Efficient and Optimal Selling Mechanisms with Private Information Acquisition Costs*

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Abstract

In auctions with private information acquisition costs, we completely characterize efficient and optimal two-stage selling mechanisms, with the first stage being the pre-screening or entry right allocation mechanism, and the second stage being the traditional private good provision mechanism. Both efficiency and optimality require the second stage mechanism to be *ex post* efficient. For the first stage of entry allocation, both efficient and optimal mechanisms admit the most efficient bidders (the bidders with the least information acquisition costs), while the optimal mechanism admits fewer entrants. The efficient entry right allocation rule maximizes the expected total surplus, while the optimal entry right allocation rule maximizes the expected "virtual" total surplus, which is the total surplus adjusted for the information rent. We show that both efficient and optimal entry right allocation rules can be truthfully implemented in dominant strategies. We also demonstrate that the optimal entry right allocation mechanism can be implemented through an all-pay auction.

Keywords: Auctions, Two-Stage Auctions, Information Acquisition, Entry, VCG Mechanism.

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1 Introduction

The earlier literature on optimal auction design and revenue comparison (e.g., Vickrey, 1961; Riley and Samuelson, 1981; Myerson, 1981; and Milgrom and Weber, 1982) generally assumes that there is an exogenously specified set of bidders and that these bidders are endowed with information about the object's valuations. By taking into account endogenous entry, the nature of optimal auctions changes dramatically (see, for example, French and McCormick, 1984; McAfee and McMillan, 1987; Tan, 1992; Engelbrecht-Wiggans, 1993; Levin and Smith, 1994; Stegeman, 1996; Tan and Yilankaya, 2006; Ye, 2004, 2007; and Lu, 2009. Also see Bergemann and Välimäki, 2006 for a very insightful survey of this growing literature). For example, in a standard symmetric independent private value setting with fixed information acquisition cost, Levin and Smith (1994) show that the optimal auction is characterized by a standard auction with free entry without imposing a reserve price strictly higher than the seller's own valuation. This is in stark contrast with the optimal auctions characterized in the seminal work of Myerson (1981).

Most recently, Moreno and Wooders (2006) and Lu (2010) extend the analysis to a setting where information acquisition costs are heterogeneous and privately known to the bidders. This setting is not just a theoretical extension, but is also more relevant in many real world settings. For example, in some complex and high-valued asset sales, many aspects of pre-bid information acquisition and analyzation are privately known to bidders (Vallen and Bullinger, 1999). As Moreno and Wooders (2006) point out, in Internet auctions a bidder's value discovery cost is the opportunity cost of her time, which also varies across bidders and is usually only known to herself.

More specifically, when potential bidders are endowed with private information about their information acquisition costs, Moreno and Wooders (2006) and Lu (2010) consider endogenous entry, in which potential bidders decide on whether to enter the auction (and incur information acquisition costs) independently and simultaneously. Their analysis is a direct extension of the important work of Levin and Smith (1994), who provide a thorough analysis of an auction setting where bidders need to incur a homogenous and publicly known information acquisition cost before learning about their valuations. Since potential bidders are identical (in terms of both value distributions and information acquisition costs) before entry, Levin and Smith (1994) focus on the symmetric entry equilibrium in which each potential bidder enters the auction with the same probability. With heterogeneous and private information acquisition costs, however, Moreno and Wooders (2006) and Lu (2010) show that a potential bidder enters the auction if and only if her entry cost is lower than some equilibrium entry threshold, which is determined by the selling mechanism (i.e., the auction format, the reserve price, the entry fee, etc.) and other commonly know parameters in the model. While Moreno and Wooders (2006) focus on the characterization of optimal auctions when entry fees are not feasible, Lu (2010) focuses on entry coordination issues and identifies conditions under which the desirable entry can be uniquely induced. Both papers provide a nice *purification* of the mixed strategies identified in Levin and Smith (1994).

In this research we take one step further trying to understand the nature of efficient and optimal auctions with independent private values and costly entry. While Moreno and Wooders (2006) and Lu (2010) consider endogenous entry where the seller does not exercise entry control, we consider the case in which the seller can pre-screen the bidders through an entry right allocation mechanism. As a result, we will effectively examine a two-stage selling mechanism, or a two-stage auction, with the first stage being the entry right allocation mechanism and the second stage being the private good provision mechanism (i.e., the traditional auction).

Exploring an optimal two-stage selling mechanism in this context is important for at least two reasons. The first reason is that a two-stage selling mechanism can implement deterministic entry to improve efficiency and expected revenue. As originally identified by Levin and Smith (1994), endogenous entry would often lead to coordination failure. Since bidders coordinate entry through their entry thresholds, the realized number of entrants is stochastic, which can be too high or too low. While a low entry will reduce competition and pose a direct cost to the seller, the possibility of high entry would reduce the entry incentive *ex ante*. As Levin and Smith (1994) demonstrate, the seller can maximize efficiency and hence expected revenue by reducing the number of potential bidders until the randomness in entry is completely eliminated.[1] In fact, Milgrom (2004, pp. 225-227) demonstrates that screening to minimize variance in participation, even if it is still random, increases expected revenue. Therefore, employing a two-stage selling mechanism to exercise entry control to reduce entry randomness is usually in the best interest of the seller. The second reason is that an optimal

[1]This will occur when the number of potential bidders is the same as the number of entrants in the efficient entry equilibrium when bidders enter the auction sequentially, as analyzed in McAfee and McMillan (1987) and Engelbretch-Wiggans (1993). Also note that in the symmetric equilibrium with a homogenous entry cost considered by Levin and Smith (1994), bidders are *ex ante* identical and their rents are driven down to zero by endogenous entry. Consequently, expected revenue is the same as expected total surplus.

two-stage selling mechanism can *uniquely* implement efficient and optimal entry. As demonstrated in Levin and Smith (1994) and Lu (2010), there are usually multiple equilibria in the entry stage with endogenous entry, so potential bidders need to coordinate over different entry equilibria (which are characterized by different entry threshold vectors). With a two-stage selling mechanism, however, we show that efficient entry and optimal entry can be uniquely implemented. Thus a two-stage selling mechanism does not only reduce entry randomness, but it also helps implement a desirable entry equilibrium.

In the traditional auction setting where bidders are passively endowed with private information about their valuations, the analysis usually focuses on optimal elicitation of private values (or signals). When costly entry is taken into account and the information acquisition costs are privately known, auction design has to additionally take into account information elicitation at the information acquisition stage. Auction design in this case has to balance information acquisition and information elicitation, which are interdependent: the second-stage selling mechanism has a direct effect on how effectively an efficient or optimal set of entrants can be induced, and the first-stage entry right allocation mechanism will in turn determine whether the final sale can be efficient or optimal.

Following a general mechanism design approach, we are able to completely characterize the efficient and optimal two-stage selling mechanisms in our setting. We demonstrate that for both efficiency and optimality, the second-stage selling mechanism must be *ex post* efficient; in particular, the optimal reserve price should be set at the same level as the seller's own valuation. For the entry screening stage, we show that for both efficiency and optimality, bidders should be admitted one by one according to their cost efficiencies, with the most efficient bidder being admitted first, the second most efficient bidder being admitted second, and so on and so forth. While efficiency requires that bidders be admitted to maximize the expected total surplus, optimality requires that bidders be admitted to maximize the expected virtual total surplus, which is equal to the total surplus adjusted for the information rent (in the spirit of Myerson's optimal auction design). A direct implication is that revenue-maximizing entry should be lower than the surplus-maximizing entry, reflecting a trade-off between efficiency and information rent extraction. The entry right allocation mechanism in our setting resembles a multi-unit auction with endogenously determined supply.[2] By constructing a VCG (Vickrey-Clark-Groves) payment rule in our setting with private information acquisition costs,

[2]See McAdams (2007) for an application of such an auction with uniform pricing rule.

we show that the efficient entry can be implemented in dominant strategies. Somewhat surprisingly, we are also able to construct an entry fee payment scheme to implement the optimal entry right allocation rule in dominant strategies. Moreover, we can show that the optimal entry right allocation mechanism can be implemented using an all-pay auction. Finally, we demonstrate that our optimal two-stage selling mechanism indeed dominates the optimal auctions with endogenous entry (studied in Moreno and Wooders, 2006; and Lu, 2010), and the optimal two-stage auction with fixed number of entrants (studied in Ye, 2007).

The paper is organized as follows. Section 2 lays out the model. Section 3 characterizes the efficient two-stage selling mechanism. Section 4 characterizes the optimal two-stage selling mechanism. Section 5 concludes.

2 The Model

There are N potential bidders interested in a single item, where N is public information. Let $\mathbf{N} = \{1, 2, ..., N\}$ denote the set of all the potential bidders, and $2^{\mathbf{N}}$ denote the collection of all the subsets (subgroup) in $\mathbf{N}$. The seller's valuation is v_0, which is public information. Bidder i has to incur an entry cost of c_i to discover her value or to prepare for a bid if she is granted the entry right. We assume that c_i is private information of bidder i, and it follows distribution $G(\cdot)$ with density $g(\cdot)$ on support $[c_l, c_h]$. After incurring c_i, each bidder observes her private value v_i. The distribution of v_i is $F(\cdot)$ with a density $f(\cdot)$ on support $[v_l, v_h]$. We assume that v_i is private information of bidder i.

Both $F(\cdot)$ and $G(\cdot)$ are public information. We assume mutual independence across i for both c_i and v_i. The seller and bidders are assumed to be risk neutral.

We consider a general mechanism design framework in which the seller also exercises entry control, so that the mechanism is conducted in two stages. The first stage is the entry right allocation mechanism, and the second stage is the private good provision mechanism.

In the first stage, given a profile of reports on their private entry costs, $\mathbf{c} = (c_i) = (c_1, c_2, ..., c_N)$, the mechanism specifies the entry right allocation rule and payment rule. More specifically, given a reported profile of $\mathbf{c}$, the entry right allocation rule specifies probabilities with which any given subset of bidders is admitted, $\mathbf{p} = (\mathbf{p}^g(\mathbf{c}))$, $\forall g \subset 2^{\mathbf{N}}$, and the payment rule $\mathbf{x}(\mathbf{c}) = (x_i(\mathbf{c}))$ specifies the payment $x_i(\mathbf{c})$, $i = 1, ..., N$, that each bidder needs to pay.

We assume that once admitted, an entrant bidder must incur her entry cost before she can participate in the second-stage selling mechanism. This is the case, for example, when the entry cost represents some legal expenses to establish one's eligibility to bid.[3]

In the second stage, a private good provision mechanism is conducted. Let this mechanism be denoted as Ω, which is a standard auction in which the object is allocated to the bidder who submits the highest bid.

Suppose a subgroup $g \subset 2^{\mathbf{N}}$ is the set of entrants who are granted the entry rights with type profile $(\mathbf{c}^g)$. Since entrants are *ex ante* symmetric in terms of their private values, the expected profit to each entrant is the same. Furthermore, given Ω, the expected revenue to the seller and the expected profit to each entrant bidder will both depend on n, the number of bidders in g rather than g, the specific set of entrants. For this reason, we can write the expected revenue to the seller $\pi_0^g(\Omega) = \pi_0(n, \Omega)$ and the expected profit to each entrant bidder $\pi^g(\Omega) = \pi(n, \Omega)$, where $n = \#(g)$. The expected total surplus generated from the second-stage auction is $S^g(\mathbf{c}^g, \Omega) = \pi_0^g(\Omega) + n \cdot \pi_0^g(\Omega) = S(n, \Omega)$. Taking entry costs into account, the expected total surplus generated from the sale is $TS^g(\mathbf{c}^g, \Omega) = S^g(\mathbf{c}^g, \Omega) - \sum\limits_{j \in g} c_j$. For brevity of notation, Ω is often suppressed so that $\pi_0(n, \Omega), \pi(n, \Omega)$, and $S(n, \Omega)$ are expressed as $\pi_0(n), \pi(n)$, and $S(n)$, respectively.

As a benchmark case, we will first characterize efficient selling mechanisms that maximize the expected total surplus of the sale.

3 Efficient Selling Mechanisms

Since the entry fee payments are monetary transfers from the bidders to the seller, the expected total surplus is entirely determined by the entry right allocation rule and the second-stage selling mechanism, which is given by

$$TS = E_{\mathbf{c}} \sum_g p^g(\mathbf{c}) \left[S^g(\mathbf{c}^g) - \sum_{j \in g} c_j \right] = E_{\mathbf{c}} \sum_g p^g(\mathbf{c}) TS^g(\mathbf{c}^g). \tag{1}$$

[3] We will also show that in equilibrium, a bidder's expected profit is greater than zero once admitted; thus on the equilibrium path there is no incentive for a bidder to enter the auction only to withdraw immediately after. In reality, the shortlisted are expected to participate. Their reputations may get hurt if they withdraw after being shortlisted.

The social planner's objective is to maximize TS subject to the usual individual rationality (IR) and incentive compatibility (IC) conditions.

For the time being, we ignore (IR) and (IC) conditions and ask what is the first best entry that can be achieved. First, it is apparent that to maximize TS, the second-stage selling mechanism must be *ex post* efficient (so that $S^g(\mathbf{c}^g)$ is maximized given any g). A direct implication is that the optimal reserve price, r^*, must be the same as the seller's own value, v_0; any other reserve price will lead to an inefficient allocation of the asset for sale with positive probability. Conditional on entry, the second-stage auction is a symmetric auction, so the revenue-equivalence theorem implies that any standard auction with reserve equal to v_0 is efficient. We thus fix the second-stage auction to be a standard auction Ω^* that has a reserve price v_0.

Given the second stage selling mechanism Ω^*, we now consider the first best efficient entry right allocation rule. Given an entry cost profile, $\mathbf{c}$, the efficient entry right allocation rule that maximizes TS should be defined as

$$\tilde{p}^*(g) = 1 \text{ if } g = \arg\max TS^g(\mathbf{c}^g, \Omega^*), \forall \mathbf{c}.$$

From equation (1), it is clear that given n, the number of entrants to be admitted, $TS^g(\mathbf{c}^g)$, is maximized if the subset g consists of the most efficient bidders (i.e., the bidders with the lowest n information acquisition costs).

It remains to determine $\tilde{n}^*$, the optimal number of entrants that maximizes the expected total surplus. It has been established that under a second-price sealed bid auction with $r^* = v_0$, the marginal contribution to the expected surplus from a new entrant equals the expected (private) gain for that entrant, i.e., $S(n) - S(n-1) = \pi(n)$,[4] which also holds under any standard auction due to the payoff equivalence theorem. Since $\pi(n)$ is decreasing in n, $S(n)$ is concave in n, which in turn implies that $\tilde{n}^*$ is uniquely determined. To summarize, the efficient entry right allocation rule, denoted by $\tilde{\mathbf{p}}^* = (\tilde{p}^{*g}(\mathbf{c}))$, is characterized as follows.

Theorem 1 *(The Efficient Entry Right Allocation Rule:) To maximize expected total surplus, bidders should be admitted one by one according to their cost efficiencies: The most efficient bidder should be admitted first, the second most efficient bidder should be admitted second, and so on and*

[4] See the proof of Proposition 1 in Engelbrech-Wiggans (1993). This result is also established in McAfee and McMillan (1987) and Levin and Smith (1994).

so forth. The last bidder admitted should be the last entrant that has positive contribution to the expected total surplus.

When $\mathbf{c}$ is privately known, it turns out that the first-best entry right allocation rule characterized in Theorem 1 can be truthfully implemented in dominant strategies by a Vickrey-Clark-Gloves (VCG) payment rule.

Let $\tilde{g}^*(\mathbf{c})$ denote the efficient set of entrants given a reported entry cost profile $\mathbf{c}$ ($\#(\tilde{g}^*(\mathbf{c})) = \tilde{n}^*$), and $c^{(\tilde{n}^*+1)}$ denote the $(\tilde{n}^*+1)$st lowest reported cost, or the reported cost type possessed by the first eliminated bidder.[5] The VCG payment rule stipulates that each agent pays the negative externality imposed to all the other agents by her presence. To pin down the VCG payment by each bidder i in our setting, we need to consider the maximized expected total surplus among all bidders but bidder i, both when bidder i is and is not present.

First we consider the case $i \notin \tilde{g}^*(\mathbf{c})$ (i is not admitted). In this case her presence does not affect the expected surplus to the rest, so her VCG payment is zero. Second we consider the case $i \in \tilde{g}^*(\mathbf{c})$ (i is admitted). When she is present, the expected surplus to all the other bidders is given by

$$\pi_0^{\tilde{g}^*} + (\tilde{n}^* - 1) \cdot \pi^{\tilde{g}^*} - \sum_{j \in \tilde{g}^* \backslash \{i\}} c_j = S(\tilde{n}^*) - \pi(\tilde{n}^*) - \sum_{j \in \tilde{g}^* \backslash \{i\}} c_j \equiv A.$$

When she is absent, there are two subcases. In the first subcase, $\pi^{\tilde{g}^*} < c^{(\tilde{n}^*+1)}$, so the efficient set of entrants is $\tilde{g}^*(\mathbf{c}) \backslash \{i\}$, in which case the expected surplus to all the other bidders is given by

$$S(\tilde{n}^* - 1) - \sum_{j \in \tilde{g}^* \backslash \{i\}} c_j \equiv B_1.$$

Thus in this case the VCG payment $B_1 - A = \pi(\tilde{n}^*) - (S(\tilde{n}^*) - S(\tilde{n}^* - 1)) = \pi(\tilde{n}^*) - \pi(\tilde{n}^*) = 0$. In the second subcase, $\pi^{\tilde{g}^*} > c^{(\tilde{n}^*+1)}$, so the efficient set of entrants is $(\tilde{g}^*(\mathbf{c}) \backslash \{i\}) \cup \{k\}$, where k is the bidder with reported type $c^{(\tilde{n}^*+1)}$.[6] In this case the expected surplus to all the other bidders is given by

$$S(\tilde{n}^*) - \sum_{j \in (\tilde{g}^* \backslash \{i\})} c_j - c^{(\tilde{n}^*+1)} \equiv B_2.$$

Thus in this case the VCG payment $B_2 - A = \pi(\tilde{n}^*) - c^{(\tilde{n}^*+1)}$.

[5] When all potential bidders are admitted, we assume that the type of the first eliminated bidder is c_h.

[6] Note that no bidder with reported type higher than $c^{(\tilde{n}^*+1)}$ can be admitted when i is absent; otherwise it contradicts the fact that $\#(\tilde{g}^*(\mathbf{c})) = \tilde{n}^*$.

The VCG payment rule in our setting can thus be summarized as follows:

$$\tilde{x}_i^*(c_i;\mathbf{c}_{-i}) = \begin{cases} \pi^{\tilde{g}^*(\mathbf{c})} - \min\{\pi^{\tilde{g}^*(\mathbf{c})}, c^{(\tilde{n}^*+1)}\} & \text{if} \quad i \in \tilde{g}^*(\mathbf{c}) \\ 0 & \text{if} \quad i \notin \tilde{g}^*(\mathbf{c}) \end{cases} \tag{2}$$

Since VCG payment rules implement efficient allocation rules in dominant strategies, the following conclusion is obtained immediately:

Theorem 2 *The first-best efficient entry right allocation rule* $\tilde{\mathbf{p}}^*$ *is truthfully implementable in dominant strategies by payment rule (2) when* $\mathbf{c}$ *is private information.*

In equilibrium, everyone who is admitted will end up with positive expected payoffs. Thus it is indeed optimal for them to pay entry fees and enter the second stage when admitted; in this regard our assumption of binding entry is not as stringent as it appears at the first glance.

4 Optimal Selling Mechanisms

Let g_i denote a generic subset in $2^{\mathbf{N}}$ that includes bidder i. Given type c_i and report c_i', bidder i's expected payoff is given by

$$\pi_i(c_i';c_i) = E_{\mathbf{c}_{-i}}\left[\sum_{g_i} p^{g_i}(c_i';\mathbf{c}_{-i})(\pi^{g_i} - c_i) - x_i(c_i';\mathbf{c}_{-i})\right].$$

We focus on incentive compatible mechanisms in which bidders report their types truthfully, i.e. $c_i' = c_i$. Thus in equilibrium,

$$\pi_i(c_i) = \pi_i(c_i;c_i) = E_{\mathbf{c}_{-i}}\left[\sum_{g_i} p^{g_i}(c_i;\mathbf{c}_{-i})(\pi^{g_i} - c_i) - x_i(c_i;\mathbf{c}_{-i})\right]. \tag{3}$$

By the envelope theorem, the incentive compatibility (IC) condition implies the following result:

$$\pi_i'(c_i) = -E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(\mathbf{c}). \tag{4}$$

Thus

$$\pi_i(c_i) = \pi_i(c_h) + \int_{c_i}^{c_h} Q_i(\tilde{c}_i)d\tilde{c}_i, \tag{5}$$

where

$$Q_i(c_i) = E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i;\mathbf{c}_{-i}) \tag{6}$$

is the probability that bidder i is admitted with type c_i.[7]

The following Lemma provides a necessary and sufficient condition for the IC condition.

LEMMA 1 *A mechanism* $(\mathbf{p}^g, \mathbf{x})$ *is incentive compatible if and only if (i)* $Q_i(c_i)$ *decreases in* c_i*, and (ii) (5) holds.*

Proof. See Appendix. ■

The seller's expected revenue is

$$E\pi_0 = E_{\mathbf{c}}\left[\sum_g p^g(\mathbf{c})\pi_0^g + \sum_{i\in\mathbf{N}} x_i(\mathbf{c})\right]. \tag{7}$$

From Lemma 1,

$$\begin{aligned} E_{c_i}\pi_i(c_i) &= \pi_i(c_h) + \int_{c_l}^{c_h}\left[\int_{c_i}^{c_h} Q_i(\tilde{c}_i)d\tilde{c}_i\right] g(c_i)dc_i \\ &= \pi_i(c_h) + \int_{c_l}^{c_h} \frac{G(c_i)}{g(c_i)} Q_i(c_i) g(c_i) dc_i \\ &= \pi_i(c_h) + E_{\mathbf{c}}\left[\frac{G(c_i)}{g(c_i)}\sum_{g_i} p^{g_i}(\mathbf{c})\right]. \end{aligned} \tag{8}$$

Using (3), we have

$$E_{c_i}\pi_i(c_i) = E_{\mathbf{c}}\left[\sum_{g_i} p^{g_i}(\mathbf{c})(\pi^{g_i} - c_i) - x_i(\mathbf{c})\right],$$

which gives

$$\sum_i E_{c_i}\pi_i(c_i) = E_{\mathbf{c}}\left[\sum_g p^g(\mathbf{c})\sum_{j\in g}(\pi^g - c_j) - \sum_i x_i(\mathbf{c})\right]. \tag{9}$$

Thus

$$E_{\mathbf{c}}\sum_i x_i(\mathbf{c}) = E_{\mathbf{c}}\sum_g p^g(\mathbf{c})\sum_{j\in g}(\pi_j^g - c_j) - \sum_i E_{c_i}\pi_i(c_i). \tag{10}$$

From (10) and (8), we have

$$E_{\mathbf{c}}\sum_i x_i(\mathbf{c}) = E_{\mathbf{c}}\sum_g p^g(\mathbf{c})\sum_{j\in g}\left[\pi^g - \frac{G(c_j)}{g(c_j)} - c_j\right] - \sum_i \pi_i(c_h). \tag{11}$$

Combining (7) and (11), we have

$$\begin{aligned} E\pi_0 &= E_{\mathbf{c}}\sum_g p^g(\mathbf{c})\left[\pi_0^g + \sum_{j\in g}\left(\pi^g - \frac{G(c_j)}{g(c_j)} - c_j\right)\right] - \sum_i \pi_i(c_h) \\ &= E_{\mathbf{c}}\sum_g p^g(\mathbf{c})\left[S^g(\mathbf{c}^g) - \sum_{j\in g}\left(c_j + \frac{G(c_j)}{g(c_j)}\right)\right] - \sum_i \pi_i(c_h). \end{aligned} \tag{12}$$

[7] If the entry right allocation rule is not identity discriminating, that is, if the entry right allocation rule depends on the profile of $\mathbf{c}$ only, we must have $Q_i(c_i) = Q(c_i)$.

Next, we consider the maximization of (12) while temporarily ignoring the IC condition of the first stage mechanism. Clearly, to maximize the expected revenue (12) while maintaining the IR condition, we should set $\pi_i(c_h) = 0$ for all i (by (5)). Thus a bidder with the highest cost type should receive zero expected profit. It is also apparent that to maximize expected revenue, the second-stage selling mechanism must be (*ex post*) efficient; otherwise $S^g(\mathbf{c}^g)$, and hence $E\pi_0$, cannot be maximized. A direct implication is that the optimal reserve price in the second-stage auction, r^*, should be v_0.

Lemma 2 *For the selling mechanism to be optimal, the second-stage auction should be an ex post efficient auction (i.e., a standard auction with reserve equal to v_0).*

Thus when entry is taken into account, the optimal auctions characterized by Myerson (1981) are no longer optimal, while efficient auctions are optimal. This result is quite robust as it is also identified in the settings with publicly known information acquisition costs (e.g., Engelbrecht-Wiggans, 1993, and Levin and Smith, 1994).

By (12) and (5), revenue equivalence follows, which says that the expected revenue and the expected payoff of bidders only depend on the entry right allocation scheme and the expected payoff for the least efficient (i.e., highest cost) type but not the specific payment schemes.

Define

$$\pi(g,\mathbf{c}) = S^g(\mathbf{c}^g) - \sum_{j\in g}\left(c_j + \frac{G(c_j)}{g(c_j)}\right) = TS^g(\mathbf{c}^g) - \sum_{j\in g}\frac{G(c_j)}{g(c_j)},$$

which is the expected total surplus generated from the sale less the informational rent to the bidders given the set of entrants, g.

To maximize $E\pi_0$, it is optimal to allocate the entry rights to the group $g^*(\mathbf{c})$, such that

$$g^*(\mathbf{c}) \in \arg\max_g \pi(g,\mathbf{c}),$$

given $\pi(g^*(\mathbf{c}),\mathbf{c}) \geq 0$.

Note that the term $\sum_{j\in g}\frac{G(c_j)}{g(c_j)}$ reflects the informational rent provision. Thus the optimal set of entrants should maximize the expected virtual total surplus in the spirit of Myerson (1981).

We denote this rule of granting entry rights by $\mathbf{p}^* = (p^{*g}(\mathbf{c}))$. To facilitate our characterization of the optimal entry right allocation rule, in what follows we maintain the following regularity condition:

Assumption 1 $H(c) = c + \frac{G(c)}{g(c)}$ *increases with c.*

Note that for Assumption 1 to hold, it is sufficient to assume that the distribution has an increasing hazard rate $\frac{G(c)}{g(c)}$, which is satisfied by most of the common distributions. The optimal entry right allocation rule is characterized below.

THEOREM 3 *(The Optimal Entry Right Allocation Rule:) Under Assumption 1, the optimal entry right allocation should admit the bidders according to their cost efficiencies: The most efficient bidder should be admitted first, the second most efficient bidder should be admitted second, and so on and so forth. The last bidder admitted should be the last entrant that has positive contribution to the expected virtual total surplus, i.e., the expected total surplus ($TS^g(\mathbf{c}^g)$) less the total information rents ($\sum_{j \in g} \frac{G(c_j)}{g(c_j)}$).*

Proof. Suppose the entry rights are awarded to any group g with n bidders. Then the expected virtual total surplus is given by

$$\pi(g, \mathbf{c}) = S^g(\mathbf{c}^g) - \sum_{j \in g} H(c_j) = S(n) - \sum_{j \in g} H(c_j).$$

By Assumption 1, if any group with size n is admitted to maximize $\pi(g, \mathbf{c})$, this group must consist of the bidders with the n lowest costs. Thus in the optimal entry right allocation mechanism, bidders should be admitted one by one according to their cost efficiencies. Furthermore, since the contribution to the expected surplus from a new entrant is decreasing in n while $H(c)$ is increasing in c, there is an optimal number of entrants, say, n^*, such that if an additional bidder were admitted, this additional entrant's contribution to the expected virtual total surplus would be negative unless $n^* = N$. ■

We next explore the properties of this optimal entry right allocation rule.

COROLLARY 1 *The least efficient type admitted must contribute positively to the total surplus $TS^g(\mathbf{c}^g)$; In other words, optimal entry is lower than efficient entry: the seller never admits more than the efficient number of bidders.*

Proof. Each admitted bidder contributes $\pi^g - H(c_i)$ to $\pi(g, \mathbf{c})$. For $\pi^g - H(c_i)$ to be positive, $\pi^g - c_j$ has to be positive, which is her contribution to the expected total surplus $TS^g(\mathbf{c}^g)$. ■

So the revenue-maximizing entry is lower than the efficiency-maximizing entry, reflecting an optimal balance between efficiency and information rent provision.

COROLLARY 2 *If a bidder with type c_i is admitted, then she will remain to be admitted with a more efficient type $c_i' < c_i$; If a bidder with type c_i is not admitted, then she will remain not to be admitted with a less efficient type $c_i' > c_i$.*

Proof. Given $\mathbf{c}_{-i}$, a decrease in c_i only affects $H(c_i)$, which increases in c_i by Assumption 1. When c_i drops, $\pi(g^*(\mathbf{c}), \mathbf{c})$ remains positive. Moreover, a reduction in c_i increases all $\pi(g_i; \mathbf{c})$ uniformly without affecting any other $\pi(g, \mathbf{c})$. Thus $g^*(\mathbf{c})$ is still the group admitted. The second part of the results follows analogously. ■

COROLLARY 3 *Given $\mathbf{c}_{-i}$, whenever bidder i is admitted, she is admitted with the same group of bidders.*

Proof. Given $\mathbf{c}_{-i}$, suppose bidder i with c_i^0 is admitted. We denote the admitted group by $g^*(\mathbf{c}^0)$, where $\#(g^*(\mathbf{c}^0)) = n$. It must be true that the first $(n-1)$ bidders with the lowest costs in $N\backslash\{i\}$ are admitted. Bidder i's cost is also among the first n lowest ones. Recall that the change in c_i uniformly changes all $\pi(g_i; \mathbf{c})$ without affecting any other $\pi(g, \mathbf{c})$. When c_i drops, $\pi(g^*(\mathbf{c}^0), \mathbf{c})$ remains to be the highest. Thus, bidder i is still included in the same group. When c_i increases, all $\pi(g_i; \mathbf{c})$ uniformly decrease but $\pi(g^*(\mathbf{c}^0), \mathbf{c})$ remains to be the highest among all $\pi(g_i; \mathbf{c})$. Thus as long as bidder i is included in any group g_i, this group must be the original $g^*(\mathbf{c}^0)$. ■

COROLLARY 4 *Suppose $i \in g^*(\mathbf{c})$ with $\#(g^*(\mathbf{c})) = n^*$, and let $\hat{c}^{g^*(\mathbf{c})}$ be such that $\pi^{g^*(\mathbf{c})} - H(\hat{c}^{g^*(\mathbf{c})}) = 0$. Then bidder i remains in $g^*(\mathbf{c})$ if and only if c_i is lower than* $\min\{\hat{c}^{g^*(\mathbf{c})}, c^{(n^*+1)}\}$.[8]

Proof. Since $i \in g^*(\mathbf{c})$ with $\#(g^*(\mathbf{c})) = n^*$, we must have $c_i \leq c^{(n^*+1)}$; otherwise, bidder i must be excluded.

If $c_i \leq c^{(n^*+1)}$ and $c_i \leq \hat{c}^{g^*(\mathbf{c})}$, then $H(c_i) \leq \pi^{g^*(\mathbf{c})}$, which implies that bidder i's contribution to the virtual total surplus is positive should $g^*(\mathbf{c})$ remain to be the group admitted. If $c_i \leq c^{(n^*+1)}$ and $c_i > \hat{c}^{g^*(\mathbf{c})}$, then $H(c_i) > \pi^{g^*(\mathbf{c})}$, which implies that bidder i's contribution to the virtual total surplus is negative should $g^*(\mathbf{c})$ remain to be the group admitted. ■

Next we explore the incentive compatible payment scheme that truthfully implements the optimal entry right allocation rule. Given that $\pi_i(c_h) = 0$, we have from (5),

$$\pi_i(c_i) = \int_{c_i}^{c_h} Q_i(\tilde{c}_i) d\tilde{c}_i = E_{\mathbf{c}_{-i}} \sum_{g_i} \int_{c_i}^{c_h} p^{g_i}(\tilde{c}_i; \mathbf{c}_{-i}) d\tilde{c}_i.$$

[8]When $\#(g^*(\mathbf{c})) = N$, we define $c^{(n^*+1)} = c_h$.

On the other hand,

$$\pi_i(c_i) = E_{\mathbf{c}_{-i}}\left[\sum_{g_i} p^{g_i}(c_i;\mathbf{c}_{-i})(\pi^{g_i} - c_i) - x_i(c_i;\mathbf{c}_{-i})\right].$$

Naturally, we select the payment rule $x_i^*(\mathbf{c})$ such that

$$x_i^*(c_i;\mathbf{c}_{-i}) = \sum_{g_i} p^{*g_i}(c_i;\mathbf{c}_{-i})(\pi^{g_i} - c_i) - \sum_{g_i}\int_{c_i}^{c_h} p^{*g_i}(\tilde{c}_i;\mathbf{c}_{-i})d\tilde{c}_i, \tag{13}$$

where $p^{*g_i}(\mathbf{c})$ is the optimal entry right allocation rule characterized above.

By (13) and Corollaries 2 – 4, when $i \in g^*(\mathbf{c})$ with $\#(g^*(\mathbf{c})) = n^*$, we have

$$x_i^*(\mathbf{c}) = [\pi^{g^*(\mathbf{c})} - c_i] - [\min\{\hat{c}^{g^*(\mathbf{c})}, c^{(n^*+1)}\} - c_i] = \pi^{g^*(\mathbf{c})} - \min\{\hat{c}^{g^*(\mathbf{c})}, c^{(n^*+1)}\}. \tag{14}$$

Without loss of generality, we assume that c_h is sufficiently large so that $\pi^{g_i} - H(c_h) < 0, \forall g_i$. Thus $\hat{c}_i^{g_i} = H^{-1}(\pi^{g_i}) < c_h$, and

$$x_i^*(\mathbf{c}) \geq \pi^{g^*(\mathbf{c})} - \hat{c}^{g^*(\mathbf{c})} \geq \pi^{g^*(\mathbf{c})} - H(\hat{c}^{g^*(\mathbf{c})}) = 0.$$

When $i \notin g^*(\mathbf{c})$, by (13) and Corollaries 2 – 4,

$$x_i^*(c_i;\mathbf{c}_{-i}) = 0.$$

To summarize, we consider the following payment rule:

$$x_i^*(c_i;\mathbf{c}_{-i}) = \begin{cases} \pi^{g^*(\mathbf{c})} - \min\{\hat{c}^{g^*(\mathbf{c})}, c^{(n^*+1)}\} & \text{if} \quad i \in g^*(\mathbf{c}) \\ 0 & \text{if} \quad i \notin g^*(\mathbf{c}) \end{cases}. \tag{15}$$

It is clear that all the admitted bidders pay a uniform entry fee that is contingent on $\mathbf{c}$. Define

$$Q_i^*(c_i) = E_{\mathbf{c}_{-i}} \sum_{g_i} p^{*g_i}(c_i;\mathbf{c}_{-i}),$$

which is bidder i's probability of being admitted with type c_i. We should show that the payment rule specified in (15) can indeed truthfully implement the optimal entry right allocation rule specified in Theorem 3. According to Lemma 1, we only need to verify (i) and (ii). Clearly (ii) is automatically satisfied by the construction of $x_i^*(\mathbf{c})$. It remains to verify (i).

It is sufficient to show that for any $\mathbf{c}_{-i}$, $\sum_{g_i} p^{*g_i}(c_i;\mathbf{c}_{-i})$ decreases in c_i. Take $c_i'' \leq c_i'$, and consider any given $\mathbf{c}_{-i}$. If reporting c_i' results in bidder i being excluded, then

$$\sum_{g_i} p^{*g_i}(c_i'';\mathbf{c}_{-i}) \geq \sum_{g_i} p^{*g_i}(c_i';\mathbf{c}_{-i}) = 0.$$

If reporting c_i' and c_i'' both results in bidder i being admitted, then bidder i must be included in the same group by Corollary 3, i.e., $\sum_{g_i} p^{*g_i}(c_i''; \mathbf{c}_{-i}) = \sum_{g_i} p^{*g_i}(c_i'; \mathbf{c}_{-i})$. Hence $Q_i^*(c_i)$ is decreasing in c_i. We have thus shown that under Assumption 1, $(\mathbf{p}^*, \mathbf{x}^*)$ constitutes an IC, optimal entry right allocation mechanism. In fact, we can show that the payment rule $\mathbf{x}^*$ even implements the optimal entry right allocation rule $\mathbf{p}^*$ in dominant strategies.

Theorem 4 *Under Assumption 1, the optimal entry right allocation rule* $\mathbf{p}^*$ *is truthfully implementable by payment rule* $\mathbf{x}^*$ *in dominant strategies.*

Proof. See Appendix. ■

So in our setting, both the efficient and optimal entry right allocation rules can be implemented in dominant strategies. Next we consider the auction implementation of the optimal mechanism. Define c^s to be the infimum of all types (in terms of costs) that will never be admitted. Define

$$x_i^*(c_i) = E_{\mathbf{c}_{-i}} x_i^*(c_i, \mathbf{c}_{-i}).$$

Then we have $x_i^*(c_i) = 0$ when $c_i \geq c^s$. When $c_i < c^s$, $x_i^*(c_i)$ is strictly decreasing. To see this, note that when c_i decreases, the probability that i is admitted increases (by Corollary 2). For any given $\mathbf{c}_{-i}$ that leads to bidder i's inclusion with her initial c_i, Corollaries 2 and 3 imply that a lower c_i would result in her being included in the same $g^*(\mathbf{c})$. Her payment thus would not change for those given $\mathbf{c}_{-i}$. For those additional $\mathbf{c}_{-i}$ that lead to bidder i's inclusion due to her lower entry cost, she has to pay a positive fee. Thus the expected payment $x_i^*(c_i)$ is strictly higher with a lower entry cost.

From Theorem 3, the entry right allocation rule is nondiscriminatory; that is, it depends on the reported profile of $\mathbf{c}$ only. It is then easily verified that $x_i^*(c_i, \mathbf{c}_{-i}) = x^*(c_i, \mathbf{c}_{-i})$ and $x_i^*(c_i) = x^*(c_i)$ for all i. Define

$$\beta(c_i) = E_{\mathbf{c}_{-i}} x^*(c_i, \mathbf{c}_{-i}) = x^*(c_i), \tag{16}$$

which is symmetric and strictly decreasing in c_i.

In view of the above results, the optimal entry allocation mechanism can be implemented using an all-pay auction.

Theorem 5 *(Implementation of Optimal Mechanisms) In the (reduced) entry right allocation all-pay auction game where bidders pay their bids regardless being admitted or not, it is a symmetric* *monotone Bayesian Nash equilibrium for each potential bidder to bid according to $\beta(\cdot)$ defined in (16). The costs can then be inverted from the bids and the optimal entry right allocation rule can be implemented.*[9]

Proof. See Appendix. ■

The idea of using an all-pay auction to implement entry right allocation is similar to Fullerton and McAfee (1999), who analyze a research tournament model with multiple contestants competing for a common prize. They show that the efficient entry can always be implemented through an all-pay auction for entry rights. In their setting, the optimal number of entry rights is fixed (which is two). This is different from our setting where the optimal number of entry rights is endogenously determined. Similarly to Fullerton and McAfee (1999), it is also not clear in our setting whether other auction formats, such as uniform-price and discriminatory-price auctions, can implement the optimal entry.[10]

By Theorem 5, the optimal selling mechanism can be implemented through a two-stage auction, with the first stage being an entry-right auction and the second stage being a standard single-object auction. If the auctioneer does not exercise control over entry, potential bidders make their own entry decisions independently and simultaneously. In that case equilibrium entry will be governed by an entry threshold, c^e, as analyzed by Moreno and Wooders (2006) and Lu (2010). Formally, the admission rule can be characterized by an entry threshold $c^e \in (c_l, c_h)$ such that the equilibrium set of entrants is $g^*(\mathbf{c}) = \{i \in \mathbf{N} | c_i \leq c^e\}$; that is, bidder i is included if and only if her cost is lower than the entry threshold c^e.

We can derive a payment rule that truthfully implements this cutoff admission rule. It is easily seen that the payment rule should take the same form as prescribed in (15) with $\min\{\hat{c}^{g^*(\mathbf{c})}, c^{(n^*+1)}\}$

[9] We define $\beta^{-1}(\emptyset) = c_h$ for bidders who do not bid, and $\beta^{-1}(b) = c_l$ for $b > \beta(c_l)$.

[10] For instance let's consider a discriminatory-price auction, where only admitted bidders need to pay for entry and each pays her own bid. Let $\beta_A(\cdot)$ denote the equilibrium bid function under an all-pay auction and $\beta_D(\cdot)$ denote the equilibrium bid function should it exist. Then by the payoff equivalence (Myerson's lemma), we must have

$$\beta_D(c) = \beta_A(c) / \Pr(\text{Being admitted with type } c)$$

While both $\beta_A(c)$ and Pr(Being admitted with type c) are strictly decreasing, we are not sure whether their ratio, and hence the candidate equilibrium bid function $\beta_D(c)$, is also strictly decreasing. Since we do not have specific expressions for Pr(Being admitted with type c), it is hard to derive conditions under which $\beta_D(c)$ is strictly monotonic.

being replaced by c^e, i.e. each admitted bidder pays $\pi^{g^*(\mathbf{c})} - c^e$, and bidders not admitted do not pay. The entrants' payment can be positive or negative depending on the size of the entrant set, with negative payments meant to demand entry subsidies. As such, the direct mechanism associated with the auction game with endogenous entry is a feasible mechanism considered in our two-stage mechanism framework. A direct implication is that it is revenue-dominated by the optimal selling mechanism that we characterize.[11]

In the two-stage auctions studied in Ye (2007), the number of bidders to be admitted is fixed and pre-announced. Let this pre-announced number be n. Then the auctioneer would admit the first n bidders with the n lowest costs. The payment rule that implements this entry admission rule can be easily derived. We define $x_i^*(\mathbf{c})$ such that

$$x_i^*(c_i; \mathbf{c}_{-i}) = \sum_{g_i} p^{*g_i}(c_i; \mathbf{c}_{-i})(\pi^{g_i} - c_i) - \sum_{g_i} \int_{c_i}^{c_h} p^{*g_i}(\tilde{c}_i; \mathbf{c}_{-i}) d\tilde{c}_i,$$

where $p^{*g_i}(\mathbf{c})$ is the entry admission rule that admits the fixed n most efficient bidders. When $i \in g^*(\mathbf{c})$ with $\#(g^*(\mathbf{c})) = n$, we have

$$\begin{aligned} x_i^*(c_i; \mathbf{c}_{-i}) &= [\pi^{g^*(\mathbf{c})} - c_i] - [c^{(n+1)} - c_i] \\ &= \pi^{g^*(\mathbf{c})} - c^{(n+1)}, \end{aligned}$$

which coincides with the highest losing bid in an all-pay auction (for entry rights) described in Ye (2007). When $i \notin g^*(\mathbf{c})$, we simply let $x_i^*(c_i; \mathbf{c}_{-i}) = 0$.

Thus the two-stage auction with the fixed-n admission rule is also feasible in our two-stage mechanism framework, and is hence dominated by the optimal selling mechanism that we characterize.

5 Concluding Remarks

This paper studies two-stage selling mechanisms with an emphasis on entry right allocation mechanisms in a setting where bidders' information acquisition costs are privately known to the bidders. Our entry right allocation mechanism resembles multi-unit auctions with endogenously determined supply. For both efficient and optimal selling mechanisms, we find that the second-stage auction must be *ex post* efficient. In the entry allocation stage, both efficiency and optimality require that

[11] In fact, by the revenue equivalence, only the admission rule matters for the revenue.

the bidders with the most efficient types (least entry costs) be admitted. However, unlike in the complete information benchmark where entry costs are publicly known, revenue-maximizing entry diverges from efficiency-maximizing entry due to the information rent consideration. We show that optimal entry is lower than efficient entry, which results from an optimal balance between efficiency and information rent extraction.

Our analysis of two-stage selling mechanism can shed new light on the practice of two-stage auctions, which is commonly used in high-valued and complex asset sales, procurements, takeovers, and merger and acquisition contests.[12] A feature common to all these two-stage auctions is that there is a pre-qualifying stage to screen the bidders,[13] which corresponds to the entry right allocation mechanism analyzed in our framework. In our current setting, bidders are heterogeneous in terms of their cost efficiencies in information acquisition. We can envision a different setting where bidders are heterogeneous in terms of their "types" (say, α_i's) before information acquisition stage, and after entry, each bidder draws a value from a distribution parameterized by her "type" α_i. Suppose that the higher this pre-entry "type", the more likely that the bidder will draw a higher value for the asset for sale. Then with costly information acquisition, the optimal selling mechanism will again take the form of a two-stage procedure, with the first stage being the pre-screening or entry right allocation mechanism. While our current model is somewhat special in the sense that the pre-entry type is simply the information acquisition cost, we believe that the general insights obtained from our current analysis are robust enough to carry over to that presumably more complicated setting. A complete characterization of optimal selling mechanisms in that setting is not straightforward, in part because the extension would introduce asymmetries in the post-entry value distributions. Despite the technical challenges, future research should extend our current analysis to more general settings.

[12] See Ye (2007) for industry examples using such a two-stage auction procedure.

[13] See Boone and Goeree (2009) for an interesting analysis of pre-qualifying auctions.

6 Appendix

Proof of Lemma 1: The necessity. We have shown (5) must hold. We next show (i) must hold. Define

$$\pi_i(c_i'; c_i) = E_{\mathbf{c}_{-i}}\left[\sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})(\pi^{g_i} - c_i) - x_i(c_i'; \mathbf{c}_{-i})\right].$$

Note $\pi_i(c_i) = \pi_i(c_i; c_i)$. IC conditions require that

$$\pi_i(c_i; c_i) \geq \pi_i(c_i'; c_i), \forall c_i, c_i'.$$

Note that

$$\pi_i(c_i'; c_i) = \pi_i(c_i'; c_i') + E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})(c_i' - c_i), \forall c_i, c_i'.$$

Thus

$$\pi_i(c_i'; c_i) - \pi_i(c_i'; c_i') \geq E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})(c_i' - c_i), \forall c_i, c_i'.$$

Similarly we have

$$\pi_i(c_i'; c_i') - \pi_i(c_i'; c_i) \geq E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i; \mathbf{c}_{-i})(c_i - c_i'), \forall c_i, c_i'.$$

These inequalities yield that

$$\begin{aligned} & E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i; \mathbf{c}_{-i})(c_i' - c_i) \\ \geq \quad & E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})(c_i' - c_i), \forall c_i, c_i'. \end{aligned}$$

or

$$\begin{aligned} & E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})(c_i - c_i') \\ \geq \quad & E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i; \mathbf{c}_{-i})(c_i - c_i'), \forall c_i, c_i'. \end{aligned}$$

This implies that $Q_i(c_i)$ decreases in c_i.

The Sufficiency. Assume (i) and (ii) hold. For $c_i' < c_i$, we have

$$\begin{aligned} \pi_i(c_i') - \pi_i(c_i) &= \int_{c_i'}^{c_i} Q_i(\tilde{c}_i) d\tilde{c}_i \\ &= \int_{c_i'}^{c_i} E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(\tilde{c}_i; \mathbf{c}_{-i}) d\tilde{c}_i \\ &\leq \int_{c_i'}^{c_i} E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i}) d\tilde{c}_i \\ &= E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})[c_i - c_i']. \end{aligned}$$

The inequality is due to (i). Thus

$$\begin{aligned}
\pi_i(c_i) &\geq \pi_i(c_i') - E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})[c_i - c_i'] \\
&= E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})[\pi^{g_i} - x_i^{g_i}(c_i'; \mathbf{c}_{-i}) - c_i'] \\
&\quad - E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})[c_i - c_i'] \\
&= E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})[\pi^{g_i} - x_i^{g_i}(c_i'; \mathbf{c}_{-i}) - c_i] \\
&= \pi_i(c_i'; c_i).
\end{aligned}$$

For $c_i' > c_i$, we have

$$\begin{aligned}
\pi_i(c_i) - \pi_i(c_i') &= \int_{c_i}^{c_i'} Q_i(\tilde{c}_i) d\tilde{c}_i \\
&= \int_{c_i}^{c_i'} E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(\tilde{c}_i; \mathbf{c}_{-i}) d\tilde{c}_i \\
&\geq \int_{c_i}^{c_i'} E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i}) d\tilde{c}_i \\
&= E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})[c_i' - c_i].
\end{aligned}$$

The last inequality is due to (i). Thus

$$\begin{aligned}
\pi_i(c_i) &\geq \pi_i(c_i') + E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})[c_i' - c_i] \\
&= E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})[\pi^{g_i} - x_i^{g_i}(c_i'; \mathbf{c}_{-i}) - c_i'] \\
&\quad - E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})[c_i - c_i'] \\
&= E_{\mathbf{c}_{-i}} \sum_{g_i} p^{g_i}(c_i'; \mathbf{c}_{-i})[\pi^{g_i} - x_i^{g_i}(c_i'; \mathbf{c}_{-i}) - c_i] \\
&= \pi_i(c_i'; c_i).
\end{aligned}$$

■

Proof of Theorem 4: Given any $\mathbf{c}_{-i}$, a profile of reported types of all but bidder i, we will show that bidder i has no incentive to misrepresent her type under the mechanism $(\mathbf{p}^*, \mathbf{x}^*)$.

If by reporting truthfully bidder i will be admitted according to $\mathbf{p}^*$, then by Corollary 4, $c_i < \min\{\hat{c}^{g^*(\mathbf{c})}, c^{(n^*+1)}\}$. In this case bidder i will have a positive expected payoff. If she misreports a cost lower than $\min\{\hat{c}^{g^*(\mathbf{c})}, c^{(n^*+1)}\}$, she will still be included in the same group by Corollaries 3 and 4 and thus pay the same amount as in the truthful reporting case. Her payoff thus remains the same.

If she misreports a cost higher than $\min\{\hat{c}^{g^*(\mathbf{c})}, c^{(n^*+1)}\}$, she will be excluded and pay zero, which results in a zero payoff. Therefore, if bidder i will be admitted by truthfully reporting her type, she has no incentive to misrepresent her type.

If by reporting truthfully bidder i will be excluded according to $\mathbf{p}^*$, then $c_i \geq \min\{\hat{c}^{g^*(\mathbf{c})}, c^{(n^*+1)}\}$, and her payoff will be zero. If she misreports a cost such that she is still excluded (and thus pays zero), her payoff remains zero. Suppose she reports a cost c_i' that is low enough for her to be included in $g^*(c_i', \mathbf{c}_{-i})$. We must have $\#(g^*(c_i', \mathbf{c}_{-i})) = \#(g^*(\mathbf{c}))$ (the previously admitted bidder with type $c^{(n^*)}$ is crowded out) or $\#(g^*(c_i', \mathbf{c}_{-i})) = \#(g^*(\mathbf{c})) + 1$ (no previously admitted bidders are crowded out). Note that these are the only two possible cases, as all the other bidders with costs higher than $c^{(n^*+1)}$ can never be admitted according to $\mathbf{p}^*$.

If $\#(g^*(c_i', \mathbf{c}_{-i})) = \#(g^*(\mathbf{c}))$, bidder i would pay $\pi^{g^*(c_i', \mathbf{c}_{-i})} - \min\{\hat{c}^{g^*(c_i', \mathbf{c}_{-i})}, c^{(n^*)}\} = \pi^{g^*(\mathbf{c})} - \min\{\hat{c}^{g^*(\mathbf{c})}, c^{(n^*)}\}$,[14] which is higher than $\pi^{g^*(\mathbf{c})} - c_i$. This results in a negative expected payoff for her.

If $\#(g^*(c_i', \mathbf{c}_{-i})) = \#(g^*(\mathbf{c})) + 1$, bidder i would pay $\pi^{g^*(c_i', \mathbf{c}_{-i})} - \min\{\hat{c}^{g^*(c_i', \mathbf{c}_{-i})}, \tilde{c}\}$, where $\tilde{c} = c^{(n^*+1)}$ if $c_i > c^{(n^*+1)}$, and $\tilde{c} = c^{(n^*+2)}$ if $c_i = c^{(n^*+1)}$. Since the bidder with $c^{(n^*+1)}$ is excluded when bidder i reports truthfully, we must have $c^{(n^*+1)} > \hat{c}^{g^*(c_i', \mathbf{c}_{-i})}$ when $\#(g^*(c_i', \mathbf{c}_{-i})) = \#(g^*(\mathbf{c})) + 1$. If $\tilde{c} = c^{(n^*+1)}$ and $c_i > c^{(n^*+1)}$, clearly $\min\{\hat{c}^{g^*(c_i', \mathbf{c}_{-i})}, \tilde{c}\} < c_i$, which results in a negative expected payoff for bidder i. If $\tilde{c} = c^{(n^*+2)}$ and $c_i = c^{(n^*+1)}$, since $c^{(n^*+1)} > \hat{c}^{g^*(c_i', \mathbf{c}_{-i})}$ we also have $\min\{\hat{c}^{g^*(c_i', \mathbf{c}_{-i})}, \tilde{c}\} < c_i$, which also results in a negative expected payoff for bidder i.

This shows that if bidder i is excluded when reporting truthfully, bidder i has no incentive to misrepresent her type either, regardless of the reports from everyone else. ■

Proof of Theorem 5: Given that everyone else bids according to $\beta(\cdot)$ defined by (16), bidder i's problem is to maximize the following objective function by choosing her bid b:

$$\Pi(b, c_i) = E_{\mathbf{c}_{-i}} I\left\{i \in g^*(\beta^{-1}(b), \mathbf{c}_{-i})\right\} \cdot \left(\pi^{g^*(\beta^{-1}(b), \mathbf{c}_{-i})} - c_i\right) - b,$$

where the indicator function $I\left\{i \in g^*(\beta^{-1}(b), \mathbf{c}_{-i})\right\} = 1$ if and only if $i \in g^*(\beta^{-1}(b), \mathbf{c}_{-i})$.

We next apply the constraint simplification theorem[15] to demonstrate that $\beta(\cdot)$ constitutes a

[14] Recall that $\pi^{g^*(\mathbf{c})}$ and $\hat{c}^{g^*(\mathbf{c})}$ are solely determined by the size of the group $g^*(\mathbf{c})$ admitted.

[15] See, e.g., Theorem 4.3 in Milgrom, 2004, pp. 105.

symmetric increasing Bayesian Nash equilibrium in this reduced entry right allocation game. This can be verified in the following steps:

1. $\beta(\cdot)$ is strictly decreasing as shown in the arguments preceding to the statement of the theorem.

2. Given that $\beta(\cdot)$ is strictly decreasing, $\partial\Pi/\partial c_i = \Pr\left\{i \in g^*(\beta^{-1}(b), \mathbf{c}_{-i})\right\}$ is strictly increasing in b, i.e., $\Pi(b, c_i)$ satisfies the strict and smooth single crossing differences property.

3. By construction, $\beta(\cdot)$ satisfies the following envelope formula:

$$\Pi(\beta(c_i), c_i) - \Pi(\beta(c_h), c_h) = -\int_{c_i}^{c_h} \Pi_2(\beta(s), s)ds.$$

4. It is also easily verified that bidding outside the range of $\beta(\cdot)$ cannot lead to higher expected payoff.

Thus all the sufficiency conditions for the constraint simplification theorem are satisfied and $\beta(\cdot)$ indeed constitutes a symmetric monotone BNE. ■

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