only if he has first mover advantage. In addition, if both players have first mover disadvantage, the valuation is just the same as that when both players have first mover advantage.

Proposition 3. *If for any $i \in N$, there exists $a_i^* \in A^i$ such that $\pi_i(a_i^*, a_{-i}) \geq \pi_i(a_i', a_{-i})$, $\forall a_i' \in A^i$, $\forall a_{-i} \in \mathscr{A}(N\backslash\{i\})$, then $\pi(a_1^*, \ldots, a_n^*)$ is a valuation of G relative to any (α, p).*

Proof. Given (α, p), let f^* be the strategy profile such that (i) each chosen player i will propose $\big(\{i\}, a_i^*\big)$; (ii) suppose the chosen player j propose (M, a^M) where $i \in M$, then i will accept this offer if and only if his expected payoff when accepting the offer is not less than that when rejecting the offer. It is easy to see that f^* is stationary. Furthermore, in each subgame of $G'(\alpha, p)$, the chosen player has no incentive to deviate from f_i^* since a_i^* is his weakly dominant action. Hence, f^* is an SSPE of $G'(\alpha, p)$, and $\pi(a_1^*, \ldots, a_n^*) = \overline{\pi}(f^*)$ is a valuation of G. □

On the other hand, there exist some games that have multiple valuations. For instance, consider the noncooperative strategic game in Figure 4. Since each player is indifferent between A and B, each payoff vector of the form $\alpha_1 H_1 + \alpha_2 H_2$ is a valuation relative to any (α, p), where $H_i \in \{(3,1), (3,2), (2,1), (2,2)\}$, $i = 1, 2$.

		Player 2 A	Player 2 B
Player 1	A	$3,1$	$2,1$
	B	$3,2$	$2,2$

Figure 4: *A game with multiple valuations*

If the valuation(s) of a game does not exist or are not unique relative to some parameters (α, p), then the outside observer's belief characterized by the underlying game $G'(\alpha, p)$ is not appropriate for the current situation, or is not sufficient to determine a unique outcome of the game. In this case, we may adjust the method to solve the game, or to impose additional restrictions to the solution. For example, we may consider other bargaining procedures in the underlying game, or solve it using some additional tie-breaking rules or refinement of SSPE.

Sometimes, there exists a unique valuation relative to some parameters (α, p). Let $\psi(\alpha, p) = \big(\psi_1(\alpha, p), \ldots, \psi_n(\alpha, p)\big)$ denote this unique valuation. If a game G has a unique valuation relative to all possible parameters (α, p), then $\psi(\alpha, p)$ can be regarded as a function of (α, p) characterizing how the valuation depends on outside observer's belief. The properties of this function can help predict the outcome of the game, since the dependence of $\psi(\alpha, p)$ on the parameters (α, p) reflects how the procedure of interactions may affect the outcome of the game.

For instance, it is possible that the players have earlier mover advantage so that for each player i, $\psi_i(\alpha, p)$ is increasing in α_i; or players may have earlier mover disadvantage so that $\psi_i(\alpha, p)$ is decreasing in α_i. The BOS game and the RPS game in Section 2 are examples of these two cases, respectively. For a game with earlier mover advantage (disadvantage), a larger α_i implies that player i's bargaining power is relatively stronger (weaker, respectively).

Moreover, sometimes the valuation may also be procedure-free; that is, $\psi(\alpha, p)$ does not depend on (α, p). For instance, the noncooperative strategic form game in Figure 5 has a unique valuation $(4, 4)$, which is independent of (α, p). Note that this valuation does not coincide with the unique Nash equilibrium payoffs $(3, 3)$. This example illustrates to us the difference between the valuation and the Nash equilibrium, even if they both provide a unique prediction for the game. A natural question is which solution is more appropriate. The answer depends on whether the game has inadequate information.

If G has a unique valuation which does not depend on (α, p), then this valuation is called the fixed valuation of G, and can be simply denoted as ψ. We are particularly interested in games with a fixed valuation, since in this case the belief of an outside observer on the inadequate information does not affect his prediction of the game. In other words, there is virtually no loss for the outside

		Player 2		
		L	K	R
	U	$4,4$	$2,0$	$0,0$
Player 1	H	$6,0$	$3,3$	$0,2$
	D	$0,0$	$0,6$	$4,4$

Figure 5: *A game with fixed valuation*

observer to predict the outcome of a game with a fixed valuation due to inadequate information.

6 Some special games

Usually, it is quite complex to calculate and analyze the valuation in a general setting. Therefore, we may start from some simple games. The valuation in several special cases are investigated in the following three subsections. We focus on the calculation of the valuation, the existence and uniqueness of the valuation, and the relationship between the valuation and some traditional solutions, such as the Nash equilibrium outcome, the (weighted) Nash bargaining solution, and the core.

6.1 Two-player zero-sum game

Consider a zero-sum strategic game $G = (S_i, u_i)_{i\in N}$. Here, $N = \{1, 2\}$ is the player set, S_i is a finite set of (pure) strategies of i, and $u_1(s)+u_2(s) = 0$ for any strategy pair $s = (s_1, s_2) \in S_1 \times S_2$. The game can be either noncooperative ($\mathscr{M} = \mathscr{M}_0$) or cooperative ($\mathscr{M} = \mathscr{M}_N$).

Let $r_1 := \max_{s_1\in S_1} \min_{s_2\in S_2} u_1(s_1, s_2)$, $r_2 := \min_{s_2\in S_2} \max_{s_1\in S_1} u_1(s_1, s_2)$. Then player 1 can guarantee that he will get no less than r_1 in G, while player 2 can guarantee that player 1 will get no more than r_2. It is easy to see that $r_1 \le r_2$.

Denote $\psi(\alpha) = \big(\psi_1(\alpha), \psi_2(\alpha)\big) := (\alpha_1 r_1 + \alpha_2 r_2, -\alpha_1 r_1 - \alpha_2 r_2)$. Let f_i^* be a stationary strategy of player i in the underlying game $G'(\alpha, p)$, such that:

(i) if i is chosen in $t = 0$, then he will refuse to be the proposer;

(ii) if $j \neq i$ has taken $\bar{s}_j$ and become inactive in $t = 0$, then in $t = 1$ i will take $s_i^* \in S_i$ so that $u_i(s_i^*, \bar{s}_j) \geq u_i(s_i', \bar{s}_j)$, $\forall s_i' \in S_i$;

(iii) if the game is cooperative and j proposes s to i in $t = 0$, then i will accept it if and only if $u_i(s) \geq p \min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j) + (1-p)\psi_i(\alpha)$.

Let $H_i(f)$ denote player i's expected payoff when players follow f and i is chosen in $t = 0$. Then given $f^* = (f_1^*, f_2^*)$, player 1's expected payoff is $H_1(f^*) = p \max_{s_1 \in S_1} \min_{s_2 \in S_2} u_1(s_1, s_2) + (1-p)\overline{\pi}_1(f^*) = pr_1 + (1-p)\overline{\pi}_1(f^*)$ if 1 is chosen in $t = 0$, while 1's payoff is $-H_2(f^*) = -p \max_{s_2 \in S_2} \min_{s_1 \in S_1} u_2(s_1, s_2) - (1-p)\overline{\pi}_2(f^*) = pr_2 + (1-p)\overline{\pi}_1(f^*)$ if 2 is chosen in $t = 0$. Thus, one has $\overline{\pi}_1(f^*) = \alpha_1[pr_1 + (1-p)\overline{\pi}_1(f^*)] + \alpha_2[pr_2 + (1-p)\overline{\pi}_1(f^*)]$, implying $\overline{\pi}_1(f^*) = \alpha_1 r_1 + \alpha_2 r_2$. Since G is zero-sum, we have $\overline{\pi}_2(f^*) = -\alpha_1 r_1 - \alpha_2 r_2$. Therefore,

$$\overline{\pi}(f^*) = \psi(\alpha). \tag{3}$$

Since $r_1 \leq r_2$, due to (3) we have $\overline{\pi}_1(f^*) \geq r_1$, $\overline{\pi}_2(f^*) \geq -r_2$. That is,

$$\overline{\pi}_i(f^*) \geq \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j), \quad i \neq j.$$

Since $H_i(f^*) = p \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j) + (1-p)\overline{\pi}_i(f^*)$, we have

$$\overline{\pi}_i(f^*) \geq H_i(f^*) \geq \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j), \quad i \neq j. \tag{4}$$

The following proposition establishes the existence and uniqueness of valuation for any two-player zero-sum game.

Proposition 4. *The unique valuation of G relative to any (α, p) is $\psi(\alpha)$.*

Proof. We shall first prove that f^* is an SSPE of $G'(\alpha, p)$. Since f^* is stationary, it remains to prove that f^* is a subgame perfect equilibrium.

If $j \neq i$ has taken $\bar{s}_j$ in $t = 0$, then it is obvious that i cannot benefit from deviating from f_i^*. If G is cooperative and j proposes s to i at $t = 0$, then when i refuses the offer, his expected payoff is $p \min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j) + (1-p)\psi_i(\alpha)$, hence i will not deviate away from f_i^*.

If i is chosen in $t = 0$, then other than f_i^*, player i can either choose $s_i \in S_i$, or, if G is cooperative, propose $s \in S_1 \times S_2$ to j.

(i) In the former case, player i will take some s_i^* so that $\min_{s_j \in S_j} u_i(s_i^*, s_j) \geq \min_{s_j \in S_j} u_i(s_i', s_j)$, $\forall s_i' \in S_i$. Then given f_j^*, i's expected payoff will be $\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j)$. Due to (4), this payoff is not larger than $H_i(f^*)$, and thus i has no incentive to deviate from f_i^*.

(ii) In the latter case, if the offer s is rejected by j, then i's expected payoff is $p \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j) + (1-p)\psi_i(\alpha) = H_i(f^*)$. If the offer s is accepted by j, then according to f_j^*, $u_j(s) \geq p \min_{s_i \in S_i} \max_{s_j \in S_j} u_j(s_i, s_j) + (1-p)\psi_j(\alpha)$, and hence $u_i(s) = -u_j(s) \leq p \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j) + (1-p)\psi_i(\alpha) = H_i(f^*)$. Again due to (4), i has no incentive to deviate from f_i^*

Thus we have proved that f^* is an SSPE of $G'(\alpha, p)$.

Furthermore, player i can guarantee that his expected payoff in $G'(\alpha, p)$ will be at least $\psi_i(\alpha)$ by using f_i^*. On the other hand, by choosing f_j^* player j can make sure that i's expected payoff will be at most $\psi_i(\alpha)$. These imply that $\psi(\alpha)$ is the only payoff outcome that can be supported by any SSPE of $G'(\alpha, p)$. □

Proposition 4 also shows that the unique valuation of G does not depend on p. But when is the valuation also independent of α? In other words, when does G have a fixed valuation? Since $\psi(\alpha) = (\alpha_1 r_1 + \alpha_2 r_2, -\alpha_1 r_1 - \alpha_2 r_2)$, one has:

Corollary 1. *G has a fixed valuation if and only if $r_1 = r_2$.*

In the literature, when $r_1 = r_2$, we say that G has a value in pure strategies, which can be denoted by $r := r_1 = r_2$. Moreover, von Neumann (1928)' Minmax Theorem shows that if players can use mixed strategies $\sigma_i \in \Delta S_i$, $i = 1, 2$, then each finite two-player zero-sum game has a value in mixed strategies:

$$\hat{r} := \max_{\sigma_1 \in \Delta S_1} \min_{\sigma_2 \in \Delta S_2} u_1(\sigma_1, \sigma_2) = \min_{\sigma_2 \in \Delta S_2} \max_{\sigma_1 \in \Delta S_1} u_1(\sigma_1, \sigma_2).$$

This value coincides with the payoff of player 1 in each Nash equilibrium of G. However, G need not have a value in pure strategies. For instance, consider again the RPS game in section 2. This game does not have a value (and Nash equilibrium) in pure strategies. Meanwhile, it is easy to verify that in the RPS game, $r_1 < r_2$, and hence this game does not have a fixed valuation.

This example motivates us to study the relationship between the valuation and the Nash equilibrium (the value). The next proposition shows that these solutions only coincide and predict the same payoff outcome when G has a fixed valuation, or when G has a pure-strategy Nash equilibrium.

Proposition 5. *The following statements are equivalent:*

(a) *G has a fixed valuation $(r, -r)$.*
(b) *G has a pure-strategy Nash equilibrium s^*, so that $u(s^*) = (r, -r)$.*

Proof. The equivalence of (b) and $r_1 = r_2 = r$ has been proved in the literature. See, for example, Maschler et al. (2013, Theorem 4.44 and Theorem 4.45). On the other hand, (a) and $r_1 = r_2 = r$ are equivalent due to Corollary 1. Hence, (a) and (b) are equivalent. □

The value for a two-player zero-sum game plays an important role in game theory. Some scholars[10] think that this is so far the only case in game theory that we can provide an unquestionable prediction for a game. However, Proposition 5 suggests that when the zero-sum game has inadequate information, the value is no longer an unquestionable solution; it is an appropriate prediction only when the game has a pure-strategy Nash equilibrium. If, on the contrary, the game does not have a Nash equilibrium in pure strategies, just as in the RPS, then the outside observer believes that the payoff outcome of the game (the valuation) should depend on α. As mentioned when analyzing RPS in section 2, in this case the value (Nash equilibrium) of the game, which is independent of α, is inconsistent with the experience of the outside observer.

6.2 Two-player pure bargaining game

Consider a pure bargaining game $G = (N, X, u^0)$ where $N = \{1, 2\}$. Suppose the feasible payoff space $X = \big\{(u_1, u_2) \in \mathbb{R}^2_+ \,|\, u_2 \leqslant g(u_1)\big\}$ is a bounded, convex, closed set, whose boundary can be denoted by $\partial X = \{(u_1, u_2) \,|\, u_2 = g(u_1), 0 \leqslant u_1 \leqslant \tau_1\} = \{(u_1, u_2) \,|\, u_1 = g^{-1}(u_2), 0 \leqslant u_2 \leqslant \tau_2\}$, where $\tau_1 = g^{-1}(0)$, $\tau_2 = g(0)$. Write $X \backslash \partial X = \{u \in X \,|\, u \notin \partial X\}$. We assume that

(h1) $g(\cdot)$ is twice continuously differentiable on $[0, \tau_1]$;
(h2) $g(x)' < 0$, $g(x)'' \leq 0$, $\forall x \in [0, \tau_1]$;

[10] For example, Aumann and Dreze (2008).

(h3) $u^0 \in X \backslash \partial X$.

The following proposition establishes the existence and uniqueness of the valuation of G. Also, it shows how to calculate the valuation.

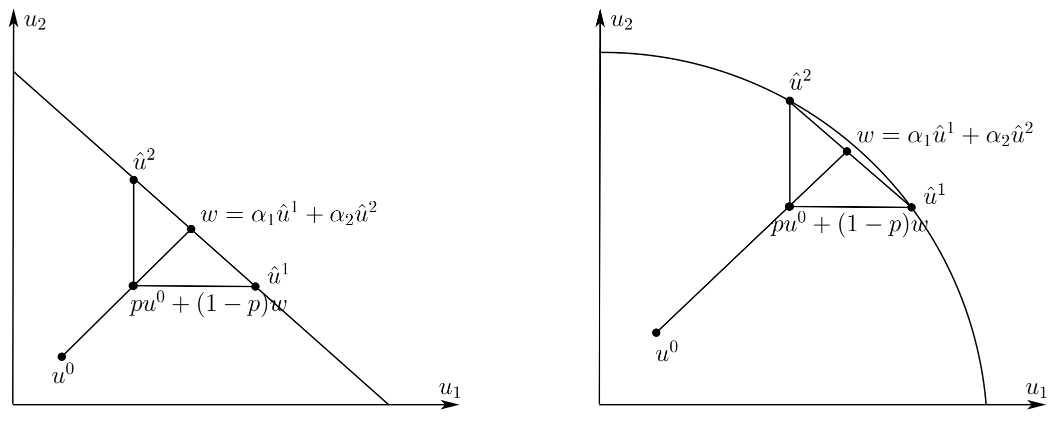
Proposition 6. *Given any* (α, p)*,* $G = (N, X, u^0)$ *has a unique valuation* $\psi(\alpha, p) = (w_1, w_2)$ *relative to* (α, p)*, satisfying*

$$w_1 = \alpha_1 g^{-1}\big((1-p)w_2 + pu_2^0\big) + \alpha_2\big[(1-p)w_1 + pu_1^0\big], \tag{5}$$

$$w_2 = \alpha_1\big[(1-p)w_2 + pu_2^0\big] + \alpha_2 g\big((1-p)w_1 + pu_1^0\big). \tag{6}$$

Proof. See Appendix. □

Note that if ∂X is a straight line, then $\psi(\alpha, p) \in \partial X$; that is, the valuation is Pareto efficient. However, the valuation is typically not Pareto efficient (i.e. $\psi(\alpha, p) \in X \backslash \partial X$) if X is strictly convex. See Figure 6(a) and 6(b) for an exhibition of these two cases, respectively. Furthermore, we shall show in Proposition 7 that as p goes to zero, the limit of valuation is Pareto efficient, and coincides with the weighted Nash bargaining solution (WNBS for short) of the game[11].



(a) ∂X is linear (b) ∂X is nonlinear

Figure 6: Valuation of two-player bargaining game

[11]This conclusion is not surprising, given the large literature on the noncooperative implementation of the (asymmetric) Nash bargaining solution. See, for examples, Hart and Mas-colell (1996), Britz et al. (2010), and Kawamori (2014).

Let $\kappa(\alpha)$ denote the WNBS of G relative to α. That is, suppose $Q(u, u^0, \alpha) = (u_1 - u_1^0)^{\alpha_1}(u_2 - u_2^0)^{\alpha_2}$ is the weighted Nash product, then $\kappa(\alpha)$ is defined as the unique solution to the optimization problem: $\max_{u \in X} Q(u, u^0, \alpha)$. The WNBS can be characterized by the following Lemma.

Lemma 2. *Suppose $v = (v_1, v_2) \in \partial X$, and the slope of the tangent to ∂X at v is $-\frac{\alpha_1}{\alpha_2} \cdot \frac{v_2 - u_2^0}{v_1 - u_1^0}$, then $v = \kappa(\alpha)$.*

Proof. The slope of the tangent to ∂X at $\kappa(\alpha)$, which is equal to the slope of the tangent to the curve $Q(u, u^0, \alpha) = Q(\kappa(\alpha), u^0, \alpha)$ at $\kappa(\alpha)$, is

$$-\frac{\partial Q(\kappa(\alpha), u^0, \alpha)/\partial u_1}{\partial Q(\kappa(\alpha), u^0, \alpha)/\partial u_2} = -\frac{\alpha_1}{\alpha_2} \cdot \frac{\kappa_2(\alpha) - u_2^0}{\kappa_1(\alpha) - u_1^0}.$$

On the other hand, if $v \in \partial X$ but $v \neq \kappa(\alpha)$, then we shall show that the corresponding slope at v cannot be $-\frac{\alpha_1}{\alpha_2} \cdot \frac{v_2 - u_2^0}{v_1 - u_1^0}$. Suppose without loss of generality that $v_1 > \kappa_1(\alpha)$, $v_2 < \kappa_2(\alpha)$, then due to (h2),

$$Slope(v) < Slope(\kappa(\alpha)) = -\frac{\alpha_1}{\alpha_2} \cdot \frac{\kappa_2(\alpha) - u_2^0}{\kappa_1(\alpha) - u_1^0} < -\frac{\alpha_1}{\alpha_2} \cdot \frac{v_2 - u_2^0}{v_1 - u_1^0}.$$

This proves our statement. □

Proposition 7. *Given any α, $\lim_{p \to 0} \psi(\alpha, p) = \kappa(\alpha)$.*

Proof. If there exists a function $\tau(\alpha)$ such that for each α, $\lim_{p \to 0} \psi(\alpha, p) = \tau(\alpha)$, then according to (6) we have $\tau_2(\alpha) = g(\tau_1(\alpha))$. Thus, $\tau(\alpha) \in \partial X$. Let

$$\begin{aligned} w^1(\alpha, p) &= \left(g^{-1}((1-p)\psi_2(\alpha, p) + p u_2^0), (1-p)\psi_2(\alpha, p) + p u_2^0\right), \\ w^2(\alpha, p) &= \left((1-p)\psi_1(\alpha, p) + p u_1^0, g((1-p)\psi_1(\alpha, p) + p u_1^0)\right), \end{aligned}$$

then $w^1(\alpha, p), w^2(\alpha, p) \in \partial X$, and $\psi(\alpha, p) = \alpha_1 w^1(\alpha, p) + \alpha_2 w^2(\alpha, p)$. The slope of the secant line passing through $w^1(\alpha, p)$ and $w^2(\alpha, p)$ is $t(\alpha, p) = \frac{w_2^2(\alpha,p) - w_2^1(\alpha,p)}{w_1^2(\alpha,p) - w_1^1(\alpha,p)}$. Due to (5) and (6), it is easy to see $t(\alpha, p) = -\frac{\alpha_1}{\alpha_2} \cdot \frac{\psi_2(\alpha,p) - u_2^0}{\psi_1(\alpha,p) - u_1^0}$. Since $\psi(\alpha, p)$ always locates on this secant line, the slope of the tangent to ∂X at $\tau(\alpha)$ is $\lim_{p \to 0} t(\alpha, p) = -\frac{\alpha_1}{\alpha_2} \cdot \frac{\tau_2(\alpha) - u_2^0}{\tau_2(\alpha) - u_1^0}$. According to Lemma 2, $\lim_{p \to 0} \psi(\alpha, p) = \tau(\alpha) = \kappa(\alpha)$.

It remains to prove that for each α, $\psi(\alpha, p)$ indeed converges as $p \to 0$. By (6) and Lagrange's mean value theorem, for any $p_1 < p_2$, there exists $\bar{p} \in [p_1, p_2]$ such that

$$\begin{aligned} \alpha_2\big(\psi_2(\alpha, p_2) - \psi_2(\alpha, p_1)\big) = & - \alpha_1\big(p_2\psi_2(\alpha, p_2) - p_1\psi_2(\alpha, p_1)\big) + \alpha_1(p_2 - p_1)u_2^0 \\ & + \alpha_2 g'\big((1-\bar{p})\psi_1(\alpha, \bar{p}) + \bar{p}u_1^0\big)(p_2 - p_1). \end{aligned} \tag{7}$$

Given α, for any $\varepsilon > 0$, we can easily find some $\delta > 0$ according to (7) such that for any p_1, p_2, $0 < p_1 < p_2 < \delta$, $|\psi_2(\alpha, p_2) - \psi_2(\alpha, p_1)| < \varepsilon$. By Cauchy convergence criterion, $\psi(\alpha, p)$ converges as $p \to 0$. $\square$

6.3 Three-player TU coalitional game

Coalitional games, where all coalitions are feasible, can be regarded as an extension of pure bargaining games. In this section, we consider a TU coalitional game (N, v) where $N = \{1, 2, 3\}$. Suppose v is superadditive, that is, $v(S) + v(T) \leq v(S \cup T)$ for any disjoint coalitions S and T. In this subsection, we sometimes abuse the notation to write i, ij, ... instead of $\{i\}$, $\{i, j\}$, ...

To calculate the valuation of this game, we introduce a strategic form game $(\overline{S}_i, \overline{u}_i)_{i \in N}$ as follows. Let $\overline{S}_i = \{0, i, ij, ik, N\}$, where $\{i, j, k\} = N$, and let $\overline{S} = \times_{i \in N} \overline{S}_i$.

Let $\overline{w}_i^{ij} := v(i) + \frac{\alpha_i}{\alpha_i + \alpha_j}\big(v(ij) - v(i) - v(j)\big)$ denote player i's payoff when the coalition $\{i, j\}$ splits surplus according to players' parameters α_i and α_j.

For any column vector $w = (w_1, w_2, w_3)^T$ and $\overline{s} = (\overline{s}_1, \overline{s}_2, \overline{s}_3) \in \overline{S}$, let

$$H^i(w, \overline{s}_i) = \big(H_1^i(w, \overline{s}_i), H_2^i(w, \overline{s}_i), H_3^i(w, \overline{s}_i)\big)^T,$$

where

$$\begin{aligned} & H_i^i(w, 0) = pv(i) + (1-p)w_i, \\ & H_j^i(w, 0) = p\overline{w}_j^{jk} + (1-p)w_j, \\ & H_i^i(w, i) = v(i), \\ & H_j^i(w, i) = \overline{w}_j^{jk}, \\ & H_i^i(w, ij) = v(ij) - p\overline{w}_j^{jk} - (1-p)w_j, \end{aligned}$$

$$
\begin{aligned}
H_j^i(w, ij) &= p\overline{w}_j^{jk} + (1-p)w_j,\\
H_k^i(w, ij) &= v(k),\\
H_i^i(w, N) &= v(N) - pv(jk) - (1-p)(w_j + w_k),\\
H_j^i(w, N) &= p\overline{w}_j^{jk} + (1-p)w_j.
\end{aligned}
$$

Finally, let $H(w, \overline{s}) = \sum_{i \in N} \alpha_i H^i(w, \overline{s}_i)$, where $\overline{s} = (\overline{s}_1, \overline{s}_2, \overline{s}_3)$. Thus for each $\overline{s}$, we have defined a mapping $H(\cdot, \overline{s})$ that associates each w with a payoff vector $H(w, \overline{s})$.

Intuitively, $\overline{S}_i$ collects the possible actions of player i when he is chosen in $t = 0$ of $G'(\alpha, p)$, where $\overline{s}_i \neq 0$ represents the offer that the coalition $\overline{s}_i$ forms and splits $v(\overline{s}_i)$ so that each $j \in \overline{s}_i \backslash i$ is indifferent between accepting and rejecting the offer, while $\overline{s}_i = 0$ represents that i refuses to be the proposer. Suppose the outside observer has an initial expectation w about the payoff outcome of $G'(\alpha, p)$. For any chosen player i at $t = 0$, the outside observer would deduce that i can enforce an expected payoff profile $H^i(w, \overline{s}_i)$. Given $\overline{s}$, $H(w, \overline{s})$ is the outside observer's predictions for the payoff outcome that the players' reactions will lead to against the initial expectation w. Hence a reasonable expectation w should be a fixed point of $H(\cdot; \overline{s})$; that is,

$$w = H(w, \overline{s}), \tag{8}$$

Lemma 3. *For any $\overline{s} \in \overline{S}$, there exists a unique vector w satisfying* (8).

Proof. See Appendix. □

Lemma 3 shows that $H(\cdot, \overline{s})$ always has a unique fixed point. Let $\overline{u}(\overline{s}) = \big(\overline{u}_1(\overline{s}), \overline{u}_2(\overline{s}), \overline{u}_3(\overline{s})\big)$ denote this fixed point. Thus, we have defined $(\overline{S}_i, \overline{u}_i)_{i \in N}$. The following proposition suggests that the valuation of (N, v) can be derived by calculating the pure-strategy Nash equilibrium of $(\overline{S}_i, \overline{u}_i)_{i \in N}$.

Proposition 8. *If a vector $\psi \in \mathbb{R}^3$ is a valuation of (N, v) relative to (α, p), then there exists a pure-strategy Nash equilibrium s^* of $(\overline{S}_i, \overline{u}_i)_{i \in N}$ such that $\psi = \overline{u}(s^*)$. Conversely, if s^* is a pure-strategy Nash equilibrium of $(\overline{S}_i, \overline{u}_i)_{i \in N}$, then $\psi = \overline{u}(s^*)$ is a valuation of (N, v) relative to (α, p).*

Proof. Suppose ψ is a valuation of (N, v) relative to (α, p), then there is an SSPE f^* of $G'(\alpha, p)$ such that $\psi = \overline{\pi}(f^*)$. A strategy profile $s^* = (s_1^*, s_2^*, s_3^*)$ of $(\overline{S}_i, \overline{u}_i)_{i \in N}$

can be derived from f^* as follows. Suppose player i is chosen in $t = 0$ of $G'(\alpha, p)$. If i refuses to be the proposer according to f_i^*, then $s_i^* = 0$. If i agrees to be the proposer and proposes an offer (M_i, a^{M_i}) according to f_i^*, then $s_i^* = M_i$. Due to Lemma 1, we might as well assume that no players in M_i will reject this offer according to f^*, since otherwise f_i^* can be replaced by a strategy in which i refuses to be the proposer. Hence, in offer (M_i, a^{M_i}), a^{M_i} will be a payoff vector such that each $j \in M_i \backslash \{i\}$ is indifferent between accepting and rejecting this offer, leading to payoff vector $H^i\big(\overline{\pi}(f^*), s_i^*\big)$. Hence, one has

$$\overline{\pi}(f^*) = \sum\nolimits_{i \in N} \alpha_i H^i\big(\overline{\pi}(f^*), s_i^*\big) = H\big(\overline{\pi}(f^*), s^*\big).$$

According to Lemma 3 and the definition of $\overline{u}$, one has $\overline{\pi}(f^*) = \overline{u}(s^*)$. Since f^* is an SSPE, no player has an incentive to unilaterally deviate his strategy from f^*. Therefore in $(\overline{S}_i, \overline{u}_i)_{i \in N}$, no player is willing to deviate from his strategy s_i^* given other players' strategies s_{-i}^*. Hence, s^* is a pure-strategy Nash equilibrium of $(\overline{S}_i, \overline{u}_i)_{i \in N}$, and $\psi = \overline{\pi}(f^*) = \overline{u}(s^*)$.

Conversely, suppose s^* is a pure-strategy Nash equilibrium of $(\overline{S}_i, \overline{u}_i)_{i \in N}$. Consider a stationary strategy profile f^* of $G'(\alpha, p)$ as follows. If i is chosen in $t = 0$, then according to f_i^*, player i will refuse to be the proposer if $s_i^* = 0$, and will propose the offer (M_i, a^{M_i}) if $s_i^* \neq 0$, where $M_i = s_i^*$, $a_j^{M_i} = H_j^i\big(\overline{u}(s^*), s_i^*\big)$, $\forall j \in M_i$. If i is chosen in some period $t > 0$ under state γ, then he will propose (M_i, a^{M_i}) where $M_i = M^\gamma$, while $(a_i^{M_i}, a_j^{M_i}) = \big(v(ij) - v(j), v(j)\big)$ if $M^\gamma = ij$, and $a_i^{M_i} = v(i)$ if $M^\gamma = i$. If $j \neq i$ is chosen in some period $t \geq 0$ under state γ and proposes (M_j, a^{M_j}) so that $i \in M_j$, then when $M^\gamma = N$, i will accept this offer if and only if $a_i^{M_j} \geq H_i^j\big(\overline{u}(s^*), s_j^*\big)$; when $M^\gamma = ij$, i will accept this offer if and only if $a_i^{M_j} \geq pv(i) + (1-p)\overline{w}_i^{ij}$. According to the definition of f^*, one has

$$\overline{\pi}(f^*) = \sum\nolimits_{i \in N} \alpha_i H^i\big(\overline{u}(s^*), s_i^*\big) = H\big(\overline{u}(s^*), s^*\big) = \overline{u}(s^*).$$

Since s^* is a pure-strategy Nash equilibrium of $(\overline{S}_i, \overline{u}_i)_{i \in N}$, f^* is a Nash equilibrium of $G'(\alpha, p)$. Furthermore, due to the definition of H^i and the superadditivity of v, it is easy to verify that f^* is subgame perfect. Therefore, $\psi = \overline{u}(s^*) = \overline{\pi}(f^*)$ is a valuation of (N, v) relative to (α, p). □

Finally, we briefly discuss some properties of the valuation through an example. We emphasize on the efficiency[12] and uniqueness of the valuation, as well as the comparison between the valuation and some other solutions, e.g. the core and the Shapley value. The core of the game is the set $C(v) = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = v(N), x_i + x_j \geq v(ij), x_k \geq v(k), \forall i, j, k \in N, i \neq j\}$. The Shapley value is a payoff vector $\phi(v) = \big(\phi_1(v), \phi_2(v), \phi_3(v)\big)$, where

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(2 - |S|)!}{6} \big[v(S \cup \{i\}) - v(S)\big], \quad \forall i \in N.$$

Example 4. Consider a game where $v(1) = v(2) = v(3) = 0$, $v(12) = v(23) = v(13) = \frac{2}{3}$, $v(N) = 1$. The Shapley value of this game is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which is also the only allocation in the core. However, when $\alpha = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, $p = 0.05$, the unique valuation of the game is $(0.4, 0.3778, 0.2222)$. This shows that the valuation of a game may be different from the core or the Shapley value. Hence, the core or the Shapley value may not be appropriate for TU games with inadequate information.

When $\alpha = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $p = 0.05$, this game has two valuations relative to (α, p): $(\frac{2}{9}, \frac{2}{9}, \frac{2}{9})$ and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. This shows that a valuation is not necessarily efficient. In addition, this example also suggests that it is possible that (N, v) has multiple valuations. □

We conclude by noting that whether a pure-strategy Nash equilibrium of $(\overline{S}_i, \overline{u}_i)_{i \in N}$, and hence the valuation of (N, v), always exists remains an open question. We are not able to show the existence of the valuation; on the other hand we also fail to find a three-player superadditive TU game in which the valuation relative to some (α, p) does not exist. We leave this existence problem for future research.

7 Conclusion

We have developed a framework in which an outside observer of a game with inadequate information can predict each player's expected payoff. A solution, namely the valuation, is defined based on this framework. We discuss the properties of this solution, especially in some special cases. The main findings are as follows.

[12]Since payoffs are transferable, a payoff vector (x_1, x_2, x_3) is efficient if $x_1 + x_2 + x_3 = v(N)$.

First, for each two-player zero-sum game or two-player pure bargaining game, there exists a unique valuation. For three-player TU coalitional games, the existence and uniqueness of the valuation are not guaranteed, but when a valuation exists, we have provided an algorithm to calculate all of the valuations.

Second, the valuation generally does not coincide with some traditional solutions of games. For example, for TU coalitional games, the valuation relative to some parameters need not coincide with the core and the Shapley value. This suggests that games with inadequate information is essentially different from traditional games. However, there are some exceptions under certain circumstances. For example, for a two-player pure bargaining game, as $p \to 0$, the limit of the valuation approaches the weighted Nash bargaining solution.

In addition, the distinction between the valuation and some traditional solutions also helps us explain some phenomena that cannot be explained by traditional theory. The BOS and the RPS in section 2 are two examples in this regard. Hence, for an outside observer of a game with inadequate information, the valuation is usually a more appropriate solution than traditional solutions.

Third, we are particularly interested in the situation when inadequate information does not affect the outside observer's prediction in any way — that is, when the game has a fixed valuation. So far, we have only found few games satisfying this property. Even a two-player zero-sum game does not have a fixed valuation, unless it has a pure-strategy Nash equilibrium.

Finally we list some works that might be worth exploring in the future research. First, some theoretical problems remain unanswered. For example, under what conditions a game has a valuation, and when the valuation is unique. Second, we can try to apply the method introduced in this paper to a variety of real world circumstances with inadequate information,[13] for example, to design a mechanism when the designer is an outside observer of the situation. Third, using the method introduced in this paper, we may build a bridge connecting game theory and econometrics. On one hand, econometrics tools can be introduced to estimate the

[13]Mao (2014) provides an application of GII that explains why a dictator, who is an outside observer of a tax-collecting game played by the government and the people, is willing to change the type of the government from dictatorship to equality, under some mild condition. In the Supplementary Material, this example is adapted and translated into English for the convenience of the readers.

parameters in the valuation, which makes the solution more accurate. On the other hand, the model of GII may help build more solid foundations for some empirical analysis.

Appendix

Proof of Proposition 6.

Let $\overline{X}(u^0) = \{(u_1, u_2) \in X \,|\, u_1 \geqslant u_1^0, u_2 \geqslant u_2^0\}$, then $\overline{X}(u^0)$ is a convex and compact set. We first show that there exists a unique vector $w = (w_1, w_2) \in \overline{X}(u^0)$ satisfying (5)(6), i.e. $w_2 = \frac{\alpha_2}{1-\alpha_1(1-p)} g\left(\frac{(1-p)\alpha_1}{1-\alpha_2(1-p)} g^{-1}\big((1-p)w_2 + pu_2^0\big) + \frac{pu_1^0}{1-\alpha_2(1-p)}\right) + \frac{\alpha_1 pu_2^0}{1-\alpha_1(1-p)}$, where $w_1 \in [u_1^0, g_{-1}(u_2^0)]$, $w_2 \in [u_2^0, g(u_1^0)]$.

Let $E(x) = -x + \frac{\alpha_2}{1-\alpha_1(1-p)} g\left(\frac{(1-p)\alpha_1}{1-\alpha_2(1-p)} g^{-1}\big((1-p)x + pu_2^0\big) + \frac{pu_1^0}{1-\alpha_2(1-p)}\right) + \frac{\alpha_1 pu_2^0}{1-\alpha_1(1-p)}$, then $E'(x) = \frac{\alpha_1\alpha_2(1-p)^2}{[1-\alpha_1(1-p)][1-\alpha_2(1-p)]} \frac{g'\left(\frac{(1-p)\alpha_1}{1-\alpha_2(1-p)} g^{-1}\big((1-p)x+pu_2^0\big) + \frac{pu_1^0}{1-\alpha_2(1-p)}\right)}{g'\Big(g^{-1}\big((1-p)x+pu_2^0\big)\Big)} - 1$, where $x \in [u_2^0, g(u_1^0)]$. According to (h3), we have $u_2^0 < g(u_1^0)$. Since $x \leqslant g(u_1^0)$, $(1-p)x + pu_2^0 < g(u_1^0)$, by (h2), $g^{-1}\big((1-p)x + pu_2^0\big) > u_1^0$. Therefore,

$$\frac{(1-p)\alpha_1}{1-\alpha_2(1-p)} g^{-1}\big((1-p)x + pu_2^0\big) + \frac{pu_1^0}{1-\alpha_2(1-p)} < g^{-1}\big((1-p)x + pu_2^0\big).$$

Again by (h2), $0 < \frac{g'\left(\frac{(1-p)\alpha_1}{1-\alpha_2(1-p)} g^{-1}\big((1-p)x+pu_2^0\big) + \frac{pu_1^0}{1-\alpha_2(1-p)}\right)}{g'\Big(g^{-1}\big((1-p)x+pu_2^0\big)\Big)} < 1$. In addition, it is obvious that $0 \leqslant \frac{\alpha_1\alpha_2(1-p)^2}{[1-\alpha_1(1-p)][1-\alpha_2(1-p)]} < 1$. Thus $E'(x) < 0$, $\forall x \in [u_2^0, g(u_1^0)]$. Hence there exists at most one $w_2 \in [u_2^0, g(u_1^0)]$ such that $E(w_2) = 0$.

Let $H(\cdot)$ be a mapping that assigns a vector $H(w) = \alpha_1 \hat{u}^1(w) + \alpha_2 \hat{u}^2(w)$ to each $w \in \overline{X}(u^0)$, where $\hat{u}^i(w) = \big(\hat{u}_1^i(w), \hat{u}_2^i(w)\big) \in \partial X$ such that

$$\hat{u}_j^i(w) = (1-p)w_j + pu_j^0, \quad j \neq i. \tag{9}$$

This mapping is obviously continuous. Since $\hat{u}^i \in \overline{X}(u^0)$, $i = 1, 2$, $H(w) \in \overline{X}(u^0)$ due to the convexity of $\overline{X}(u^0)$. By Brouwer's fixed point Theorem, there exists at least one $w \in \overline{X}(u^0)$ such that $w = H(w)$. It is obvious that $w = H(w)$ is equivalent to (5) and (6). Hence, there exists at least one w satisfying (5) and (6).

In sum, there exists a unique $w^* \in \overline{X}(u^0)$ satisfying (5) and (6).

Now consider the following stationary strategy profile f^* of the underlying game $G'(\alpha, p)$: if $\gamma = \{i\}$, then i chooses to stay independent which leads to u^0; if $\gamma = N$ and i is chosen, then i will agree to be the proposer and suggest $\big(N, \hat{u}^i(w^*)\big)$ where $\hat{u}^i(w^*) = \big(\hat{u}^i_1(w^*), \hat{u}^i_2(w^*)\big) \in \partial X$, $\hat{u}^i_j(w^*) = (1-p)w^*_j + pu^0_j$, and $j \neq i$ will accept an offer if and only if his payoff in the offer is not less than $\hat{u}^i_j(w^*)$. We have $\overline{\pi}(f^*) = \alpha_1 \hat{u}^1(w^*) + \alpha_2 \hat{u}^2(w^*) = H(w^*) = w^*$.

It is easy to verify that f^* is a subgame perfect equilibrium, and thus is an SSPE. In fact, the case $\gamma = \{i\}$ is trivial. Now consider the case $\gamma = N$ and i is chosen. Given f^*_i, if j rejects i's offer, than j's expected payoff will be $(1-p)w^*_j + pu^0_j$, which is exactly $\hat{u}^i_j(w^*)$ according to (9). Given f^*_j, if i refuses to be the proposer, or to propose an offer that will be rejected by j, then i's expected payoff will be $(1-p)w^*_i + pu^0_i$, which is smaller than $\hat{u}^i_i(w^*)$ since $(1-p)w^* + pu^0 \in X \backslash \partial X$. Hence, no player has an incentive to deviate from f^* unilaterally. Therefore, we have proved that f^* is an SSPE, and thus w^* is a valuation of G.

Now we shall show that w^* is the only valuation of G. Assume on the contrary that G has another valuation $w' \neq w^*$, then there is an SSPE f' such that $\overline{\pi}(f') = w'$. Suppose according to f', the chosen player i at $\gamma = N$ will propose $\big(N, \bar{u}^i\big)$ where $\bar{u}^i \in \partial X$, and j will accept the offer if and only if he will get no less than $\bar{u}^i_j$. For f' to be SSPE, we have $\bar{u}^i_j = pu^0_j + (1-p)\overline{\pi}_j(f') = pu^0_j + (1-p)w'_j = \hat{u}^i_j(w')$. Therefore, $\bar{u}^i = u^i(w')$. This implies $w' = \alpha_1 \bar{u}^1 + \alpha_2 \bar{u}^2 = \alpha_1 u^1(w') + \alpha_2 u^2(w') = H(w')$, which contradicts the uniqueness of the fixed point of $H(w)$. □

Proof of Lemma 3.

Note that (8) can be rewritten in terms of matrices as follows:

$$T(\overline{s})w = c(\overline{s}), \tag{10}$$

where $c(\overline{s})$ is a column vector, and $T(\overline{s}) = \big(T_{i,j}(\overline{s})\big)$ is a square matrix. Due to the definition of $H(\cdot; \overline{s})$, we have $T(\overline{s}) = I_3 - \sum_{i \in N} \alpha_i T^i(\overline{s}_i)$, where I_3 is the identity matrix of size 3, while $T^i(0) = (1-p)I_3$, $T^i(i)$ is 3×3 zero matrix, and $T^i(\overline{s}_i)$ is a matrix whose entries are either 0 or $\pm(1-p)$ so that the sum of the entries in each column is zero if $\overline{s}_i \neq 0, i$. For example,

$$T^1(12) = \begin{pmatrix} 0 & p-1 & 0 \\ 0 & 1-p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2(12) = \begin{pmatrix} 1-p & 0 & 0 \\ p-1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T^3(13) = \begin{pmatrix} 1-p & 0 & 0 \\ 0 & 0 & 0 \\ p-1 & 0 & 0 \end{pmatrix}, \quad T^1(N) = \begin{pmatrix} 0 & p-1 & p-1 \\ 0 & 1-p & 0 \\ 0 & 0 & 1-p \end{pmatrix},$$

$$T^2(N) = \begin{pmatrix} 1-p & 0 & 0 \\ p-1 & 0 & p-1 \\ 0 & 0 & 1-p \end{pmatrix}, \quad T^3(N) = \begin{pmatrix} 1-p & 0 & 0 \\ 0 & 1-p & 0 \\ p-1 & p-1 & 0 \end{pmatrix}.$$

It is easy to verify that for any $\overline{s}$, $T(\overline{s})$ satisfies: (i) the sums of the entries in each column of $T(\overline{s})$, $\sum_{i=1}^{3} T_{i,j}(\overline{s}) \in (0,1]$, are identical across all columns, (ii) each entry $T_{i,j}(\overline{s}) \geqslant 0$, and (iii) the non-zero entries off the main diagonal of $T(\overline{s})$ are of the form $T_{i,j}(\overline{s}) = \alpha_i(1-p)$, $i \neq j$.

We can use some elementary row operations to transform $T(\overline{s})$ into $T'(\overline{s})$ as follows. First, add row two and row three to row one, and multiply row one by some appropriate number to transform this row into $(1,1,1)$. Next, multiply row 1 by $-T_{i,1}(\overline{s})$ and add it to row i, $i = 2,3$, then the matrix has been transformed into the form:

$$T'(\overline{s}) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & T'_{2,2} & T'_{2,3} \\ 0 & T'_{2,3} & T'_{3,3} \end{pmatrix}.$$

We can prove by exhaustion that $\det[T'(\overline{s})] = T'_{2,2}T'_{3,3} - T'_{2,2}T'_{3,3} > 0$. For examples, $\det[T'(0,12,13)] = p[1-\alpha_1(1-p)] > 0$, $\det[T'(N,N,N)] = p^2 > 0$, etc.

Therefore we have found a nonsingular matrix P such that $T(\overline{s}) = T'(\overline{s})P$. Hence, $\det[T(\overline{s})] = \det[T'(\overline{s})]\det(P) \neq 0$. Thus there exists a unique vector w satisfying (10). □

Supplementary Material

The following example is adapted from Mao (2014). Basically, this is a two-stage game with three players. The player who moves in stage one is an outside observer of the stage-two game, which is a noncooperative game played by the other two players. Hence, in stage one the moving player must first solve the stage-two game with inadequate information before choosing his action, which will affect the setting and the expected outcome of the stage-two game.

Once upon a time, there is a kingdom. The king relies on the government to govern a group of people. The people (represented by a single person) can produce e unit of output with $e^2/2$ unit of cost. The government decides the tax rate $\theta \in [0, 1]$, and transfers all tax income θe to the king. Given e and θ, the king's payoff is $u = \theta e$, and the people's payoff is $v = (1 - \theta)e - \frac{1}{2}e^2$.

There are two possible types of the government: dictatorship and equality. Let $t = a$ and $t = b$ denote these two types, respectively. If $t = a$, the government only concerns the interests of the king, and hence the government's payoff is $w^a = u$. If $t = b$, the government treats the people as important as the king, and the government's payoff is $w^b = \min\{u, v\}$.

The game is noncooperative, and proceeds as follows:

- Stage one: The king chooses the type of the government, $t \in \{a, b\}$.
- Stage two: The people and the government decide e and θ respectively in a game with inadequate information, which follows (g1)–(g5). Suppose the parameters are (α_1, p), where $p = 1$, and $\alpha_1 \in (0, 1)$ is the probability that the government is chosen to move first.

Assume that each player seeks to maximize his expected payoff. The king will choose equality if he is indifferent between two types.

If $t = a$, and the government moves first in stage two and chooses θ, then by backward induction we know $e = \frac{1}{2}$, $\theta = \frac{1}{2}$, and the payoffs are $(u_1^a, v_1^a, w_1^a) = (\frac{1}{4}, \frac{1}{8}, \frac{1}{4})$. When the people move first in stage two and choose e, we have $e = 0$, $\theta = 1$, and the payoffs are $(u_2^a, v_2^a, w_2^a) = (0, 0, 0)$. Thus, the expected payoffs are $(u^a, v^a, w^a) = \left(\frac{1}{4}\alpha_1, \frac{1}{8}\alpha_1, \frac{1}{4}\alpha_1\right)$.

If $t = b$, we can similarly obtain payoffs when the government and the people are chosen to move first respectively: $(u_1^b, v_1^b, w_1^b) = (\frac{2}{9}, \frac{2}{9}, \frac{2}{9})$, $(u_2^b, v_2^b, w_2^b) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. The expected payoffs are $(u^b, v^b, w^b) = \left(\frac{1}{4} - \frac{1}{36}\alpha_1, \frac{1}{4} - \frac{1}{36}\alpha_1, \frac{1}{4} - \frac{1}{36}\alpha_1\right)$.

The king will choose equality type if and only $u^a \leq u^b$; that is, $\alpha_1 \leq 0.9$. In other words, when the chance that the government moving earlier than the people is not very large (sometimes this implies that the government's bargaining power relative to the people is not too weak), the king is willing to give up dictatorship and introduce a more equal government out of his own interests. This result may help explain the transition of political systems in the history.

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