May 3 2006

Imperfect Signaling
and the Local Credibility Test

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Abstract

In this paper we study equilibrium refinement in signaling models. We propose a Local Credibility Test (LCT) that is somewhat stronger than the Cho and Kreps Intuitive Criterion but weaker than the refinement concept proposed by Grossman and Perry. Allowing deviations by a pool of "nearby" types, the LCT gives consistent solutions for any positive, though not necessarily perfect correlation between the signal sender's true types (e.g., signaling cost) and the value to the signal receiver (e.g., marginal product). Furthermore, it avoids selecting separating equilibria when they do not make sense. We identify conditions for an equilibrium to satisfy the LCT in both the finite and continuous type cases, and demonstrate that the conditions are identical as we take the limit in the finite type case. Intuitively, the conditions for an equilibrium to survive our LCT test require that a measure of signaling "effectiveness" is sufficiently high for every type and that the type distribution is not tilted upwards too much. We then apply the characterization results to several signaling applications.

*UCLA, UCLA, and The Ohio State University. We would like to thank seminar participants at Arizona University, Illinois Workshop on Economic Theory, Ohio State University, Penn State University, Rutgers University, UC Riverside, UC Santa Barbara, and Case Western Reserve University, for helpful comments and suggestions. All remaining errors are our own.

1. Introduction

Since the seminal work of Cho and Kreps (1987), various refinement concepts have been proposed to rank different equilibria in signaling games in terms of their "reasonableness". However, the mission is still far from being completed. In many applications, signals are "imperfect" in the sense that there is a positive yet imperfect correlation between the signal sender's true type (e.g., signaling cost) and the signal receiver's expected value (which then determines her response), see Riley (2001, 2002). [1] Consider a situation in which two of the sender types have a same signaling cost but quite different values to the receiver. If these two types do not observe their values to the receiver, they are effectively the same type, so the existing refinement concepts, such as the Cho and Kreps Intuitive Criterion, apply in the usual way. However, if these two types do observe their different values to the receiver, then the Intuitive Criterion is unable to rank equilibria. The reason is that if one of the two types likes a deviation, the other also likes it, hence no deviation is credible by a single type. This is highly unsatisfactory because the two cases are observationally equivalent.

The reason for the inconsistent solutions in the above example is that the existing refinement concepts focus on deviations by a single type only and do not consider deviations by a pool of types. Grossman and Perry (1986a,b), in a bargaining context, propose an equilibrium refinement concept strengthening the Cho-Kreps Intuitive criterion to allow pooling deviations. In this paper, in a general signaling model, we weaken the Grossman-Perry Criterion, and propose a "Local Credibility Test" (LCT) in which a possible deviation is interpreted as coming from one or more types whose equilibrium actions are nearby. We consider only local pooling deviations, first because

[1] Imperfect correlation between the signaling cost type and the receiver's expected value naturally arises when the sender's "physical" characteristics are multidimensional. The sender and the receiver may have similar preferences over the characteristics, but place different weights on its different dimensions. In the Spence education signaling model, a worker's characteristics may consist of her analytical skills and social skills. Both skills can be important to her education cost as well as to her marginal product in the workplace, but their relative importance in the two clearly can differ. In the reserve price signaling model of Cai, Riley and Ye (2004), characteristics of an artwork in an auction include its quality, rarity, history, etc. The seller (who has private information about its characteristics and signals the information with reserve prices) may be mostly concerned with characteristics related to the artwork's secondary market value, while potential buyers (who buy for self consumption) may care more about its impact in the setting for which it is intended.

they seem to us the most natural, second because they have much of the power of global pooling deviations, and third because they are more easily analyzed.

Consider an equilibrium of a signaling game. Suppose an out-of-equilibrium signal is observed. By the Cho-Kreps Intuitive Criterion, if one sender type can be strictly better off deviating to this signal from his equilibrium signal but all other types cannot, then such a deviation is credible for this type. The equilibrium is said to fail the Intuitive Criterion if there exists such a credible deviation. In addition to the requirement of the Intuitive Criterion, the Local Credibility Test allows the possibility of pooling deviations. Specifically, imagine that those types whose equilibrium signals are nearby the observed out-of-equilibrium signal deviate to this signal from their equilibrium signals but all other types do not, and the receiver correctly "anticipates" such a pooling deviation and holds the right perception about the expected type of the pool. If under the receiver's right perception, all the nearby types can be strictly better off from the deviation but all other types cannot, then such a pooling deviation is credible and we say that the equilibrium fails the LCT test. By allowing pooling deviations, the LCT test can be easily applied to situations with imperfect correlation between the signaling cost type and the receiver's expected value.

More importantly, the Local Credibility Test does not always rule out pooling equilibria in favor of separating equilibria. We will argue that in some situations separating equilibria seem unreasonable while pooling equilibria can be rather appealing. Precisely in such situations, the LCT avoids selecting the unreasonable separating equilibria. Thus, unlike the existing refinement concepts that always rank separating equilibria above pooling equilibria, the LCT selects separating equilibria only when they are reasonable. Consider a simple two type education-signaling model, in which the high type must take a quite costly signal (e.g., many years of unproductive education) to separate from the low type. Now suppose there is only one low type agent in every 5 million high type agents. In such a situation separation seems highly unreasonable, because without taking the costly signaling action an agent should not be perceived much differently from being the high type. By the LCT, it is easy to show that in any separating equilibrium a pooling deviation to some sufficiently low cost level of the signal is profitable to both types, so no separating equilibrium satisfies the LCT.

Many signaling applications are formulated in models with continuous types. Another advantage of the LCT is that it can be applied to both finite and continuous type cases equally well. We begin by formulating the concept of the LCT for the finite type models first, since the intuition is easier to present. Then we consider a discretization of the continuous type model, and take the limit as the discretization becomes finer. Later we study a family of continuous type models of which many commonly studied signaling applications such as the Spence education signaling model are members. We demonstrate that the conditions for an equilibrium to satisfy the LCT are identical in these two cases.

Another innovation of our analysis is to consider explicitly the sender's decision to participate in signaling. Economically this is important because potential entrants can influence signaling behavior of active senders in real world applications. Analytically, the existence of potential entrants helps ensure that senders of types slightly above the minimum signaling type do not want to deviate collectively to the minimum signal. We show that the only candidate equilibrium that can survive the LCT is a separating equilibrium that satisfies simple "upward" constraints and has the "right" minimum signaling type and the associated minimum signal. We then characterize conditions under which this equilibrium satisfies the LCT. The required conditions are intuitive. As long as a measure of signaling "effectiveness" is sufficiently high for every type above the minimum signaling type and the type distribution is not tilted upwards too much, the candidate equilibrium can survive our LCT test.

In the continuous type case, the set of equilibrium signals is dense so that out-of-equilibrium signals can be only found outside the set of equilibrium signals. However, thinking of the continuous type case as the limiting case of the finite type case with many close types, it is natural to generalize the concept of the LCT to the continuous type case. An equilibrium survives the LCT if no change in perception is credible in the following sense: for any possible signal (on- or off-equilibrium), if the revised perception is that the signal is from types of a small neighborhood of the immediate equilibrium type, it is profitable for the types in this neighborhood to deviate to this signal, but unprofitable for types out of this neighborhood to do so. Another way of thinking about this credibility test in the continuous type case is the following. If, for an on-equilibrium signal, there is such a deviation-perception pair, then those nearby types can credibly deviate to the particular on-equilibrium signal by throwing away ε amount of money. Since no other types would be willing to do so, this could convince the receiver that the deviating sender is indeed one of those nearby types, thus making the deviation-perception credible.

We derive conditions under which the LCT is satisfied by an equilibrium in the continuous type case. The conditions are exactly the same as in the limiting finite type case. This is satisfactory, because models with continuous types and models with finitely many types are theoretical tools for analyzing the same kind of real world problems. Put differently, it would be highly unsatisfactory if an equilibrium refinement concept applies to one case but not the other, or gives different answers for the two cases.

The paper is structured as follows. The next section uses simple examples to illustrate the basic idea of the LCT. Then Section 3 presents the general signaling model and formulates the concept of the LCT for the finite type case. We then provide a general characterization of the equilibrium satisfying the LCT. In Section 4, we derive conditions under which the LCT is satisfied by the candidate separating equilibrium in a limiting many type case. Section 5 generalizes the formulation of the LCT to the continuous type case, and shows that the conditions for the LCT are exactly the same as in the finite type case. Concluding remarks are in Section 6.

2. Examples

A consultant (s_i, v_j) has a signaling cost type s_i and a marginal product of v_j, where $s_1 < s_2 < ... < s_n$ and $v_1 < v_2 < ... < v_m$. She can signal at level *z* at a cost of $C(z, s_i)$, where $z \in [0, \bar{z}]$. We suppose that $\frac{\partial C}{\partial s}(z, s) < 0$ so that a higher type has a lower signaling cost. If paid a wage *w*, her payoff is $U(s_i, w, z) = w - C(z, s_i)$. In a competitive labor market for consultants, her wage will be her marginal product perceived by the market. Activity *z* is a potential signal because the marginal cost of signaling, $\frac{\partial C}{\partial z}(z, s_i)$, is a decreasing function of s_i. The probability of each type, $\pi(s_i, v_j)$, is positive and is common knowledge.[2] We suppose the two characteristics are affiliated. Then the conditional expectation $v(s_i) = E\{v \mid s_i\}$ is an increasing function of s_i, that is, $v(s_1) < v(s_2) < ... < v(s_n)$.

Initially we assume that each consultant observes her own signaling cost type but not her productivity. There is a continuum of separating Nash equilibria in this signaling game. A separating Nash equilibrium with three signaling cost types is depicted in Figure 2.1. Each curve is an indifference curve for some signaling cost type. A less heavy curve indicates a lower signaling cost type. In a separating equilibrium, the market can infer the consultant's signaling cost s_i from the signal she sends and thus pays her a wage equal to the expected marginal product $v(s_i)$. Note that the equilibrium choice for each type s_i (indicated by a shaded dot) is preferred over the choices of the other types.

Such an equilibrium fails the Intuitive Criterion proposed by Cho and Kreps (1987).[3] To see this, suppose a consultant chooses the signal $\hat{z}$ and argues that she is type s_2. Is this credible? If the consultant is believed, her wage will be bid up to $v(s_2)$ so she earns the same wage as in the separating equilibrium but incurs a lower signaling cost. Moreover, $(\hat{z}, v(s_2))$ is strictly worse than $(z_1, v(s_1))$ for type s_1 and strictly worse than $(z_3, v(s_3))$ for type s_3. Thus the claim is indeed credible, hence this separating equilibrium does not survive the Intuitive Criterion.

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[2] If $\pi(s_i, v_j) = 0$ for all $i \neq j$, the model reduces to the usual Spence model in which the negative correlation between signaling cost and value to receivers is perfect. While we assume $\pi(s_i, v_j) > 0$ for all i and j, the analysis applies generally.

[3] As noted by Cho and Kreps (1987), with more than two types, it is necessary to modify their original Intuitive Criterion or it loses much of its power. For the modified Intuitive Criterion the question is whether any particular type is uniquely able to benefit from some out-of-equilibrium signal if the signal receivers correctly infer the signaler's type.

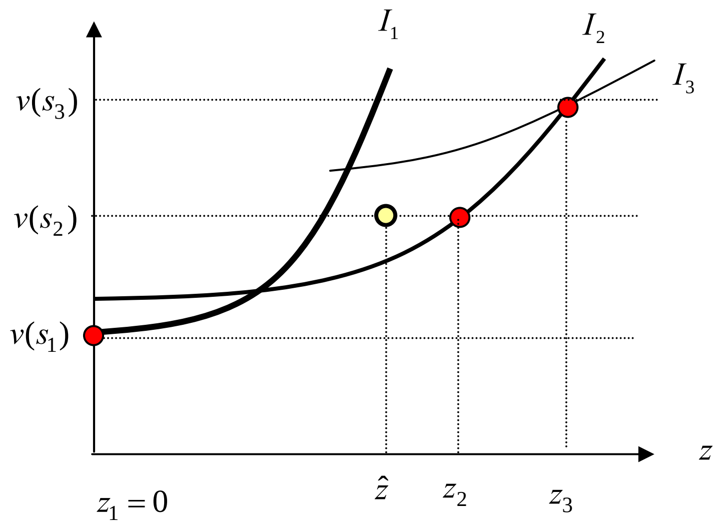

Fig: 2.1: Separating Nash Equilibria

Similar arguments rule out any Nash equilibrium where different signaling cost types are pooled. Thus the only equilibrium that satisfies the Intuitive Criterion is the *Pareto dominant separating equilibrium* (i.e., the Riley outcome) in which the lowest type chooses the smallest signal ($z = 0$) and each "local upward incentive constraint" is binding.

Next suppose that each consultant knows both her signaling cost type and her marginal product. Again consider the separating Nash Equilibrium depicted above. Suppose in this equilibrium three different types are pooled at each signal level. Consider the three types (s_2, v_1), (s_2, v_2), (s_2, v_3) pooled at z_2 with the expected marginal product $v(s_2)$. Suppose a consultant chooses $\hat{z}$ and claims to be type (s_2, v_3). Is this credible? If the claim is believed, the consultant's wage will rise from $v(s_2)$ to v_3 thus the consultant is indeed better off. But any offer that makes type (s_2, v_3) better off also make types (s_2, v_1) and (s_2, v_2) better off, since they have the same signaling cost. Thus there is no credible claim that type (s_2, v_3) *alone* can make. A similar argument holds for each of the other types. Thus any Nash separating equilibrium satisfies the Intuitive Criterion. An almost identical argument establishes that any Nash Equilibrium with (partial) pooling satisfies the Intuitive Criterion as well.

Since all the types with the same signaling cost are observationally equivalent, it seems to us that any argument for ranking the equilibria in the first model (productivity unknown) should also be applicable to the second model (productivity known) as well. The discussion also makes clear that a solution that achieves this goal should allow the possibility of pooling deviations in addition to deviations by single types. That is, if a pool of two or more types can credibly deviate to an out of equilibrium signal so that they can be better off while other types cannot, then the equilibrium fails the refinement test. In the above example, if an out of equilibrium signal $\hat{z}$ is observed, the receiver should allow the possibility that the sender can be any of the three types $(s_2, v_1),\ (s_2, v_2),\ (s_2, v_3)$. The question is what belief should the receiver have? Consistent with Cho and Kreps' original idea, one way to generalize their Intuitive Criterion (while allowing pooling deviations) is to suppose that the receiver has the most conservative belief that the sender is the lowest type from the pool. However, this generalization does not have power in the above example, because not all three types $(s_2, v_1),\ (s_2, v_2),\ (s_2, v_3)$ would be better off deviating to $\hat{z}$ if the receiver's belief is v_1.[4]

Given that upon observing the out of equilibrium signal $\hat{z}$ the receiver thinks that it can be any of the three types $(s_2, v_1),\ (s_2, v_2),\ (s_2, v_3)$, it is natural that she uses the Bayes Rule so her expected marginal product should be $v(s_2)$. Under this belief, a deviation to $\hat{z}$ by the pool of $(s_2, v_1),\ (s_2, v_2),\ (s_2, v_3)$ is clearly credible: any type in this pool is better off but types not in the pool are worse off from such a deviation. Then once again, the unique Nash Equilibrium satisfying this refinement test is the Pareto Dominant separating equilibrium.

We now introduce the formal definition of the Local Credibility Test.

[4] Similarly, it can be verified that the Cho and Sobel (1990)'s refinement concept of "divinity", which is built on the idea of stability of Kohlberg and Mertens (1986) and can be considered as a logic offspring of the Intuitive Criterion, does not have power either in the above example. Ramey (1996) extends the Cho and Sobel's divinity concept to the case of a continuum of types. Like the Intuitive Criterion, divinity faces the same problem of distinguishing types $(s_2, v_1),\ (s_2, v_2),\ (s_2, v_3)$ to interpret a possible deviation, while these types have the same incentives to deviate. Riley (2001) discusses in greater details these and other refinement concepts.

Local Credibility Test (LCT):

Suppose that an out-of-equilibrium signal $\hat{z}$ is observed and that z^- is the largest Nash Equilibrium signal less than $\hat{z}$ and z^+ is the smallest Nash Equilibrium signal greater than $\hat{z}$, if they exist. Let $\hat{S}$ be the subset of signaling cost types choosing z^- or z^+ with positive probability. For each $S \subset \hat{S}$, define $\overline{v}(S) = E[v \mid s \in S]$. Then the equilibrium passes the Local Credibility Test (LCT) if there is no ($\hat{z}$, S) such that $(\hat{z}, \overline{v}(S))$ is strictly preferred over the Nash Equilibrium outcome if and only if $s \in S$.

Heuristically, the sender who chooses the out of equilibrium signal $\hat{z}$ can make the following statement to the receiver: "I am in the subset S and you should believe me, because if you do and apply the Bayes Rule to update your belief, every type in S will be better off and all other types will be worse off than in the equilibrium." If there exists such a pair $(\hat{z}, S)$, the equilibrium fails the LCT.

Note that if $\hat{z}$ is smaller (greater) than all equilibrium signals, then z^- (z^+) does not exist and z^+ (z^-) is the smallest (largest) equilibrium signal. By the above definition, $\hat{S}$ is the subset of types choosing z^+ (z^-). Also note that by considering a subset of $\hat{S}$ to be the singleton set of a single type choosing z^+ or z^-, the definition of the Local Credibility Test allows deviations by single types. It follows that the LCT test is stronger than the Intuitive Criterion and hence only the Pareto Dominant separating equilibrium can pass the LCT.

On the other hand, the idea of the Local Credibility Test is weaker than the refinement concept proposed by Grossman and Perry (1986a,b) in bargaining models. For any out-of-equilibrium signal $\hat{z}$, their criterion considers *any* subset of types as a potential deviating pool. An equilibrium fails the Grossman and Perry test if $\hat{z}$ is credible for one subset of types. In signaling models the Grossman and Perry test is often too strong because no equilibrium can pass the test, especially when the type space is large. Here we restrict attention to local deviations. This makes the analysis more tractable and, we believe, more plausible.

More importantly, we now argue that the Pareto Dominant separating equilibrium survives the LCT test only when it makes sense. To see that the Pareto Dominant separating equilibrium can sometimes defy common sense, consider the following example. Suppose there are two signaling cost types. For those with a high signaling cost ($s = s_1$), the cost of signaling is $c_1(z)$ with $c_1(0) = 0$ and $c_1' > 0$, and the expected marginal product is 100. For those with a low signaling cost $(s = s_2)$, the signaling cost is $c_2(z) = (1 - \frac{\varepsilon}{100})c_1(z)$ and the expected marginal product is 200. The Pareto dominant separating equilibrium is depicted below in Figure 2.2.

The low type must be indifferent between $(0, v(s_1))$ and the choice of type s_2, that is $(z_2, v(s_2))$. Therefore,

$$U_1 = 100 = 200 - c_1(z_2) \text{ and so } c_1(z_2) = 100.$$

The payoff for type s_2 is therefore

$$U_2 = v(s_2) - c_2(z_2) = 200 - (1 - \frac{\varepsilon}{100})c_1(z_2) = 100 + \varepsilon.$$

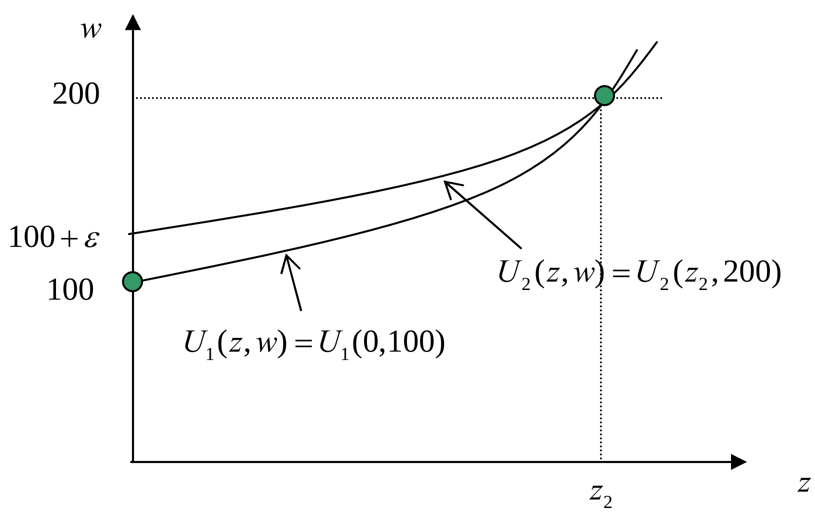

Fig. 2.2: Separating equilibrium with a gain of ε

Suppose that only 1 in 100 consultants is of type s_1. Then the unconditional mean marginal product is 199. Thus essentially all the social surplus generated by the high types is dissipated by signaling and both types have an income which is approximately half the income they would have in the Nash pooling equilibrium! We believe a good criterion for ranking equilibria should not rule out pooling equilibria in such circumstances. Applying the LCT test, it is easy to see that the Pareto dominant separating equilibrium does not survive the test. Consider a pooling deviation that both types deviate to an out of equilibrium signal $\hat{z}$ sufficiently close to zero. Then they both will be better off since the expected marginal product (and hence the wage) is 199 while the signaling cost is very small. Since no equilibrium survives the LCT, the more reasonable pooling equilibrium is not ruled out.

In this simple example, it is easy to determine when the Pareto dominant separating equilibrium passes the LCT. For concreteness, suppose $c_1(z) = 100z$, $c_2(z) = (100-\varepsilon)z$, and the probability of a consultant being type s_1 is q. It can be verified that the equilibrium survives the LCT if and only if $q > 1 - 0.01\varepsilon$.[5] Therefore, the larger the proportion of the low type is, or the greater the marginal cost difference between types is (i.e., the stronger the signal is), the more likely the Pareto dominant separating equilibrium passes the LCT test.

Another advantage of the LCT is that it can be applied consistently in both the finite type and the continuous type cases. To illustrate this, we now show in a simple Spence education signaling model how the LCT can be applied when there are many types and, in the limit, a continuum of types. Let the set of signaling cost types be $S = \{s_1, ..., s_n\}$ where $s_{i+1} - s_i = \delta > 0$, with probabilities $\{g_1, ..., g_n\}$ where $\sum_i g_i = 1$.

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[5] In the equilibrium, $z_1 = 0$, $z_2 = 1$, $U_1 = 100$ and $U_2 = 100 + \varepsilon$. By the definition of the LCT, consider $S = \hat{S} = \{s_1, s_2\}$. The average productivity of these two signaling cost types is $\overline{v} = 200 - 100q$. For $\hat{z} = \lambda \in (0,1)$ to be a credible deviation by S, both types must strictly prefer $(\hat{z}, \overline{v})$. Note that $U_1(\hat{z}, \overline{v}) = 200 - 100q - 100\lambda$ and $U_2(\hat{z}, \overline{v}) = 200 - 100q - (100-\varepsilon)\lambda$. Thus, if there exists a λ such that $U_1(\hat{z}, \overline{v}) = 200 - 100q - 100\lambda > 100 = U_1$ and $U_2(\hat{z}, \overline{v}) = 200 - 100q - (100-\varepsilon)\lambda > 100 + \varepsilon = U_2$, then both types are indeed better off choosing the out-of-equilibrium signal $\hat{z} = \lambda$ and the equilibrium will fail the LCT test. For the equilibrium to satisfy the LCT, it must be that $q > 1 - 0.01\varepsilon$.

Suppose type s_i has a signaling cost $C(z,s_i)=\dfrac{c(z)}{s_i}$. We also assume that those of type s_i have an expected marginal product of $v(s_i)=s_i$.

We seek conditions under which the Pareto dominant separating equilibrium passes the LCT. In this equilibrium, the local upward incentive constraints are binding. Therefore, as depicted below, those with signaling cost type s_{i-1} are indifferent between (z_{i-1},s_{i-1}) and (z_i,s_i). (The indifference curve is labeled I_{i-1}.) We construct the signal levels $\hat{z}$ and z_{i+1} as follows. Choose $\underline{z}$ so that those with signal type s_{i-1} are indifferent between (z_i,s_i) and $(\underline{z},\frac{1}{2}s_i+\frac{1}{2}s_{i+1})$. In the Figure 2.3 below, these are the points C_i and $\underline{C}$.

Then choose z_{i+1} so that those with signal cost type s_{i+1} are indifferent between $(\underline{z},\frac{1}{2}s_i+\frac{1}{2}s_{i+1})$ and (z_{i+1},s_{i+1}). In Figure 2.3, these are the points $\underline{C}$ and C_{i+1}. We will argue that type s_i must be indifferent between C_i and C_{i+1} as depicted. That is, C_{i+1} is the efficient separating contract for those with signaling cost type s_{i+1}.

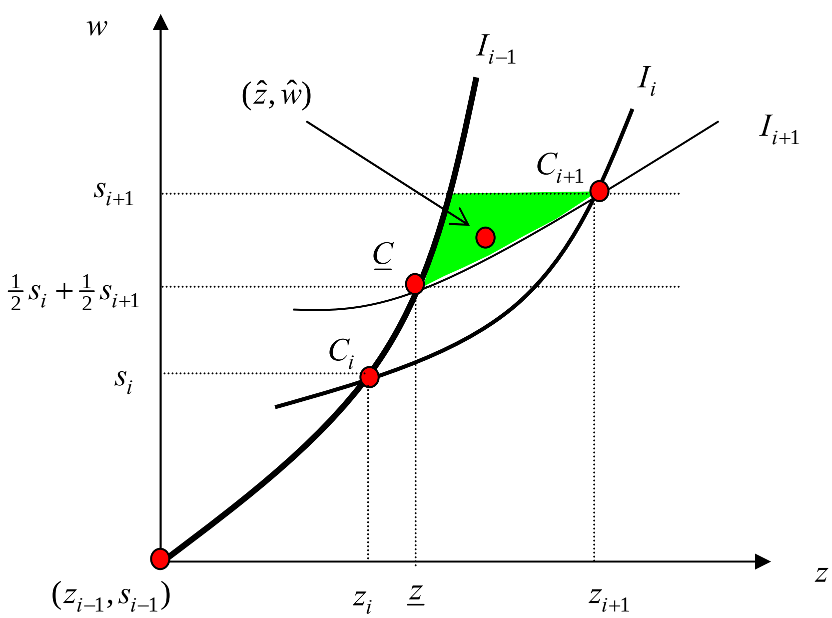

Fig. 2.3: Applying the LCT with many types

A type s_j consultant is indifferent between (z,w) and (z',w') if and only if $w-\frac{c(z)}{s_j}=w'-\frac{c(z')}{s_j}$, that is, if

$$c(z')-c(z)=s_j(w'-w) \tag{2.1}$$

By construction, a type s_{i-1} consultant is indifferent between (z_i,s_i) and $(\underline{z},\frac{1}{2}z_i+\frac{1}{2}z_{i+1})$. Appealing to (2.1),

$$c(\underline{z})-c(z_i)=s_{i-1}(\tfrac{1}{2}s_i+\tfrac{1}{2}s_{i+1}-s_i)=\tfrac{1}{2}s_{i-1}(s_{i+1}-s_i)\,. \tag{2.2}$$

Also, a type s_{i+1} consultant is indifferent between $(\underline{z},\frac{1}{2}z_i+\frac{1}{2}z_{i+1})$ and (z_{i+1},s_{i+1}). Again appealing to (2.1),

$$c(z_{i+1})-c(\underline{z})=s_{i+1}(s_{i+1}-\tfrac{1}{2}s_i-\tfrac{1}{2}s_{i+1})=\tfrac{1}{2}s_{i+1}(s_{i+1}-s_i)\,. \tag{2.3}$$

Adding equations (2.2) and (2.3) and noting that, by hypothesis, $s_i=\frac{1}{2}s_{i-1}+\frac{1}{2}s_{i+1}$, it follows that

$$c(z_{i+1})-c(z_i)=s_i(s_{i+1}-s_i)\,.$$

Appealing, finally, to (2.1), it follows that type s_i is indeed indifferent between (z_i,s_i) and (z_{i+1},s_{i+1}). Thus the point C_{i+1} is the Pareto dominant separating Nash Equilibrium contract for type s_{i+1}.

Note that for any wage $\hat{w}$ above $\frac{1}{2}s_i+\frac{1}{2}s_{i+1}$, there is a signal $\hat{z}$ between $\underline{z}$ and z_{i+1} that is strictly preferred by those with signaling cost type s_i and s_{i+1} but is not preferred by type s_{i-1} (or lower types). If $g_i<g_{i+1}$, the expected marginal product of these two types, $\frac{g_is_i+g_{i+1}s_{i+1}}{g_i+g_{i+1}}$ exceeds $\frac{1}{2}s_i+\frac{1}{2}s_{i+1}$ and, we can choose $\hat{w}$ to be equal to the expected marginal product. Then if types s_i and s_{i+1} choose the out-of-equilibrium signal $\hat{z}$, they can expect to be paid $\hat{w}$. The out-of-equilibrium signal is therefore credible and so the separating equilibrium fails the LCT. Conversely if $g_i>g_{i+1}$, the expected marginal product of these two types is less than $\frac{1}{2}s_i + \frac{1}{2}s_{i+1}$, the out-of-equilibrium signal $\hat{z}$ by the two types is not credible. If this holds for all *i*, that is $g_i > g_{i+1}$, $i = 1,...,n-1$, the Pareto dominant separating equilibrium passes the LCT.

Note that this condition is independent of the step size δ between signaling cost types. Treating the continuum of types as the limit when $\delta \to 0$, it follows that the Pareto dominant separating equilibrium passes the LCT if the density function $g(s)$ is everywhere decreasing. This is exactly the same as in the Spence education signaling model with a continuum of types, which we discuss in Example 1 of Section 5.

3. A General Model and Characterization

We consider the following signaling environment. A player, the sender, has a signaling cost characteristic $s_i \in S = \{s_1, s_2, ...s_n\}$ and a characteristic $v_j \in V = \{v_1, v_2, ...v_m\}$ valued by a receiver, where $s_i < s_{i+1}$ and $v_j < v_{j+1}$. The joint probability distribution of signaling cost and value characteristic is given by $\pi(s_i, v_j) \geq 0$, $i = 1,...,n$, $j = 1,...,m$, where $\sum_{i,j} \pi(s_i, v_j) = 1$. We shall refer to the sender with a specific vector of characteristics as type (s, v) and all those with a particular signaling cost as type s. Let the unconditional probability distribution of s be $g(s_i)$, where $\sum_i g(s_i) = 1$, and let $G(\cdot)$ denote the associated cumulative distribution function. We assume that the two characteristics are affiliated. Thus, the conditional expectation $v(s) = E[v \mid s]$ is an increasing function. The standard signaling model usually considers the special case in which s and v are perfectly correlated.

The sender knows her type and chooses an action, $y \in \{\phi\} \cup Y$, where $Y = [\underline{y}, \infty)$ is the set of feasible signals. When the sender chooses not to signal, $y = \phi$. The receiver does not know either characteristic of the sender but knows the distribution $\pi(s_i, v_j)$. He observes the sender's action and forms belief about her type. We assume that the receiver's payoff is linear in v so that it is only the perceived expected value $\hat{v}$ which determines the receiver's response. We can then write the sender's payoff as a function $U(s, \hat{v}, y)$ of her true type s, the receiver's expected value $\hat{v}$ and the sender's signal y.

If the sender chooses not to signal, she gets her reservation payoff $U^R(s)$, and without loss of generality the receiver gets a payoff of zero. Most signaling models do not consider the sender's incentives to participate in the signaling game. We believe the sender's participation is an important issue. For example, suppose an MBA degree is the minimum signal required in the consulting industry (and degrees from more expensive MBA programs are higher signals). A worker who has a good job opportunity in other professions (e.g., accounting) can easily decide not to pursue an MBA degree at all. For simplicity, we assume that the sender's reservation utility $U^R(s)$ is a function only of her signaling cost type and is non-decreasing. Furthermore, there exists a sender type s^* such that $U^R(s) \le U(s,s,z^*(s))$ if and only if $s \ge s^*$ and $U^R(s) < U(s,s,z^*(s))$ for all $s > s^*$, that is, only types higher than s^* are willing to participate in the game under complete information.

We make the following standard assumptions. The sender's payoff $U(s,\hat{v},y)$ is third order differentiable in all its elements and is increasing in s and $\hat{v}$: $U_1(s,\hat{v},y) > 0$, $U_2(s,\hat{v},y) > 0$. Furthermore, the sender's payoff function under full information $U(s,v(s),y)$ is quasi-concave in y, so that there is a unique optimal signal z_i^* for each type s_i if the sender's type is known to the receiver. To rule out trivial cases, we assume that $U(s_i,v(s_{i+1}),z_{i+1}^*) > U(s_i,v(s_i),z_i^*), \ \forall i$. If this condition does not hold for every type, then it is a separating equilibrium in which each type chooses her complete information optimal signal. This condition must be satisfied if the types are sufficiently close.[6] The last standard assumption we make is the single crossing condition:

$$\frac{\partial}{\partial s}\frac{d\hat{v}}{dy}\bigg|_U = -\frac{\partial}{\partial s}\frac{U_3}{U_2} = -\frac{U_{13}U_2 - U_{12}U_3}{U_2^2} < 0$$

[6] Let $\Delta s = s_{i+1} - s_i \to 0$, then, since $U_3(s,v(s),z^*(s)) = 0$, $[U(s_i,v(s_{i+1}),z_{i+1}^*) - U(s_i,v(s_i),z_i^*)]\big/\Delta s \to U_1 + U_2 v'(s) > 0$.

The slope of the indifference curve through any pair $(\hat{v}, y)$ is $\left.\frac{d\hat{v}}{dy}\right|_U = -\frac{U_3}{U_2}$. The single crossing condition requires that this should decrease with type.

For simplicity, we make the following technical assumptions:

B1: $U_{12} = 0$ for all $(s, \hat{v}, y)$,

B2: $U_{22} = 0$ for all $(s, \hat{v}, y)$.

B1 and B2 are satisfied by many signaling models. They are not crucial for our main results to hold. Assumption (B1) and the single crossing condition imply that $U_{13} > 0$.

Note that senders with a same signaling cost s and different value characteristics will always choose the same action in equilibrium. Thus, without loss of generality, we can focus on sender strategies that vary only by signaling cost type. As discussed in Section 2, our analysis applies equally whether or not the sender knows her value characteristic v.

A strategy for the type s sender is a probability vector $p(s) = (p_1(s), ..., p_L(s))$, where $Z = \{z_1, ..., z_L\}$ is the set of signals chosen with positive probability by at least one of the types, and $p_l(s)$ is the probability of the type s sender choosing signal z_l.[7] If the sum of the probabilities is less than 1, type s chooses her outside option (i.e., $z = \phi$) with probability $p_0(s) = 1 - \sum_{l=1}^{L} p_l(s)$. Given the strategies of all types, the expected value for those sender types choosing each level of the signal is given by $\bar{v}_l = E[v \mid z = z_l]$.

[7] We do not consider mixed strategies by the senders that have a positive support over an interval of signals. As will be clear below, any equilibrium that can pass the LCT must be separating. In a separating equilibrium, a sender will not mix over different signals because he cannot be indifferent between them.

A *Nash Equilibrium* of the signaling game satisfies the following conditions: (i) Incentive Compatibility: (a) for any i, l, if $p_l(s_i) > 0$, then $U(s_i, \overline{v}_l, z_l) \geq U(s_i, \overline{v}_j, z_j)$ for all $j \in \{1, 2, ..., L\}$; (b) for any i, if $p_0(s_i) > 0$, then $U^R(s_i) \geq U(s_i, \overline{v}_j, z_j)$ for all $j \in \{1, 2, ..., L\}$. (ii) Participation constraint: for any i, l, *if* $p_l(s_i) > 0$, then $U(s_i, \overline{v}_l, z_l) \geq U^R(s_i)$. We define $U^E(s)$ to be the NE payoff of type s.

We restate the definition of the Local Credibility Test here.

Local Credibility Test

Suppose that an out-of-equilibrium signal $\hat{z} \notin Z$ is observed and that z^- is the largest Nash Equilibrium signal less than $\hat{z}$ and z^+ is the smallest Nash Equilibrium signal greater than $\hat{z}$, if they exist. Let $\hat{S}$ be the subset of signaling cost types choosing z^- or z^+ with positive probability. For each $S \subset \hat{S}$, define $\overline{v}(S) = E[v \mid s \in S]$. Then the Nash Equilibrium satisfies the LCT if there exists no ($\hat{z}$, S) such that (i) $U(s, \overline{v}(S), \hat{z}) > U^E(s), \;\; s \in S$ and (ii) $U(s, \overline{v}(S), \hat{z}) < U^E(s), \;\; s \notin S$.

It proves useful to partition the types into a subset T_S whose members choose to signal and a subset T_o of types that prefer their outside alternatives. Proposition 1 gives necessary conditions for an equilibrium to pass the LCT. The idea behind it should be clear from the discussions in Section 2.

Proposition 1: *A Nash Equilibrium that passes the LCT test must have these properties:*

(1) There can be no pooling of signaling cost types $t \in T_S$.

(2) Binding upward constraints: For each type $t \in T_S$, *either the upward incentive constraint is binding for the next lowest type in* T_s *or the participation constraint is binding for some type* $s < t, \;\; s \in T_o$.

(3) The payoff for the lowest type s_1 *is the same as with full information:* $U^E(s_1)$ *equals* $Max\{U^R(s_1), U(s_1, v(s_1), z_1^*)\}$, *where* z_1^* *maximizes* $U(s_1, v(s_1), z)$.

Proof: See the Appendix.

Let us call a separating equilibrium that satisfies the binding upward constraint condition a *tight separating equilibrium*. By Proposition 1, only tight separating equilibria can pass the LCT test. An example of such an equilibrium is depicted in Fig. 3.1 below. Each of the square boxes is a reservation payoff. Type 1 senders earn their marginal product and choose the smallest signal $\underline{y}$. Type 2 senders earn their marginal product and signal at z_2. Any lower signal would attract type 1 so the tightness property holds for type 2. The minimum signal for type 3 (see the unshaded dot) yields a lower utility than the reservation payoff U_3^R. Thus type 3 senders do not signal. Type 4 senders must separate from all the lower types. As depicted, type 4 senders should choose signal z_4 so the participation constraint is binding for type 3.

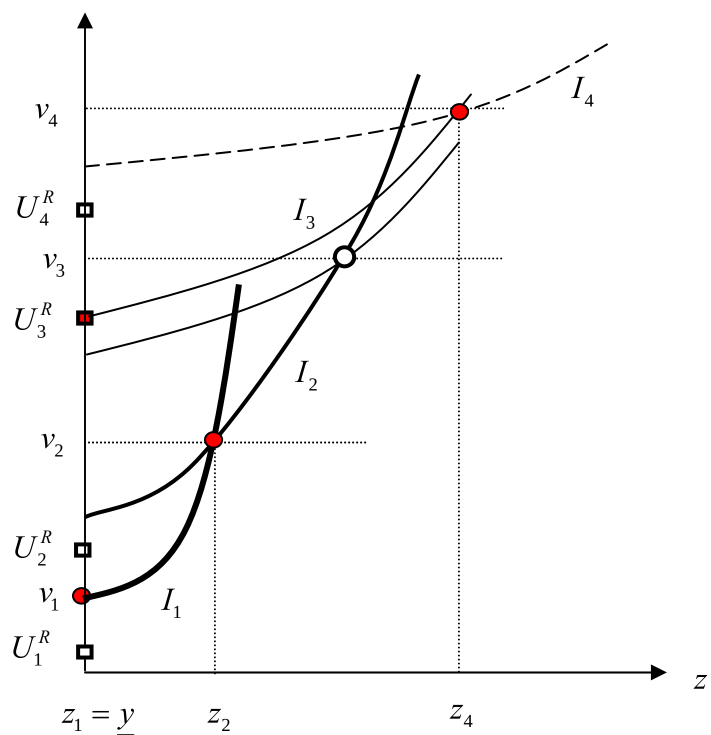

Fig. 3.1: Tight Separating Equilibrium

Within the class of tight separating equilibria, each equilibrium is identified by a minimum signaling type and a minimum signal this type chooses. The rest of the signaling schedule can be pinned down by the binding upward incentive constraints. Proposition 2 below characterizes the unique tight separating equilibrium that can possibly pass the LCT.

Proposition 2: *If an equilibrium satisfies the LCT test, it must be the tight separating equilibrium characterized as follows.*

(i) When $U^R(s_1) \le U(s_1, v(s_1), z_1^*)$*, the lowest type* s_1 *is the minimum signaling type (*$s_{\tilde{i}} = s_1$*), who chooses her optimal signal under complete information* z_1^*.

(ii) When $U^R(s_1) > U(s_1, v(s_1), z_1^*)$*, the lowest type* s_1 *does not signal (*$z_1 = \phi$*). The minimum type to signal is the smallest* $s_{\tilde{i}}$ *such that there is a signal* $z_{\tilde{i}}$ *satisfying (a)* $U^R(s_{\tilde{i}-1}) = U(s_{\tilde{i}-1}, v(s_{\tilde{i}}), z_{\tilde{i}})$ *and (b)* $U^R(s_{\tilde{i}}) \le U(s_{\tilde{i}}, v(s_{\tilde{i}}), z_{\tilde{i}})$.

(iii) Suppose that $U_1(s, v(s), z(s)) \ge dU^R(s)/ds$ *for* $s \ge s_{\tilde{i}}$*. For* $i = \tilde{i}+1, \ldots, n$*, type* s_i *chooses signal* z_i *such that* $U^E(s_{i-1}) = U(s_{i-1}, v(s_i), z_i)$.

Proof: See the Appendix.

Proposition 2 (i) and (ii) identify the minimum signaling type and associated minimum signal for the candidate equilibrium that can possibly survive the LCT. If the lowest type can get a payoff greater than her reservation payoff by choosing her complete information optimal signal, then she will participate in signaling and by definition will be the minimum signaling type. Otherwise, the minimum signaling type must choose a signal such that the type just below her must be indifferent between choosing the minimum signal to pretend to be the minimum signaling type and getting the reservation payoff. Of course, the minimum signaling type must get a payoff greater than her reservation payoff by choosing the minimum signal. An important fact to note is that when the lowest type does not signal, the minimum signaling type does not choose her complete information optimal signal because her signal must make the type just below her not willing to mimic.

The regularity assumption in (iii) guarantees that the local upward incentive constraints are binding, that is, if one type participates in signaling in the tight separating equilibrium, all higher types want to do so as well.[8] Under this condition Part (iii) of Proposition 2 characterizes the equilibrium signaling schedule. In addition, it can be easily checked that a sufficient condition for the assumption to hold is $U_1(s, v(s), z^*(s)) \geq dU^R(s)/ds$, where $z^*(s)$ maximizes $U(s, v(s), z)$. This says that the sender's optimal payoff under complete information increases in types faster than her reservation payoff.

Proposition 2 gives the only candidate equilibrium that can possibly pass the LCT test. We now investigate when this equilibrium indeed passes the LCT test.

First consider the case where the lowest type s_1 chooses to signal. Her signal and type s_2's signal are depicted in Figure 3.2 below.

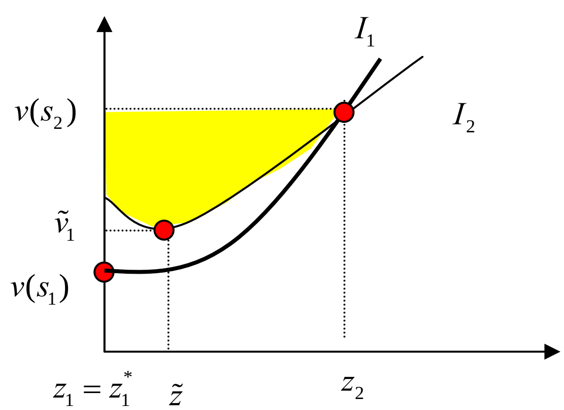

Figure 3.2

Let $(\tilde{z}, \tilde{v}_1)$ solve

$$\underset{(z,v)}{Min}\{v \mid U(s_1, v(s_1), z_1) = U(s_1, v(s_2), z_2) \text{ and } U(s_2, v, z) \geq U(s_2, v(s_2), z_2)\}$$

That is, $(\tilde{z}, \tilde{v}_1)$ is the lowest point on the indifference curve I_2 shown in Figure 3.2.

For any $s_i > s_1$, the equilibrium signals for s_i and s_{i+1} are depicted in Figure 3.3 below.

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[8] The condition thus rules out situations as depicted in Figure 3.1 in which the participation constraints bind for arbitrary types. However, it does not raise any additional conceptual issues to apply the LCT to such situations.

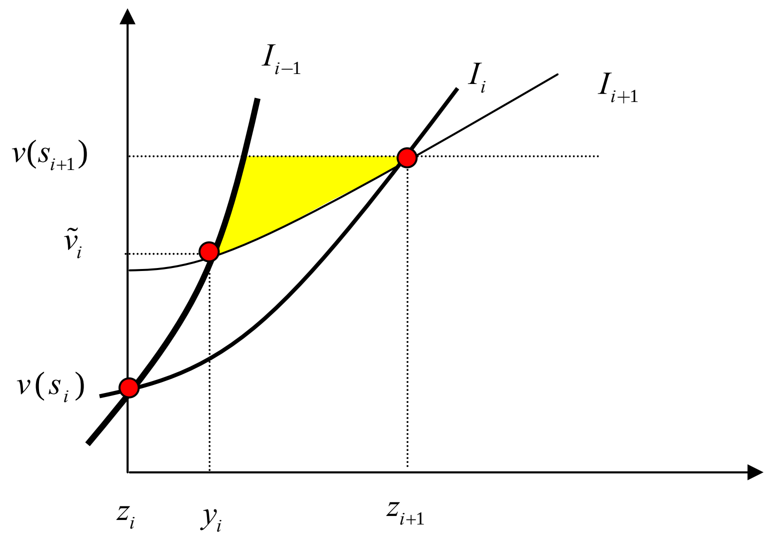

Figure 3.3

Define $(\tilde{v}_i, y_i)$ to be the intercept of indifference curves I_{i-1} and I_{i+1} depicted in Figure 3.3. Then

$$\begin{cases} U(s_{i-1}, \tilde{v}_i, y_i) = U(s_{i-1}, v(s_i), z_i) \\ U(s_{i+1}, \tilde{v}_i, y_i) = U(s_{i+1}, v(s_{i+1}), z_{i+1}) \\ U(s_i, v(s_i), z_i) = U(s_i, v(s_{i+1}), z_{i+1}) \end{cases} \tag{3.1}$$

For all i, let $\overline{v}_i$ be the average type of s_i and s_{i+1}, that is,

$$\overline{v}_i = \frac{[G(s_i) - G(s_{i-1})]s_i + [G(s_{i+1}) - G(s_i)]s_{i+1}}{G(s_{i+1}) - G(s_{i-1})} \tag{3.2}$$

In the other case where the minimum signaling type $s_{\tilde{i}}$ is greater than s_1, we define $\tilde{v}_i$ just as in (3.1) for $i \geq \tilde{i}$. For $i = \tilde{i} - 1 > 1$, $\tilde{v}_i$ can still be defined as in (3.1) except that $U(s_i, v(s_i), z_i)$ in the last equation in (3.1) is replaced with $U^R(s_i)$. For $i < \tilde{i} - 1$, we let $\tilde{v}_i = \overline{v}_i$. If $i = \tilde{i} - 1 = 1$, $\tilde{v}_1$ is defined as in Fig. 3.2 except that the constraint for type s_1 is his participation constraint. Thus $(\tilde{z}, \tilde{v}_1)$ solves

$$\underset{(z,v)}{Min}\{v \mid U^R(s_1) = U(s_1, v(s_2), z_2) \text{ and } U(s_2, v, z) \geq U(s_2, v(s_2), z_2)\}$$

Theorem 1: *The tight separating equilibrium characterized in Proposition 2 satisfies the LCT test if and only if for all* $i \le n-1$, $\tilde{v}_i > \overline{v}_i$.

<u>Proof</u>: Consider first the case where s_1 chooses the signal $z_1 = z_1^*$. Note that only points in the shaded area in Figures 3.2 are preferred by both types of s_1 and s_2 to their respective equilibrium points. If $\tilde{v}_1 > \overline{v}_1$, then there is no signal between z_1 and z_2 to which both types of s_1 and s_2 are willing to deviate if the receiver has the perception of $\overline{v}_1$. So the LCT is satisfied. On the other hand, if the LCT is satisfied, then it must be the case that $\tilde{v}_1 > \overline{v}_1$. Otherwise, it is clear from Figures 3.2 that there are signals that both types of s_1 and s_2 are willing to deviate to but other types do not. The same argument applies to higher types except for a minor modification in the definition of $\tilde{v}_i$. In defining $\tilde{v}_i$, note that only points in the shaded areas in Figures 3.3 are preferred by both types of s_i and s_{i+1}, *but not preferred by type* s_{i-1}, to their respective equilibrium points.

The same argument also works when the minimum signaling type is not s_1. For those types who participate in signaling, the LCT test is exactly the same as above. The only minor difference arises around the minimum signaling type. Type $s_{\tilde{i}}$ and type $s_{\tilde{i}-1}$ (who does not signal in the equilibrium) could both be better off deviating to a signal $y_{\tilde{i}-1} < z_{\tilde{i}}$ as long as the receiver's perception is at least $\tilde{v}_{\tilde{i}-1}$. When there are types lower than $s_{\tilde{i}-1}$ (i.e., $\tilde{i} > 2$), this $\tilde{v}_{\tilde{i}-1}$ is determined such that under this perception type $s_{\tilde{i}-2}$ would not be better off choosing $y_{\tilde{i}-1}$ than her reservation payoff $U^R(s_{\tilde{i}-2})$ (in Figure 3.3 the indifference curve I_{i-1} should represent $s_{\tilde{i}-2}$'s reservation payoff). When $\tilde{i} = 2$, then $\tilde{v}_1$ is determined as the minimum perception for type $s_{\tilde{i}} = s_2$ to be better off deviating to $z_1 = z_1^*$ (Figure 3.2). **Q.E.D**

Theorem 1 gives a complete characterization of the existence of equilibrium satisfying the LCT in any finite type model. The idea of Theorem 1 is simple. $\tilde{v}_i$ is the minimum perception by the receiver such that two types s_i and s_{i+1} can find a common profitable deviation that is not attractive to any other types. The receiver's correct perception about the pool of s_i and s_{i+1} is $\overline{v}_i$. If $\overline{v}_i < \tilde{v}_i$ for all $i \le n-1$, then no pair of types can find a local pooled deviation such that under the correct perception by the receiver, it is profitable only to them but not any other type. If that is the case, the LCT is satisfied by the separating equilibrium.

4. The Limiting Case of Many Types

One purpose of this research is to propose an effective equilibrium refinement that works for signaling models with continuous types. To make the connection to the continuous type case where the type space is $S = [\underline{s}, \overline{s}]$ and the cumulative probability distribution function is $G(s)$ with $G'(s) > 0$ z for all s, we consider the following finite type version of the signaling model. Suppose there are N types where $s_1 = \underline{s},\ s_{i+1} = s_i + \delta,\quad s_N = \overline{s}$ and $\delta = \dfrac{\overline{s} - \underline{s}}{N-1}$. For this finite type case, given any δ let $G(s_i) = prob.\{s \le s_i\}$ so that when $\delta \to 0$ the cumulative probability distribution approximates the cumulative probability distribution in the continuous type case. We let $N = 2^k - 1$ for $k \ge 2$, that is, as k increases by one, each interval is divided into two even ones. We are interested in the limit case when $k \to \infty$, or $\delta \to 0$, as an approximation of the continuous type case.

To simplify notation, we suppose $v(s) = s$. This is without loss of generality, because we can rewrite the sender's payoff function $U(s_i, v(s_i), z)$ as $u(s_i, s_i, z)$ by a change of variable (recall that $v(s)$ is strictly increasing in s). Let $\hat{s}$ be the receiver's perception of the sender's type. The sender's expected payoff is now $U(s, \hat{s}, y)$.

Without the participation constraints, a tight separating equilibrium $z_i = z(s_i)$ satisfies $U(s_i, s_{i,} z_i) = U(s_i, s_{i+1,} z_{i+1})$ for every s_i. Fix any s_i and let $\delta \to 0$, it must be that $(s_{i+1} - s_i)U_2 + (z_{i+1} - z_i)U_3 \to 0$. Thus, the equilibrium signaling schedule $z_i = z(s_i)$ satisfies $z'(s) = -U_2(s, s, z) / U_3(s, s, z)$.

If $U^R(\underline{s}) \le U(\underline{s}, \underline{s}, \underline{z}^*)$, where $\underline{z}^*$ is the lowest type's optimal signal under complete information, then by Proposition 2(i), the only candidate equilibrium that can satisfy the LCT is such that the lowest type chooses signal $\underline{z}^*$ and the equilibrium signal schedule as $\delta \to 0$ is given by $z'(s) = -U_2(s,s,z)/U_3(s,s,z)$.

If $U^R(\underline{s}) > U(\underline{s}, \underline{s}, \underline{z}^*)$, then the lowest type chooses not to signal. As before, let $s_{\tilde{i}}$ be the lowest type to signal. Applying Proposition 2 (ii) and taking the limit, we have

Lemma 1: *Suppose* $U^R(\underline{s}) > U(\underline{s}, \underline{s}, \underline{z}^*)$. *When* $\delta \to 0$, *the minimum signaling type* $\tilde{s}$ *and her equilibrium signal* $\tilde{z}$ *must satisfy*

$$U^R(s) = U(s,s,z) \tag{4.1}$$

and

$$\frac{d}{ds} U^R(s) = U_1(s,s,z)\,. \tag{4.2}$$

<u>Proof</u>: See the Appendix.

Note that by (4.1) and by the assumption that $U^R(s) \le U(s,s,z^*(s))$ if and only if $s \ge s^*$, we have $s_{\tilde{i}} \ge s^* > \underline{s}$, that is, the minimum signaling type is greater than the minimum type that is willing to participate under complete information.

To apply Theorem 1, for any k and $n \in \{\tilde{i}, \tilde{i}+1, \ldots 2^k - 1\}$, fix $s_n = s$. As k increases, the nearest types to s, $s_{n-1} = s - \delta(k)$ and $s_{n+1} = s + \delta(k)$, both get closer to s. Let $\overline{v}(s,\delta) = \overline{v}_n = E[s \mid s = s_n \text{ or } s_{n+1}]$ as defined in (3.2), and $v(s,\delta) = \tilde{v}_n$ be the solution to (3.1).

Lemma 2: *When* $\delta \to 0$, *(i)* $\overline{v}(s,\delta) \to s$; *(ii)* $\overline{v}_2(s,\delta) \to \dfrac{1}{2}$; *(iii)* $\overline{v}_{22}(s,\delta) \to \dfrac{G''(s)}{2G'(s)}$.

<u>Proof</u>: See the Appendix.

Lemma 3: *Suppose conditions* ***B1*** *and* ***B2*** *hold. When* $\delta \to 0$*, (i)* $v(s,\delta) \to s$*; (ii)* $v_2(s,\delta) \to \frac{1}{2}$*; (iii)* $v_{22}(s,\delta) \to -\frac{U_2 U_{13} U_{33} + U_2 U_3 U_{133} + 2U_3 U_{13} U_{23} - 2U_3^2 U_{113} + 4U_3 U_{13}^2}{4U_3^2 U_{13}}$.

Proof: See the Appendix.

We have our main characterization result for the limiting finite type case.

Theorem 2: *Suppose assumptions* ***B1*** *and* ***B2*** *hold. When* $\delta \to 0$*, a tight separating equilibrium satisfies the LCT if and only if the following conditions hold*

(i) $U^R(\underline{s}) > U(\underline{s}, \underline{s}, \underline{z}^*)$.

(ii) There is a solution $(\tilde{s}, \tilde{z})$ *to* (4.1) *and* (4.2) *such that* $\tilde{s} \neq s^*$ *or* $\tilde{s} = s^*$ *but* $U_3(s^*, s^*, z^*(s^*)) \neq 0$*, which defines the minimal signaling type and her signal.*

(iii) For all signaling types $s > \tilde{s}$*, suppose that* $U_1(s,s,z(s)) \geq dU^R(s)/ds$*. The equilibrium signaling schedule is given by* $z'(s) = -U_2(s,s,z)/U_3(s,s,z)$.

(iv) For any $s \in (\tilde{s}, \overline{s}]$,

$$-\frac{-U_2 U_{13} U_{33} + U_2 U_3 U_{133} + 2U_3 U_{13} U_{23} - 2U_3^2 U_{113} + 4U_3 U_{13}^2}{2U_3^2 U_{13}} > \frac{G''(s)}{G'(s)}. \tag{4.3}$$

Proof: See the Appendix.

Theorem 2 gives a complete characterization of the conditions under which there exists an equilibrium satisfying the LCT test for the limiting case of many types. Parts (ii) and (iii) are the limiting version of the tight separating equilibrium of Proposition 2(ii). Part (i) says that for a tight separating equilibrium to satisfy the LCT, the lowest type must not participate in signaling. To understand why this must be the case, consider the lowest type $s_1 = \underline{s}$ and type s_2. For simplicity suppose that $U_3(s_1, s_1, z) < 0$ so that the optimal signal for the lowest type, $z_1^* = \underline{y}$. If the lowest type s_1 participates in signaling, in a tight separating equilibrium the local upward constraint for this type is binding. Hence $U(s_1, s_1, z_1^*) = U(s_1, s_2, z_2)$. This is depicted below.

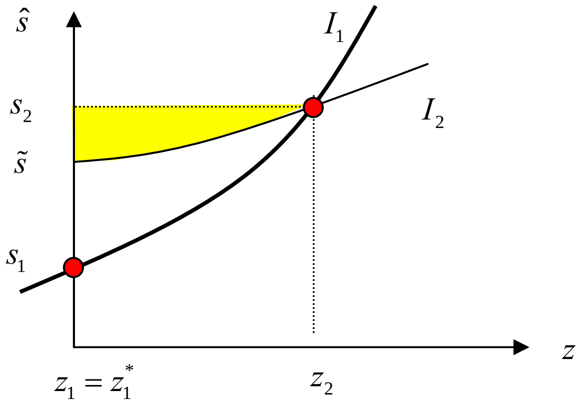

Fig. 4.1: The two lowest types

Both types strictly prefer the interior of the shaded area, where $\tilde{s}$ satisfies the following constraint: $U(s_2, \tilde{s}, z_1^*) = U(s_2, s_2, z_2)$. Taking Taylor approximations of both constraints around (s_2, z_2),

$$U_2(s_1, s_2, z_2)(s_2 - s_1) + U_3(s_1, s_2, z_2)(z_2 - z_1^*) = 0$$

and

$$U_2(s_2, s_2, z_2)(s_2 - \tilde{s}) + U_3(s_2, s_2, z_2)(z_2 - z_1^*) = 0 .$$

Substituting for $z_2 - z_1$,

$$\frac{s_2 - \tilde{s}}{s_2 - s_1} = \frac{U_3(s_2, s_2, z_2)}{U_3(s_1, s_2, z_2)} \frac{U_2(s_1, s_2, z_2)}{U_2(s_2, s_2, z_2)} .$$

In the limit the right hand side approaches 1. Hence $\dfrac{\tilde{s} - s_1}{s_2 - s_1} = 1 - \dfrac{s_2 - \tilde{s}}{s_2 - s_1}$ approaches zero and so for all sufficiently small differences in types, $\dfrac{\bar{s} - s_1}{s_2 - s_1} = \dfrac{g(s_2)}{g(s_1) + g(s_2)} > \dfrac{\tilde{s} - s_1}{s_2 - s_1}$.

Then $\bar{s} > \tilde{s}$. This problem does not arise when the minimum signaling type is not the lowest type, because a credible deviation by the pool of the minimum signaling type and the type just above her must make the type just below her unwilling to join a deviation (see Figure 3.3). The fact that the LCT test implies that the lowest type does not participate in signaling is not very restrictive. Once we explicitly take into account the sender's participation decisions in situations with a large set of potential sender types, it seems rather natural that some low types find it not worthwhile to signal. Then it is important to consider the effects of potential entrants on equilibrium signaling behavior of the active senders.

The idea for Theorem 2(iv) is as follows. Consider the consulting example discussed in Section 2, where we showed that one reason for non-existence is that the type distribution is tilted upwards too much. Nonexistence also arises if the indifference curves differ too little across types. Since the marginal rate of substitution between signal z and perception v is similar for the different types in that example, the indifference maps are similar and so indifference curves are close together. As a result, both types are better off deviating to $\hat{z}$ if the receiver believes that both may be choosing to deviate, thus violating the requirement of no credible deviation. However, if the marginal rate of substitution declines sufficiently rapidly with type, the indifference curve I_{i+1} in Figure 3.3 will be flat relative to the indifference curve I_i and I_{i-1}. Then the intercept of I_{i+1} and I_{i-1}, $\tilde{v}_i$, will be above $\overline{v}$, the average type. In this case there is no deviation by types s_i and s_{i+1} such that they will be better off under the perception of $\overline{v}$ but not any other types.

Intuitively, the rate at which the marginal rate of substitution declines with s is a measure of signaling effectiveness. Thus Theorem 2(iv) suggests that when signaling effectiveness is sufficiently large, the separating equilibrium will survive the LCT. This intuition is reflected in conditions (4.3). Note that the slope of the indifference map is given by $MRS = -U_3/U_2$, and by Assumption **B1,** $\frac{\partial}{\partial s}MRS = \frac{-U_{13}}{U_2}$. The last term of the LHS of (4.3) (over the denominator), $-\frac{2U_{13}}{U_3} = -\frac{2U_{13}/U_2}{U_3/U_2}$, has exactly the same interpretation: a measure of how rapidly the MRS declines with s. From Figure 3.3, the critical value of $\tilde{v}_i$ depends on how rapidly the curve I_{i-1} increases with z and how slowly the curve I_{i+1} increases with z. Note that $\frac{\partial}{\partial v}MRS(s,v,z(s))\Big|_{v=s} = \frac{-U_{23}}{U_2}$ by

Assumption **B2** and $\frac{\partial}{\partial z}MRS(s,s,z)=\frac{-U_{33}U_2+U_3U_{23}}{(U_2)^2}$. The first and third terms of the LHS of (4.3) can be rewritten as

$$-\frac{U_{13}(U_3U_{23}-U_2U_{33})+U_3U_{13}U_{23}}{2U_3^2U_{13}}=-\frac{U_3U_{23}-U_2U_{33}}{2U_3^2}-\frac{U_{23}}{2U_3}=-\frac{\partial MRS/\partial z}{MRS^2}+\frac{\partial MRS/\partial v}{MRS}$$

Figuratively, when the curve I_{i-1} is more straight-up (large $\partial MRS/\partial v$) and the curve I_{i+1} is more flat (small $\partial MRS/\partial z$), there will be no credible deviation with the perception at $\bar{v}$. The RHS of (4.3) is the concavity of the distribution function of s, $G(s)$, normalized by its density function. Intuitively, the more concave $G(s)$ is (i.e., the smaller G'' is), the more probability mass on smaller s in any set of types, thus the smaller the expected value of any set of types. Consequently, the smaller G'' is, the less likely a deviation is credible.

In summary, Theorem 2(iv) says that the tight separating equilibrium will satisfy the LCT if signals are effective in distinguishing types (i.e., MRS declining fast with types) and the type distribution is not too tilted upwards.

5. The Case of Continuous Types

In this section we work with continuous types directly and derive conditions under which a signaling equilibrium satisfies the LCT. Now the type space is $S=[\underline{s},\overline{s}]$ and the cumulative probability distribution function is $G(s)$ with $G'(s)>0$ for all s. We will show that the conditions for a signaling equilibrium to satisfy LCT will be exactly the same as those derived in the limiting finite type case studied in the preceding section.

As in the finite type case, we can focus on separating equilibria. The standard result in the literature (Riley, 1979; Mailath, 1987) shows that a separating equilibrium satisfies the following differential equation for active senders who choose to signal:

$$z'(s)=-\frac{U_2(s,s,z)}{U_3(s,s,z)} \tag{5.1}$$

Note that this is identical to part (iii) of Theorem 2.

As before, the lowest type $\underline{s}$ chooses to signal if and only if $U^R(\underline{s}) \le U(\underline{s}, \underline{s}, \underline{z}^*)$. For the same issue of credible deviations to the lowest boundary point as in the finite type case, part (i) of Theorem 2 is needed, that is, $U^R(\underline{s}) > U(\underline{s}, \underline{s}, \underline{z}^*)$. When this holds, by continuity, the minimum signaling type $\tilde{s}$ should satisfy $U^R(s) = U(s, s, z(s))$. Total differentiating this and using (5.1), we have $d\,U^R(s)/ds = U_1(s, s, z(s))$. These two conditions give the minimum signaling type $\tilde{s}$ and her signal $\tilde{z}$. This is identical to part (ii) of Theorem 2.

A conceptual issue arises when equilibrium refinements are considered for continuous types. In the continuous type case, any signal $y \in [\tilde{z}, \overline{z}]$ is "on-equilibrium," which leaves no room for considering out of equilibrium beliefs in the conventional approaches to equilibrium refinements. However, thinking of the continuous type case as the limit of the finite type case with many very close types, it is easy to understand how an "on-equilibrium" signal can be alternatively interpreted as a deviating signal and how to check credibility of such deviations. Consider any "on-equilibrium" signal $\hat{y}$, and suppose the type of sender for this signal in the separating equilibrium is s_0. Suppose in a finite type version of the model, $s_n < s_0 < s_{n+1}$ where s_n and s_{n+1} are two consecutive types. Suppose the nearby types $s \in S_0 = \{s_n, s_{n+1}\}$ deviate to this signal, and this is correctly perceived by the receiver and so the perception of the average type is $\hat{s} = E[s \mid s \in S_0]$. The LCT requires that if given the perception $\hat{s}$, all the deviating types in S_0 can gain relative to their equilibrium payoffs while all other types cannot, then those nearby types can credibly deviate to $\hat{y}$. If a separating equilibrium does not allow any such credible deviations, then it satisfies the LCT. Another way of thinking about on-equilibrium deviations is as follows. If for an on-equilibrium signal there is such a deviation-perception pair $(\hat{y}, \hat{s})$ as described above, then those nearby types can credibly deviate to $\hat{y}$ by throwing away a small amount of money. Since other types will not gain by mimicking, throwing away money by types in S_0 can convey to the receiver that they are the types deviating to $\hat{y}$.

Formally, the LCT test in the continuous type case can be stated as follows

Local Credibility Test:

Consider a separating equilibrium schedule $z(s):[\tilde{s},\bar{s}]\to[\tilde{z},\bar{z}]$. For any signal $\hat{y}\in[\tilde{z},\bar{z}]$, let $z(s_0)=\hat{y}$ and consider a small neighborhood of s_0, $S_0\subset S$. Let $\hat{s}=E[s\mid s\in S_0]$. If

(*i*) $U(s,\hat{s},\hat{y})>U(s,s,z(s))$, for all $s\in \operatorname{int} S_o$, and

(*ii*) $U(s,\hat{s},\hat{y})<U(s,s,z(s))$, for all $s\notin S_o$.

Then the signal-perception $(\hat{y},\hat{s})$ is credible. If the separating equilibrium $z(s)$ survives the LCT, then there cannot exist any credible signal-perception pair.

Since the continuous type case is viewed as an approximation of the case of many close finite types, we only need to check whether there are any credible deviations for very small intervals, in the sense that will be made precise below.

For any two types s and s' where $\tilde{s}\le s<s'$, suppose those in the interval $[s,s']$ pool at a certain signal y. Let $\bar{v}(s,s')$ be the expected type of this pool:

$\bar{v}(s,s')=\dfrac{\int_s^{s'}xdG(x)}{G(s')-G(s)}$. Let $v(s,s')\in[s,s']$ and $y(s,s')\in Y$ be a solution to

$$\begin{cases}U(s',v,y)=U(s',s',z(s'))\\ U(s,v,y)=U(s,s,z(s))\end{cases} \tag{5.2}$$

The point $(y(s,s'),v(s,s'))$ is depicted below in Figure 5.1. Given this signal-perception pair, all those types in $[s,s']$ prefer the pool to their separating equilibrium payoff.

The types in $[s,s']$ cannot find a credible deviation if and only if the signal-perception pair of $(y(s,s'),\bar{v}(s,s'))$ is not credible. Therefore, the separating equilibrium $z(s)$ satisfies LCT if for any $s\in[\tilde{s},\bar{s})$ and $s'>s$, $\bar{v}(s,s')<v(s,s')$ as $s'\to s$. Note that for any $s\in[\tilde{s},\bar{s})$, $\bar{v}(s,s)=v(s,s)=s$. Furthermore, we have

Lemma 4: *For any* $s\in[\tilde{s},\bar{s})$ *, (i)* $\bar{v}_2(s,s)=1/2$ *; (ii)* $\bar{v}_{22}(s,s)=\dfrac{1}{6}\dfrac{G''(s)}{G'(s)}$.

Proof: See the Appendix.

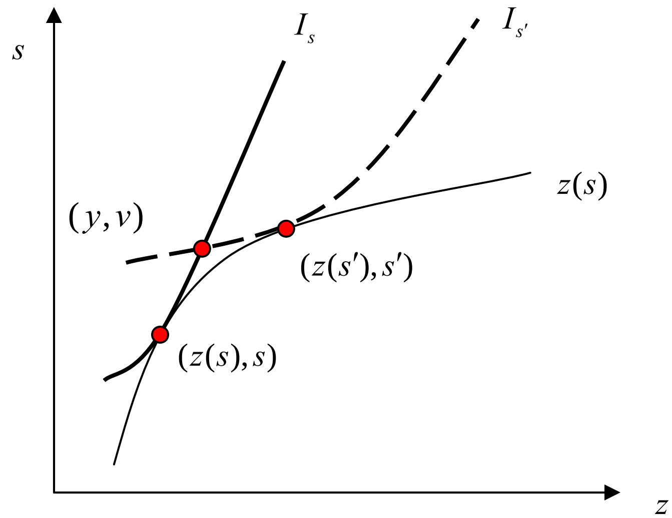

Fig. 5.1: Pool of types in $[s,s']$

Lemma 5: *Suppose conditions **B1** and **B2** hold. (i)* $v_2(s,s)=1/2$ *; (ii)*

$$v_{22}(s,s)=-\frac{-U_2U_{13}U_{33}+U_2U_3U_{133}+2U_3U_{13}U_{23}-2U_3^2U_{113}+4U_3U_{13}^2}{12U_3^2U_{13}}$$

<u>Proof</u>: See the Appendix.

Theorem 3 below shows that the characterization result of Theorem 2 applies equally well to the continuous type.

Theorem 3: *Suppose assumptions **B1** and **B2** hold. A separating equilibrium satisfies the LCT if and only if all the conditions of Theorem 2 hold.*
<u>Proof</u>: See the Appendix.

Therefore, Theorems 2 and 3 show that the concept of the LCT can be applied to the continuous type model exactly as in the finite type model. While the continuous type model is easier to work with analytically in terms of characterizing the separating equilibrium, it should be viewed as an approximation of the situation with many close finite types. Our position is that it should be subject to the same scrutiny of credibility as finite type models, even though signals are literally "on-equilibrium" in the continuous type model. The equivalence of Theorems 2 and 3 demonstrates that this approach is valid.

Below we use three examples to illustrate how to apply our results to signaling models.

Example 1: The Spence education signaling model

A worker knows her own personal skill level or productivity, denoted by s. The labor market knows that s is drawn from distribution $G(s)$ on $[0,1]$. Her expected payoff is $U(s,\hat{s},y)=\hat{s}-C(s,y)$, where s is her productivity unknown to firms, $\hat{s}$ is her productivity perceived by firms and hence is also the wage offered to her by competing firms, and $y\in[0,\overline{y}]$ is the education signal the worker can choose. It is typically assumed that for all (s,y), (i) $C_1(s,y)<0$; (ii) $C_2(s,y)>0$; and (iii) $C_{12}(s,y)<0$. It can be verified that the single crossing and conditions B1 and B2 are satisfied.

To further simplify things, suppose $C(s,y)=ys^{-a}$, $a>0$. The marginal cost of signaling is $MC(s)=s^{-a}$ so the parameter a is the elasticity of the marginal cost of signaling with respect to type. Note that under complete information, workers of all types choose the minimal signal $\underline{y}=0$. Suppose the worker's reservation payoff is $U^R=\alpha+\beta s$, where $\alpha,\beta>0$. If $\alpha+\beta\geq 1$, no type can be better off in a separating equilibrium than if she accepts her outside alternative. Thus we assume that $\alpha+\beta<1$. Note that the lowest type in this case will not signal since $U^R(\underline{s})=\alpha>0=U(\underline{s},\underline{s},\underline{y})$. Moreover, with complete information a worker chooses not to signal if and only if $\alpha+\beta s<s$, that is, $s<s^*=\alpha/(1-\beta)$.[9]

To determine the minimum signaling type $\tilde{s}$ and the corresponding signal $\tilde{y}$, by part (ii) of Theorem 2 we have $\beta=U_1=ays^{-a-1}$ and $\alpha+\beta s=s-ys^{-a}$. Thus, $\tilde{s}=\alpha/(1-(\frac{a+1}{a})\beta)$. For $\tilde{s}<1$, we need to assume that $\alpha+\frac{1+a}{a}\beta<1$, That is, the elasticity of the signaling cost with respect to type (the parameter a) must be sufficiently

[9] With complete information a worker will choose $y^*=0$ should she decide to signal.

large. Note that $\tilde{s} > s^*$, i.e., fewer types participate in the signaling market than under complete information. Moreover, the higher the elasticity of the marginal cost of signaling is (the larger a is), the lower is the minimum type who signals.

By part (iii) of Theorem 2, a separating equilibrium satisfies

$$z'(s) = -\frac{U_2}{U_3} = \frac{1}{C_2} = s^a$$

Thus the signaling schedule is given by $z(s) = \frac{s^{a+1}}{a+1} + (\frac{\beta}{a} - \frac{1}{a+1})\tilde{s}^{a+1}$. Substituting for $z(s)$, it can be verified that for $s \in (\tilde{s},1]$, $U_1(s,s,z(s)) = az(s)/s^{1+a}$ is increasing in s, thus $U_1(s,s,z(s)) > \beta = dU^R/ds$. Hence the participation constraint is satisfied for all $s \in (\tilde{s},1]$.

Since $C(s,y) = ys^{-a}$, it can be checked that condition (4.3) can be simplified to

$$\frac{a-1}{s} > \frac{G''(s)}{G'(s)}$$

Therefore, by Theorem 2, the LCT will be satisfied if the signal effectiveness measured by a is sufficiently large and the type distribution is not tilted upward too much. When s is uniformly distributed, the right hand side is zero. Then when $a > 1$, the tight separating equilibrium satisfies the LCT.

<u>Example 2: The reserve price signaling model</u>

Cai, Riley and Ye (2004) study reserve price signaling in a fairly general auction environment allowing bidders' signals to be affiliated. A simpler version of the model is as follows. A seller of an indivisible good has private information about certain characteristics of the good that potential bidders do not know. Let $\theta \in \Theta \subseteq R^n$ be the seller's private information. The seller's own valuation of the good is $\gamma s(\theta)$, and the common value component of the bidders' valuations is $t(\theta) = s(\theta)$, where $\gamma > 0$. We normalize the range of $s(\theta)$ so that $s \in [0,1]$. Ex ante, the distribution of θ induces a distribution, $G(\cdot)$, for s.

Bidder i's valuation is $t + x_i$, where $x_i \in [0,\overline{x}]$ is the private value component that is known to himself only. The bidders' private signals $\{x_i\}$ are i.i.d. random variables with a distribution function $F(\bullet)$ and an everywhere positive density function $f(\bullet)$. Suppose the seller uses a sealed-bid second-price auction to sell the good; and she sets a reserve price r which determines m, the minimum type of bidder who enters the auction. Let $\hat{s}$ be the perceived type of the seller, i.e., the perceived common value component in bidders' valuations, then $m = r - \hat{s}$. Since the reserve price schedule can be recovered from the minimum type schedule through $r(s) = m(s) + s$, Cai, Riley and Ye (2004) focus on the minimum type schedule $m(\cdot)$ to characterize the signaling equilibrium. First, given the signal m and the perceived common value $\hat{s}$, the seller's expected payoff can be expressed as

$$U(s,\hat{s},m) = \gamma s F_{(1)}(m) + \hat{s}(1 - F_{(1)}(m)) + B(m) \tag{5.3}$$

where $F_{(1)}(\bullet)$ is the distribution function of the first order statistics, and

$B(m) = m(F_{(2)}(m) - F_{(1)}(m)) + \int_m^{\bar{x}} x dF_{(2)}(x)$ and $F_{(2)}(\bullet)$ is the distribution function of the second order statistics. Thus the model fits into the standard signaling framework. It can be verified that the single crossing and **B1** and **B2** conditions are all satisfied.

If the seller does not sell the item in the auction, she may sell the item by some other means such as posted price or bargaining. Suppose the payoff implied by the best outside option is given by $U^R(s) = \alpha + \beta s$, where $\alpha, \beta > 0,$ and $\beta < \gamma$. To determine the minimum signaling type $\tilde{s}$ and the corresponding signal $\tilde{m}$, by part (ii) of Theorem 2 we have $\beta = \gamma F_{(1)}(m)$ and $\alpha + \beta s = \gamma s F_{(1)}(m) + s(1 - F_{(1)}(m)) + B(m)$. Solving this equation system we have $\tilde{m} = F_{(1)}^{-1}(\beta/\gamma)$ and $\tilde{s} = \left(1 - \beta/\gamma\right)^{-1}\left(\alpha - B(\tilde{m})\right)$. For $\tilde{s} \in (0,1)$, we need to assume that $\left(1 - \beta/\gamma\right)^{-1}\left(\alpha - B(\tilde{m})\right) \in (0,1)$.

By part (iii) of Theorem 2, a separating equilibrium satisfies

$$m'(s) = -\frac{U_2(s,s,m(s))}{U_3(s,s,m(s))} = \frac{1 - F_{(1)}(m)}{(J(m) - (\gamma - 1)s) f_{(1)}(m)}$$

where $J(m) = m - \left(1 - F(m)\right)/f(m)$. From (5.3), it can be verified that condition (4.3) in this reserve price signaling model amounts to

$$m'(s)\left[\frac{J'(m)}{2\left((\gamma-1)s - J(m)\right)} + (2\gamma-1)\frac{f_{(1)}(m)}{1-F_{(1)}(m)}\right] > \frac{G''(s)}{G'(s)}. \qquad (5.4)$$

In the special case where there are two bidders, each bidder's private value signal is distributed uniformly on [0,1], and $\gamma \in (1,2)$,[10] it can be verified that the solution to (4.1) and (4.2) is given by $\tilde{m} = \sqrt{\beta/\gamma}$ and $\tilde{s} = \left(1-\beta/\gamma\right)^{-1}\left[\alpha - \frac{1}{3} - \beta/\gamma + \frac{4}{3}(\beta/\gamma)^{3/2}\right]$. In this case, for $\tilde{s} < 1$, it requires that $(\beta/\gamma)^{3/2} < 1 - 3\alpha/4$. Thus, if either α or β/γ is too large, there is no signaling by any type, which is intuitive as then the outside option would be too attractive. Since $F(x)$ is uniform on [0, 1], by substituting $F_{(1)}(m) = m^2$, $f_{(1)}(m) = 2m$, and $J(m) = 2m - 1$ into (5.4), we have

$$m'(s)\left[-\frac{1}{2m-1-(\gamma-1)s} + (2\gamma-1)\frac{2m}{1-m^2}\right] > \frac{G''(s)}{G'(s)}$$

If $G(\cdot)$ is concave, then the above inequality holds if the LHS is strictly positive. It can be verified that for a fixed β, this holds for relatively small γ within the relevant range (such that $\tilde{s} < 1$). Thus, else being equal, the smaller γ, the more likely that the signaling equilibrium can satisfy our LCT (when $G(\cdot)$ is concave).

In the two examples above, both of the regularity conditions **B1** and **B2** hold so we can directly apply Theorem 2 (iv). Below we illustrate via an example that our approach is still applicable even when those regularity conditions fail.

Example 3: An advertising signaling model

[10] Restricting γ within $(1,2)$ ensures that the equilibrium signaling schedule would not be truncated (see Theorem 3 in Cai, Riley, and Ye (2004)).

This example is adopted from Milgrom and Roberts (1986). A monopolistic firm can produce a good with a constant marginal cost c. It sells to a unit mass of consumers. The firm knows its product quality, denoted by s. Among the consumers, $a<1$ are informed about s. The rest $1-a$ of consumers are uninformed of s, and their belief is given by the distribution $G(s)$ on $[\underline{s},\overline{s}]$. For products of quality s, the consumers' inverse demand function is $p=s-bq$, where b is a positive parameter and q is the quantity. So, given price p, the demand of an informed consumer is $q_I=(s-p)/b$, and that of an uninformed consumer is $q_U=(\hat{s}-p)/b$, where $\hat{s}$ is his perception of s. The total demand is then $q=aq_I+(1-a)q_U$.

Suppose the firm spends z on advertising, which leads to a perception of $\hat{s}$ by the uninformed consumers. By choosing an optimal price given $\hat{s}$, the firm's maximum profit is $U=\dfrac{[as+(1-a)\hat{s}-c]^2}{4b}$. Consider a possible separating signaling schedule $z(s)$. The firm's payoff function is

$$U(s,\hat{s},z)=\frac{[as+(1-a)\hat{s}-c]^2}{4b}-z$$

Suppose the firm's reservation payoff is $U^R=\alpha+\beta s$, where $\alpha,\beta>0$. When $\alpha+\beta\underline{s}>(\underline{s}-c)^2/(4b)$, the lowest type will not signal. The minimal signaling type in equilibrium can be solved from (4.2), which gives $\beta=U_1(s,s,z(s))=a(s-c)/(2b)$, or $\tilde{s}=c+2b\beta/a$. The associated minimum signal can be found from (4.1), $\tilde{z}=(\tilde{s}-c)^2/(4b)-\alpha-\beta\tilde{s}$.

It can be checked that the single crossing condition holds. From part (iii) of Theorem 2,

$$z'(s)=-\frac{U_2(s,s,z(s))}{U_3(s,s,z(s))}=\frac{(1-a)(s-c)}{2b}$$

Since $U_1(s,s,z(s))$ is increasing in s, thus $U_1(s,s,z(s))>\beta=dU^R/ds$ for $s\in(\tilde{s},1]$. Hence the participation constraint is satisfied for all types above $\tilde{s}$.

From the firm's payoff function, we have $U_{12}=a(1-a)/2b$ and $U_{22}=(1-a)^2/2b$. Since this model does not satisfy conditions **B1** and **B2**, we cannot apply Theorem 2 (iv) directly. However, using the same method that we follow in showing Theorem 2 (iv), it is not difficult to derive conditions under which the LCT is satisfied in this example. We summarize the conditions in the following proposition:

Proposition 3: *In the advertising signaling model, the separating equilibrium satisfies the LCT if* $G''(s) < 0$ *for* $s \in [\tilde{s}, \overline{s}]$ *, where* $\tilde{s} = c + 2b\beta / a$ *is the minimal signaling type.*

Proof: See the Appendix.

Because of the quadratic function form of the firm's payoff function and other special features of this model (e.g., many cross derivatives are zero), the measure of signal effectiveness is constant and equals zero. So whether the tight separating equilibrium satisfies the LCT only depends on the type distribution. A special case commonly studied in applications is when the type is uniformly distributed. Then $G'' = 0$ for $s > \underline{s}$. By our definition, however, the LCT is not satisfied.

6. Concluding Remarks

Except in the special case of perfect correlation between the sender's true type and the value to the signal receiver, standard refinements (Intuitive Criterion, Divinity, Stability) are not applicable. We argue that to have any "bite" at all, a refinement is needed in which the signal receiver takes into account the way sender types are distributed and allows the possibility of deviations by a pool of sender types (in addition to single-type deviations). We then propose a Local Credibility Test which is somewhat stronger than the Cho and Kreps Intuitive Criterion but milder than the Grossman-Perry Criterion. Allowing deviations by a pool of "nearby" types, the LCT gives consistent solutions for any positive, though not necessarily perfect correlation between the signal sender's signaling cost types and the receiver's expected values. Besides this, the LCT has two additional advantages. It avoids selecting separating equilibria when they do not make sense, thus providing economically sensible answers to the equilibrium selection problem in signaling models. Moreover, it applies equally well in cases of finite and continuous types, making it applicable to many signaling applications that are formulated in continuous type models. In this paper we provide conditions under which a tight separating equilibrium satisfies the LCT. These conditions are the more likely to be met, (a) the less rapidly the density increases or the more rapidly the density decreases with type, and (b) the more rapidly the marginal cost of signaling decreases with type. We illustrate the applicability of the LCT in three examples of signaling models.

What is the “right” equilibrium when our conditions are not met? This is a challenging question for which we have no satisfactory answer. As can be seen from the example depicted in Figure 2.2, our LCT test shows that the Pareto dominant separating equilibrium is no longer more reasonable than other equilibria. We conjecture that pooling or partial pooling must be a part of any more complete analysis of signaling.

Appendix: Proofs

Proof of Proposition 1: (1) Suppose a group of types choose signal z_l with positive probability. Let $\vec{s}$ be the largest in the group. Then the receiver's conditional expected value $\overline{v}_l = E\{v \mid z = z_l\}$ is smaller than $v(\vec{s}) = E[v \mid \vec{s}\,]$. By the single crossing condition, there exists some signal higher than z_l such that only $\vec{s}$ in the group can be better off deviating to z_l when the receiver beliefs that the deviating sender is type $\vec{s}$. Thus, any equilibrium involving pooling on a signal fails the LCT.

(2) A separating equilibrium in which the local upward incentive constraint is not binding is depicted in Figure 2.1. As argued in Section 2, a type for whom none of the local upward incentive constraints are binding has an incentive to deviate to a signal lower than her equilibrium signal. There exists such a deviation such that only she will be better off, which fails the LCT.

(3) With full information s_1 has a payoff of $Max\{U^R(s_1), \underset{z}{Max}\, U(s_1, s_1, z)\}$. With asymmetric information she knows that beliefs about her type can never be lower than s_1. Thus her payoff is bounded from below by the full information payoff. But to satisfy the LCT, the equilibrium must be separating. So this is also the upper bound. ***Q.E.D***

Proof of Proposition 2: (i) The equilibrium payoff for type s_1 and the equilibrium signals by other types follow directly from Proposition 1.

(ii) Suppose $s_{\tilde{i}-1}$ chooses not to signal. If type $s_{\tilde{i}}$ wants to signal, then she should choose a signal satisfying (a). Type $s_{\tilde{i}-1}$ would deviate to $z_{\tilde{i}}$ if $U^R(s_{\tilde{i}-1}) < U(s_{\tilde{i}-1}, v(s_{\tilde{i}}), z_{\tilde{i}})$. If $U^R(s_{\tilde{i}-1}) > U(s_{\tilde{i}-1}, v(s_{\tilde{i}}), z_{\tilde{i}})$, type $s_{\tilde{i}}$ would be better off making a profitable deviation while keeping $s_{\tilde{i}-1}$ choose not to signal, violating the LCT test. Condition (b) clearly must hold since it is the participation constraint for $s_{\tilde{i}}$.

(iii). What needs to be checked here is whether all other types want to participate in signaling. Suppose $s_{\tilde{i}} = s_1$ and consider type s_2. By the tightness requirement, z_2 satisfies

$$U^E(s_1) = U(s_1, v(s_1), z_1^*) = U(s_1, v(s_2), z_2) \tag{7.1}$$

Since $U_1(s, v(s), z(s)) \geq dU^R(s)/ds$, $U(s_2, v(s_2), z_2) - U(s_1, v(s_2), z_2) \geq U^R(s_2) - U^R(s_1)$. By (7.1) we have $U(s_2, v(s_2), z_2) - U^E(s_1) \geq U^R(s_2) - U^R(s_1)$. Since $U^E(s_1) \geq U^R(s_1)$, we have $U(s_2, v(s_2), z_2) \geq U^R(s_2)$. The argument applies to all other types as well. ***Q.E.D***

Proof of Lemma 1: By the tightness requirement, $U^R(s_{\tilde{i}-1}) = U(s_{\tilde{i}-1}, s_{\tilde{i}}, z_{\tilde{i}})$. By the participation constraint, we have $U^R(s_{\tilde{i}}) \leq U(s_{\tilde{i}}, s_{\tilde{i}}, z_{\tilde{i}})$. Since $U^R(\cdot)$ is nondecreasing, we have

$$U(s_{\tilde{i}-1}, s_{\tilde{i}}, z_{\tilde{i}}) = U^R(s_{\tilde{i}-1}) \leq U^R(s_{\tilde{i}}) \leq U(s_{\tilde{i}}, s_{\tilde{i}}, z_{\tilde{i}}) \tag{7.2}$$

Letting $\delta \to 0$, the minimum signaling type satisfies $U^R(s) = U(s, s, z(s))$, which is (4.1). (7.2) implies that $U^R(s_{\tilde{i}}) - U^R(s_{\tilde{i}-1}) \leq U(s_{\tilde{i}}, s_{\tilde{i}}, z_{\tilde{i}}) - U(s_{\tilde{i}-1}, s_{\tilde{i}}, z_{\tilde{i}})$. Letting $\delta \to 0$, we have

$$\frac{dU^R(s)}{ds} \leq U_1(s, s, z(s)).$$

Since U is quasi-concave in z, for any s_i such that $U^R(s_i) \leq U(s_i, s_i, z_i^*)$, there exists $z_i^R \geq z_i^*$ such that $U^R(s_i) = U(s_i, s_i, z_i^R)$. Since $U(s_{\tilde{i}}, s_{\tilde{i}}, z_{\tilde{i}}^R) = U^R(s_{\tilde{i}}) \leq U(s_{\tilde{i}}, s_{\tilde{i}}, z_{\tilde{i}})$, it must be that $z_{\tilde{i}}^R \geq z_{\tilde{i}}$.

Define $\hat{z}$ such that $U^R(s_{\tilde{i}-2}) = U(s_{\tilde{i}-2}, s_{\tilde{i}-1}, \hat{z})$. Then it must be that

$$U(s_{\tilde{i}-1}, s_{\tilde{i}-1}, z_{\tilde{i}-1}^R) = U^R(s_{\tilde{i}-1}) \geq U(s_{\tilde{i}-1}, s_{\tilde{i}-1}, \hat{z}) \tag{7.3}$$

Otherwise, by Proposition 2(ii), type $s_{\tilde{i}-1}$ should signal by choosing $\hat{z}$. Thus, $z_{\tilde{i}-1}^R \leq \hat{z}$. On the other hand, since type $s_{\tilde{i}-2}$ stays out, we have

$U^R(s_{\tilde{i}-2}) = U(s_{\tilde{i}-2}, s_{\tilde{i}-1}, \hat{z}) \geq U(s_{\tilde{i}-2}, s_{\tilde{i}}, z_{\tilde{i}}) \geq U(s_{\tilde{i}-2}, s_{\tilde{i}-1}, z_{\tilde{i}})$. Thus, $\hat{z} \leq z_{\tilde{i}}$. Summarizing, we have $z_{\tilde{i}-1}^R \leq \hat{z} \leq z_{\tilde{i}} \leq z_{\tilde{i}}^R$. In the limit, $\hat{z} \to z_{\tilde{i}}$.

From (7.3), we have

$$U^R(s_{\tilde{i}-1}) - U^R(s_{\tilde{i}-2}) \geq U(s_{\tilde{i}-1}, s_{\tilde{i}-1}, \hat{z}) - U^R(s_{\tilde{i}-2}) = U(s_{\tilde{i}-1}, s_{\tilde{i}-1}, \hat{z}) - U(s_{\tilde{i}-2}, s_{\tilde{i}-1}, \hat{z})$$

Letting $\delta \to 0$, and using $\hat{z} \to z(s)$, we have

$$\frac{dU^R(s)}{ds} \geq U_1(s, s, z(s))$$

Therefore, in the limit, the minimum signaling type should satisfy $\frac{dU^R(s)}{ds} = U_1(s, s, z(s))$, which is (4.2). ***Q.E.D***

Proof of Lemma 2: Using $s_n = s,\ s_{n-1} = s - \delta,\ s_{n+1} = s + \delta$, we rewrite (3.2) as follows:

$$\overline{v}(s,\delta)(G(s_{n+1}) - G(s_{n-1})) = [G(s) - G(s_{n-1})]s + [G(s_{n+1}) - G(s)]s_{n+1}$$

Differentiating with respect to δ on both sides, we have

$$\overline{v}_2(s,\delta)[G(s_{n+1}) - G(s_{n-1})] + \overline{v}(s,\delta)[G'(s_{n+1}) + G'(s_{n-1})] = sG'(s_{n-1}) + s_{n+1}G'(s_{n+1}) + G(s_{n+1}) - G(s)$$

$$\begin{aligned}&\overline{v}_{22}(s,\delta)[G(s_{n+1}) - G(s_{n-1})] + 2\overline{v}_2(s,\delta)[G'(s_{n+1}) + G'(s_{n-1})] + \overline{v}(s,\delta)[G'(s_{n+1}) - G'(s_{n-1})] \\ &\quad = -sG''(s_{n-1}) + s_{n+1}G''(s_{n+1}) + 2G'(s_{n+1})\end{aligned}$$

$$\begin{aligned}&\overline{v}_{222}(s,\delta)[G(s_{n+1}) - G(s_{n-1})] + 3\overline{v}_{22}(s,\delta)[G'(s_{n+1}) + G'(s_{n-1})] \\ &\quad + 3\overline{v}_2(s,\delta)[G''(s_{n+1}) - G''(s_{n-1})] + \overline{v}(s,\delta)[G'''(s_{n+1}) + G'''(s_{n-1})] \\ &\quad = sG'''(s_{n-1}) + s_{n+1}G'''(s_{n+1}) + 3G''(s_{n+1})\end{aligned}$$

Letting $\delta \to 0$ we obtain: $\overline{v}(s,\delta) \to s$; $\overline{v}_2(s,\delta) \to 1/2$; $\overline{v}_{22}(s,\delta) \to \frac{G''(s)}{2G'(s)}$. ***Q.E.D***

Proof of Lemma 3: First according to our new notation let's rewrite (3.1) into the following:

$$\begin{cases} U(s_{n-1}, v, y) = U(s_{n-1}, s_n, z_n) \\ U(s_{n+1}, v, y) = U(s_{n+1}, s_{n+1}, z_{n+1}) \\ U(s_n, s_n, z_n) = U(s_n, s_{n+1}, z_{n+1}) \end{cases} \tag{7.4}$$

By the continuity of U we have $v(s,0) = s$ from (7.4). Differentiating (7.4) with respect to δ, we have

$$-U_1(s_{n-1}, v, y) + U_2(s_{n-1}, v, y)\frac{dv}{d\delta} + U_3(s_{n-1}, v, y)\frac{dy}{d\delta} = -U_1(s_{n-1}, s_n, z_n) \tag{7.5}$$

$$\begin{aligned} &U_1(s_{n+1}, v, y) + U_2(s_{n+1}, v, y)\frac{dv}{d\delta} + U_3(s_{n+1}, v, y)\frac{dy}{d\delta} \\ &\quad = U_1(s_{n+1}, s_{n+1}, z_{n+1}) + U_2(s_{n+1}, s_{n+1}, z_{n+1}) + U_3(s_{n+1}, s_{n+1}, z_{n+1})\frac{dz_{n+1}}{d\delta} \end{aligned} \tag{7.6}$$

$$0 = U_2(s_n, s_{n+1}, z_{n+1}) + U_3(s_n, s_{n+1}, z_{n+1})\frac{dz_{n+1}}{d\delta} \tag{7.7}$$

To save on notation, let $z' = dz_{n+1}/d\delta$. From (7.7) we have

$$z' = -\frac{U_2(s_n, s_{n+1}, z_{n+1})}{U_3(s_n, s_{n+1}, z_{n+1})} \tag{7.8}$$

We can obtain the higher order derivatives for z_{n+1}:

$$z'' = \frac{d^2 z_{n+1}}{d\delta^2} = -\frac{U_2(U_2 U_{33} - 2U_3 U_{23})}{U_3^3} \tag{7.9}$$

$$z''' = \frac{d^3 z_{n+1}}{d\delta^3} = -\frac{U_2}{U_3^5}\left[6U_3^2 U_{23}^2 - 9U_2 U_3 U_{23} U_{33} + 3U_2^2 U_{33}^2 + 3U_2 U_3^2 U_{233} - U_2^2 U_3 U_{333}\right] \tag{7.10}$$

where $U = U(s_n, s_{n+1}, z_{n+1})$.

Write $v' = dv/d\delta = v_2(s,\delta)$ and $y' = dy/d\delta = y_2(s,\delta)$. From (7.5) and (7.6) we have

$$v' = \frac{\Delta_1}{\Delta} \quad \text{and} \quad y' = \frac{\Delta_2}{\Delta} \quad \text{where} \tag{7.11}$$

$$\Delta = U_2(s_{n-1}, v, y)U_3(s_{n+1}, v, y) - U_2(s_{n+1}, v, y)U_3(s_{n-1}, v, y)$$

$$\Delta_1 = U_3(s_{n+1}, v, y)[U_1(s_{n-1}, v, y) - U_1(s_{n-1}, s_n, z_n)]$$
$$- U_3(s_{n-1}, v, y)[U_1(s_{n+1}, s_{n+1}, z_{n+1}) - U_1(s_{n+1}, v, y) + U_2(s_{n+1}, s_{n+1}, z_{n+1}) + U_3(s_{n+1}, s_{n+1}, z_{n+1})z']$$

$$\Delta_2 = U_2(s_{n-1}, v, y)[U_1(s_{n+1}, s_{n+1}, z_{n+1}) - U_1(s_{n+1}, v, y) + U_2(s_{n+1}, s_{n+1}, z_{n+1}) + U_3(s_{n+1}, s_{n+1}, z_{n+1})z']$$
$$- U_2(s_{n+1}, v, y)[U_1(s_{n-1}, v, y) - U_1(s_{n-1}, s_n, z_n)]$$

Differentiating the above equations, and using **B1**, **B2** and (7.8) - (7.10), we can derive the following derivatives evaluated at $\delta = 0$:

$$\Delta' = \frac{d\Delta}{d\delta}|_{\delta=0} = 2U_2U_{13}$$

$$\Delta'' = \frac{d^2\Delta}{d\delta^2}|_{\delta=0} = 4(U_{13}U_{23} + U_2U_{133})y'$$

$$\Delta_1' = \frac{d\Delta_1}{d\delta}|_{\delta=0} = 2U_3U_{13}\cdot y' + 2U_2U_{13}$$

$$\begin{aligned}\Delta_1'' = \frac{d^2\Delta_1}{d\delta^2}|_{\delta=0} &= 2(U_{13} + U_{23}v' + U_{33}y')U_{13}\cdot y' + U_3[-U_{113}y' + (-U_{113} + U_{133}y')y' + U_{13}y''] \\ &- 2(-U_{13} + U_{23}v' + U_{33}y')[U_{13}(z' - y') + U_{23}z' + (U_{13} + U_{23} + U_{33}z')z' + U_3z''] \\ &- U_3[U_{113}(z' - y') + (U_{113} + U_{133}z')z' + U_{13}z'' - (U_{113} + U_{133}y')y' - U_{13}y'' \\ &+ U_{233}z'^2 + U_{23}z'' + (U_{13} + U_{23} + U_{33}z')z'' \\ &+ (U_{113} + 2U_{133}z' + 2U_{233}z' + U_{333}z'^2 + U_{33}z'')z' + U_3z''' + (U_{13} + U_{23} + U_{33}z')z'']\end{aligned}$$

$$\Delta_2' = \frac{d\Delta_2}{d\delta}|_{\delta=0} = -2\frac{U_2^2U_{13}}{U_3} - 2U_2U_{13}v'$$

$$\begin{aligned}\Delta_2'' = \frac{d^2\Delta}{d\delta^2}|_{\delta=0} &= 2U_{23}y'[(2U_{13} + 2U_{23} + U_{33}z')z' + U_3z'' - 2U_{13}y'] \\ &+ U_2[(3U_{113} + (3U_{133} + 3U_{233} + U_{333}z')z' + U_{33}z'')z' + (3U_{13} + 3U_{23} + 2U_{33}z')z'' \\ &+ U_3z''' - 2U_{133}y'^2 - 2U_{13}y''] + (U_{233}y'^2 + U_{23}y'')(U_2 + U_3z')\end{aligned}$$

Using L'Hopital's rule, we have

$$\begin{cases} v' = \dfrac{\Delta_1'}{\Delta'} = \dfrac{2U_3U_{13}y' + 2U_2U_{13}}{2U_2U_{13}} = 1 + \dfrac{U_3}{U_2}y' \\ y' = \dfrac{\Delta_2'}{\Delta'} = \dfrac{-2U_2^2U_{13} - 2U_2U_{13}y'}{2U_2U_{13}} = -\dfrac{U_2}{U_3} - y' \end{cases}$$

which implies $v' = v_2(s,0) = 1/2$ and $y' = y_2(s,0) = -U_2/2U_3$. Taking second order derivatives, we have

$$\frac{d^2v}{d\delta^2} = \frac{d}{d\delta}\left(\frac{\Delta_1}{\Delta}\right) = \frac{d\Delta_1/d\delta - (dv/d\delta)\cdot(d\Delta/d\delta)}{\Delta}$$

$$\frac{d^2y}{d\delta^2} = \frac{d}{d\delta}\left(\frac{\Delta_2}{\Delta}\right) = \frac{d\Delta_2/d\delta - (dy/d\delta)\cdot(d\Delta/d\delta)}{\Delta}$$

Using L'Hospital's rule again we have

$$\begin{cases} v_{22}(s,0) = \dfrac{d^2v}{d\delta^2}\Big|_{\delta=0} = \dfrac{\Delta_1'' - v''\cdot\Delta' - v'\cdot\Delta''}{\Delta'} \\ y_{22}(s,0) = \dfrac{d^2y}{d\delta^2}\Big|_{\delta=0} = \dfrac{\Delta_2'' - y''\cdot\Delta' - y'\cdot\Delta''}{\Delta'} \end{cases} \tag{7.12}$$

Substituting the expressions of $v', y', \Delta', \Delta'', \Delta_1'', \Delta_2''$ derived above into (7.12), we can solve for values of $v_{22}(s,0)$ as follows:

$$v'' = v_{22}(s,0) = -\frac{-U_2U_{13}U_{33} + U_2U_3U_{133} + 2U_3U_{13}U_{23} - 2U_3^2U_{113} + 4U_3U_{13}^2}{4U_3^2U_{13}}$$ ***Q.E.D.***

Proof of Theorem 2: (i) If $U^R(\underline{s}) \le U(\underline{s}, \underline{s}, \underline{z}^*)$, then the lowest type $\underline{s}$ chooses the signal $\underline{z}^*$. By Proposition 2(i) and Figure 3.2, let $v(\underline{s},\delta) = \tilde{v}_1$ be the solution to $U(\underline{s}+\delta, v(\underline{s},\delta), \underline{z}^*) = U(\underline{s}+\delta, \underline{s}+\delta, z_2)$, where z_2 is such that $U(\underline{s}, \underline{s}, \underline{z}^*) = U(\underline{s}, \underline{s}+\delta, z_2)$. Note that since $G(\underline{s}) = 0$,

$\bar{v}(\underline{s},\delta) = \dfrac{\underline{s}G(\underline{s}) + (\underline{s}+\delta)[G(\underline{s}+\delta) - G(\underline{s})]}{G(\underline{s}+\delta)} = \underline{s}+\delta$, which means that in the limit $v(\underline{s},\delta) = \bar{v}(\underline{s},\delta)$ and hence the condition of Theorem 1 is violated if $\underline{s}$ participates in signaling. This proves part (i).

Lemma 1 establishes that the minimum signaling type and her signal must satisfy (4.1) and (4.2). If these two questions do not have a solution, then the LCT cannot be satisfied. If the solution happens to be $\tilde{s} = s^*, \tilde{z} = z^*(\tilde{s})$, then we need to require that $U_3(s^*, s^*, z^*(s^*)) \neq 0$ (e.g., signals are unproductive), otherwise the equilibrium signaling schedule $z'(s)$ and condition (4.3) will be un-defined. When the solution is interior (i.e., $\tilde{s} \in (s^*, \overline{s})$), then by (4.1), $U^R(\tilde{s}) = U(\tilde{s}, \tilde{s}, \tilde{z}) < U(\tilde{s}, \tilde{s}, z^*(\tilde{s})),$ hence $\tilde{z} > z^*(\tilde{s})$. This guarantees that at $s = \tilde{s}$, $U_3 \neq 0$. This proves Part (ii). [11]

Part (iii) is shown in the main text. Part (iv) follows directly from Lemmas 2 and 3. The only thing remains to be checked is that for $s > \tilde{s}$, $U_3 \neq 0$; otherwise $\tilde{v}_{22}(s, \delta)$ is not well defined. At $\tilde{s}$, since $U^R(s) = U(s, s, z(s))$, we have

$$\frac{dU^R(s)}{ds} = U_1 + U_2 + U_3 z'(s)$$

By (4.2), $U_2 + U_3 z'(s) = 0$. Since $U_2 < 0$, it must be that $U_3 < 0$ at $\tilde{s}$. ***Q.E.D.***

Proof of Lemma 4: By definition, $\overline{v}(s, s') = 1/[G(s') - G(s)] \times \int_s^{s'} x dG(x)$. Multiplying both sides by $G(s') - G(s)$ and then differentiating by s', we have

$$\overline{v}_2(s, s')(G(s') - G(s)) + \overline{v}(s, s')G'(s') = s'G'(s').$$

Differentiating by s' again,

$$\overline{v}_{22}(s, s')(G(s') - G(s)) + 2\overline{v}_2(s, s')G'(s') + \overline{v}(s, s')G''(s') = G'(s') + s'G''(s') \quad (7.13)$$

Setting $s' = s$, it follows immediately that $\overline{v}_2(s, s) = 1/2$.

Differentiating (7.13) by s' again,

$$\overline{v}_{222}(s, s')(G(s') - G(s)) + 3\overline{v}_{22}(s, s')G'(s') + 3\overline{v}_2(s, s')G''(s') + \overline{v}(s, s')G'''(s') = 2G''(s') + s'G'''(s')$$

Since $\overline{v}(s, s) = s$ and $\overline{v}_2(s, s) = 1/2$, setting $s' = s$ we obtain $\overline{v}_{22}(s, s) = \frac{1}{6}\frac{G''(s)}{G'(s)}$. ***Q.E.D.***

Proof of Lemma 5: Total differentiating (5.2) gives

[11] It is easy to find sufficient conditions that guarantee the existence of an interior solution to (4.1) and (4.2). Let z_1 be the solution to $U^R(\overline{s}) = U(\overline{s}, \overline{s}, z)$, z_2 be the solution to $dU^R(s)/ds\,|_{s^*} = U_1(s^*, s^*, z)$, and z_3 be the solution to $dU^R(s)/ds\,|_{\overline{s}} = U_1(\overline{s}, \overline{s}, z)$. Then an interior solution exists when $z_2 > z^*(s^*)$ and $z_3 < z_1$.

$$U_1(s',v,y)ds' + U_2(s',v,y)dv + U_3(s',v,y)dy = U_1(s',s',z(s'))ds'$$
$$U_2(s,v,y)dv + U_3(s,v,y)dy = 0$$

Solving the equations, we have

$$\frac{dv}{ds'} = \frac{\Delta_1}{\Delta}, \quad \frac{\mathrm{dy}}{ds'} = \frac{\Delta_2}{\Delta}$$

where

$$\Delta = U_2(s',v,y)U_3(s,v,y) - U_2(s,v,y)U_3(s',v,y),$$
$$\Delta_1 = U_3(s,v,y)[U_1(s',s',z(s')) - U_1(s',v,y)],$$
$$\Delta_2 = -U_2(s,v,y)[U_1(s',s',z(s')) - U_1(s',v,y)].$$

Under Assumption **B1**, we have

$$\frac{dy}{ds'} = \frac{\Delta_2}{\Delta} = \frac{U_1(s',v,y) - U_1(s',s',z(s'))}{U_3(s,v,y) - U_3(s',v,y)} \tag{7.14}$$

Fix any s, as $s' \to s$. It must be that $v \to s$, $z(s') \to z(s)$, and $y \to z(s)$. For the simplicity of notation, write $v'(s') = v_2(s,s')$ and $y'(s') = y_2(s,s')$. Applying the I'Hopital's rule, as $s' \to s$, we get

$$\left.\frac{dy}{ds'}\right|_{s'\to s}$$
$$= \lim_{s'\to s} \frac{U_{11}(s',v,y) + U_{12}(s',v,y)v'(s') + U_{13}(s',v,y)y'(s') - U_{11}(s',s',z(s')) - U_{12}(s',s',z(s')) - U_{13}(s',s',z(s'))z'(s')}{U_{23}(s,v,y)v'(s') + U_{33}(s,v,y)y'(s') - U_{23}(s',v,y)v'(s') - U_{33}(s',v,y)y'(s') - U_{13}(s',v,y)}$$
$$= \lim_{s'\to s} \frac{U_{13}(s',v,y)y'(s') - U_{13}(s',s',z(s'))z'(s')}{-U_{13}(s',v,y)}$$
$$= z'(s) - \left.\frac{dy}{ds'}\right|_{s'\to s}$$

Hence as $s' \to s$, $\frac{dy}{ds'} \to 0.5z'(s)$ as long as $z'(s) = -U_2(s,s,z(s))/U_3(s,s,z(s))$ is defined at s, or $U_3(s,s,z(s)) \neq 0$ at s.

Since

$$\frac{dv}{ds'} = \frac{\Delta_{11}}{\Delta} = \frac{\Delta_{11}}{\Delta_{21}}\frac{dy}{ds'} = -\frac{U_3(s,v,y)}{U_2(s,v,y)}\frac{dy}{ds'} \tag{7.15}$$

we have

$$\left.\frac{dv}{ds'}\right|_{s'\to s} = \lim_{s'\to s} -\frac{U_3(s,v,y)}{U_2(s,v,y)}\frac{dy}{ds'} = -\frac{U_3(s,s,z(s))}{U_2(s,s,z(s))}\lim_{s'\to s}\frac{dy}{ds'}$$
$$= -0.5\frac{U_3(s,s,z(s))}{U_2(s,s,z(s))}z'(s) = 0.5$$

for any s such that $U_3(s,s,z(s)) \neq 0$. This proves part (i).

For part (ii), first note that from $z'(s) = -U_2(s,s,z(s))/U_3(s,s,z(s))$,

$$z''(s) = -\frac{U_{22}}{U_3} - z'(s)\frac{U_{13}+2U_{23}+U_{33}z'(s)}{U_3}$$

From (7.14), and by Assumption **B1**, we have

$$\frac{d^2y}{ds'^2} = \frac{U_{11}(s',v,y)+U_{13}(s',v,y)\frac{dy}{ds'} - U_{11}(s',s',z(s')) - U_{13}(s',s',z(s'))z'(s')}{U_3(s,v,y)-U_3(s',v,y)}$$
$$-\frac{U_{33}(s,v,y)\frac{dy}{ds'} - U_{33}(s',v,y)\frac{dy}{ds'} - U_{13}(s',v,y)}{U_3(s,v,y)-U_3(s',v,y)}\frac{dy}{ds'}$$
$$= \frac{U_{11}(s',v,y)-U_{11}(s',s',z(s'))}{U_3(s,v,y)-U_3(s',v,y)} + \frac{U_{33}(s',v,y)-U_{33}(s,v,y)}{U_3(s,v,y)-U_3(s',v,y)}\left(\frac{dy}{ds'}\right)^2$$

$$+\frac{2U_{13}(s',v,y)\frac{dy}{ds'} - U_{13}(s',s',z(s'))z'(s')}{U_3(s,v,y)-U_3(s',v,y)} \tag{7.16}$$

Let $L_i(s,s')$ be the ith term on the right hand side of the above equation. For any s such that $U_3(s,s,z(s)) \neq 0$, it can be checked that

$$\lim_{s'\to s} L_1 = \frac{1}{2}\frac{U_{113}(s,s,z(s))z'(s)}{U_{13}(s,s,z(s))}$$
$$\lim_{s'\to s} L_2 = -\frac{1}{4}\frac{U_{133}(s,s,z(s))}{U_{13}(s,s,z(s))}[z'(s)]^2$$
$$\lim_{s'\to s} L_3 = z''(s) - 2\left.\frac{d^2y}{ds'^2}\right|_{s'\to s} + 0.5\frac{U_{133}(s,s,z(s))[z'(s)]^2}{U_{13}(s,s,z(s))}$$

Therefore,

$$6\left.\frac{d^2y}{ds'^2}\right|_{s'\to s} = 2z''(s) + \frac{U_{113}(s,s,z(s)) + 0.5U_{133}(s,s,z(s))z'(s)}{U_{13}(s,s,z(s))}z'(s)$$

From (7.15), and using Assumption **B2**, we have

$$\frac{d^2v}{ds'^2} = -\frac{U_3(s,v,y)}{U_2(s,v,y)}\frac{d^2y}{ds'^2} - \left[\frac{U_{33}(s,v,y)}{U_2(s,v,y)} - \frac{2U_3(s,v,y)U_{23}(s,v,y)}{U_2^2(s,v,y)}\right]\left(\frac{dy}{ds'}\right)^2 \qquad (7.17)$$

As $s' \to s$, we know that $\frac{dv}{ds'} \to 0.5$ and $U_3(s,v,y)\frac{dy}{ds'} \to 0.5U_3 z'(s) = -0.5U_2$. So,

$$\left.\frac{d^2v}{ds'^2}\right|_{s'\to s} = -\frac{U_3(s,s,z(s))}{U_2(s,s,z(s))}\left.\frac{d^2y}{ds'^2}\right|_{s'\to s} - 0.5\frac{U_{23}(s,s,z(s)) + 0.5U_{33}(s,s,z(s))z'(s)}{U_2(s,s,z(s))}z'(s)$$

$$= \frac{\left.\frac{d^2y}{ds'^2}\right|_{s'\to s}}{z'(s)} - 0.5\frac{U_{23} + 0.5U_{33}z'(s)}{U_2}z'(s)$$

$$= \frac{-1}{3}[\frac{U_{13} + 2U_{23} + U_{33}z'(s)}{U_3}] + \frac{1}{6}\frac{U_{113} + 0.5U_{133}z'(s)}{U_{13}} - 0.5\frac{U_{23} + 0.5U_{33}z'(s)}{U_2}z'(s)$$

$$= \frac{1}{3}[\frac{U_{13} + 2U_{23} + U_{33}z'(s)}{U_2}z'(s)] + \frac{1}{6}\frac{U_{113} + 0.5U_{133}z'(s)}{U_{13}} - 0.5\frac{U_{23} + 0.5U_{33}z'(s)}{U_2}z'(s)$$

$$= \frac{1}{6}\frac{U_{113}}{U_{13}} + \frac{1}{12}[\frac{4U_{13} + 2U_{23}}{U_2} + \frac{U_{133}}{U_{13}}]z'(s) + \frac{1}{12}\frac{U_{33}}{U_2}[z'(s)]^2$$

$$= -\frac{-U_2U_{13}U_{33} + U_2U_3U_{133} + 2U_3U_{13}U_{23} - 2U_3^2U_{113} + 4U_3U_{13}^2}{12U_3^2U_{13}}$$

This proves part (ii). ***Q.E.D.***

Proof of Theorem 3: In the continuous type case, part (iv) of Theorem 2 follows directly from Lemmas 4 and 5. The only thing remains to be checked is part (i) that the lowest type does not signal if the separating equilibrium satisfies the LCT. Suppose the lowest type signals. For any type $s' > \underline{s}$, suppose those in the interval $[\underline{s}, s']$ all choose $\underline{z} = z(\underline{s})$, the equilibrium signal by $\underline{s}$. Let $\overline{v}(\underline{s}, s')$ be the expected type of this pool. Define $\overline{v}(\underline{s}, s') = \frac{\int_s^{s'} xdG(x)}{G(s')}$. Let $v(\underline{s}, s') \in [\underline{s}, s']$ be a solution to $U(s', v, \underline{z}) = U(s', s', z(s'))$. In order for the separating equilibrium characterized by $z(s)$ to satisfy the LCT, it must be that for s' close to s, any such signal-perception pair of $(\underline{z}, \overline{v}(s, s'))$ is not credible. That is, $\overline{v}(\underline{s}, s') < v(\underline{s}, s')$ as $s' \to \underline{s}$. Note that $\overline{v}(\underline{s}, \underline{s}) = v(\underline{s}, \underline{s}) = \underline{s}$. However, it can be verified that as $s' \to \underline{s}$, $v_2(\underline{s}, s') \to 0$ while $\overline{v}_2(\underline{s}, s') \to 0.5$. So in the neighborhood of $\underline{s}$,

$\overline{v}(\underline{s}, s') > v(\underline{s}, s')$. Therefore, there is always a credible boundary deviation at the lowest signal $\underline{y} = z(\underline{s})$. This proves part (i) of Theorem 2 for the continuous type case. ***Q.E.D.***

Proof of Proposition 3: For $s \in [\tilde{s}, \overline{s}]$, following the same notation as before, we have

$$\begin{aligned}\frac{dv}{ds'} = \frac{\Delta_{11}}{\Delta} &= \frac{U_3(s,v,y)[U_1(s',s',z(s')) - U_1(s',v,y)]}{U_2(s',v,y)U_3(s,v,y) - U_2(s,v,y)U_3(s',v,y)} \\ &= \frac{-(a/2b)(1-a)(s'-v)}{-(1-a)/2b \cdot a(s'-s)} = \frac{s'-v}{s'-s}\end{aligned} \tag{7.18}$$

Using L'Hospital's rule, we have $v_2(s,s) = \lim_{s'\to s} \frac{dv}{ds'} = 1 - \lim_{s'\to s} \frac{dv}{ds'}$, which implies $v_2(s,s) = 1/2.$ Differentiating (7.18) with respect to s', we have

$$\frac{d^2v}{ds'^2} = \frac{(1-dv/ds')(s'-s) - (s'-v)}{(s'-s)^2} \tag{7.19}$$

Using L'Hospital's rule again, we have

$$\lim_{s'\to s} \frac{d^2v}{ds'^2} = \lim_{s'\to s} \frac{-d^2v/ds'^2(s'-s)}{2(s'-s)} = -\frac{1}{2}\lim_{s'\to s}\frac{d^2v}{ds'^2},$$

which implies $v_{22}(s,s) = 0$. Thus, there is no credible deviation if $0 > G''(s)/G'(s)$, or if $G(\cdot)$ is strictly concave. ***Q.E.D.***

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