Reserve Price Signaling*

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Abstract

In a general auction model in which bidders' signals are affiliated, we characterize the unique separating equilibrium in which the seller can use reserve prices to credibly signal her private information. When the buyers' signals are independent, the optimal reserve price is shown to be increasing in the number of bidders under certain conditions. We also demonstrate that the probability that the item is sold at the reserve price can increase as the number of bidders increases, which indicates a more central role for reserve prices than perceived in the standard auction models.

Keywords: auctions, reserve price, signaling

JEL: D44, D80, D82

1. Introduction

In this paper we consider an auction environment in which a seller of a single object has private information about the object's characteristics. These characteristics affect the seller's valuation of the object and the common valuation for a group of potential buyers, each of whom also has a private signal about the object. For example, a seller of an artwork (e.g., an auction house) may know better than potential buyers the conditions (quality, rarity, history, etc.) and the secondary market value of the artwork. Similarly, a government agency auctioning procurement of a public project may have better information than bidding firms about certain factors (e.g., environmental impacts and regulations) that affect both its valuation of the project and project costs common to all bidding firms.

If direct verification of the seller's information is costless, it is incentive compatible for the seller to truthfully reveal her information for the following reason. For any subset of types, those sellers with the most favorable estimates of the item's value have an incentive to reveal this information since it raises buyers' valuations and hence their equilibrium bids. Thus there can be no equilibrium pooling of types and so the unique Nash equilibrium involves full revelation of the seller's private information. However, in many auction settings, a costless revelation technology (e.g., a perfectly objective evaluation by a third party) is not available to the seller. In such cases, the seller's announcement of her information to the potential buyers is not credible as she faces the adverse selection problem, that is, she always wants to claim the highest possible value to the buyers. A natural way to credibly reveal the private information is through signaling, and a natural signaling instrument in this environment is the reserve price: a high type seller has an incentive to signal this to the buyers by setting a high reserve price.

In this paper we introduce a reserve price signaling model in which the buyers' private signals are affiliated. The key observation is that a higher reserve price makes it unprofitable for a larger set of buyer types to bid. As the minimum bidder type rises, so does the probability that the item will not be sold. Then the marginal cost of raising the reserve price is lower for a seller with a more favorable signal, since his assessment of the use value of the item is higher. From this observation, we are able to fit the signaling model into the standard signaling framework Riley [12], and the analysis is greatly simplified. We characterize the unique separating equilibrium in which the lowest type seller sets a reserve price that is optimal under complete information. We then show that when the buyers' signals are independent, the equilibrium reserve price is increasing in the number of bidders under fairly general specifications of buyers' valuations. Thus our results show that a reserve price can play a more central role than perceived by the traditional literature. In the standard private value auction model, the seller's optimal reserve price is set to capture additional revenue when there is only one buyer who has a valuation much higher than her own. This optimal reserve price is independent of the number of bidders. Therefore, unless the number of bidders is small, the probability that the item is sold at the reserve price is small and hence the extra profit captured by setting a reserve price is also low. In contrast, when the reserve price plays a signaling role, our results indicate that the probability that the item is sold at the reserve price need not decrease as the number of bidders becomes large.

After analyzing the general signaling model, we study a linear valuation model in which each buyer's valuation is the sum of his own private signal and a common value component which is the seller's private information. The seller's own valuation for the object is proportional to her private signal. We solve for an analytical solution of the reserve price schedule in the separating equilibrium.

A simultaneous and independent paper by Jullien and Mariotti [6] is closely related to ours. Working on essentially the same model but using somewhat different approaches, their paper and ours arrive at the same characterization of the unique separating equilibrium of the model.[1] The differences between the two papers are as follows. First, the two papers focus on different economic applications of the model. Jullien and Mariotti [6] compare the decentralized signaling equilibrium outcome with the optimal mechanism for a monopoly broker who buys from the seller and sells to the buyers. We focus on the signaling role of the reserve price and study how it changes with the number of bidders.[2] Secondly, Jullien and Mariotti [6] consider the case in which the buyers have independent signals and the seller's valuation is always greater than the buyers' common value component, while we allow affiliation of the buyers' signals and the possibility that the seller's valuation can sometimes be smaller than the buyer's common value component. Thirdly, Jullien and Mariotti [6] consider the two bidder case, while we allow any number of bidders. In fact, one of our main results is to show that under reasonable conditions, the reserve price as a signaling instrument increases in the number of bidders (Theorem 2).

[1] Jullien and Mariotti [6] also analyze pooling and partial pooling equilibria, which we do not.

[2] In Cai, Riley and Ye [1], we also present three other economic applications of the model to outside certification, to an analysis of relative importance of private values, and to a Lemons market analysis.

The paper is organized as follows. We introduce our basic model in Section 2, followed by the equilibrium characterization in Section 3. Section 4 studies a linear valuation model and solves for the equilibrium analytically. In Section 5 we discuss possible extensions and offer concluding remarks.

2. The Model

We consider a second-price sealed bid auction in which n symmetric buyers bid for a single, indivisible object.[3] The seller observes a signal s, which is not observed by the buyers. The seller's signal s is drawn from a distribution function $G(\cdot)$ with support $S=[\underline{s},\overline{s}]$ and density function $g(s)>0$ for all $s\in[\underline{s},\overline{s}]$. The seller's own valuation for the item is $\xi(s)$, which is strictly increasing in s. Each buyer i observes a signal X_i, which is his private information. The n buyers' signals have a joint distribution with density function $\overline{f}(z)$, where $z=(x_1,x_2,...,x_n)\in[\underline{x},\overline{x}]^n$. Since buyers are symmetric, the marginal distributions of $\overline{f}(z)$ for all x_i are identical. Ex ante, buyers' signals are affiliated, with independence as a special case. We assume that the seller's signal s is independent of the buyers' signals.[4]

Given s and $z=(x_1,x_2,...,x_n)$, buyer i's valuation for the item is given by $V_i=u_i(s,x_1,...,x_n)$. We assume that there is a function u on $\mathbb{R}^{1+n}$ such that for all i, $u_i(s,x_1,...,x_n)=u(s,x_i,x_{-i})$. Therefore, all the buyers' valuations depend on s in the

[3] We consider the second-price auctions mainly for simplicity of analysis. The insights of our main results should still carry over to other auction formats. However, with affiliated valuations revenue equivalence usually does not hold among different auction formats.

[4] This implies that even if bidders can communicate about their signals, they cannot infer the seller's private information s.

same manner and each buyer's valuation is a symmetric function of the other bidders' signals. Moreover, we assume that the function u is nonnegative, and is continuous and increasing in all its arguments. It is also integrable so $EV_i < \infty$.

Let $X_{(1)}^{-i}$ denote the highest signal among all but buyer i's signals. We define a function $v_i : \mathbb{R}^2 \to \mathbb{R}$ by $v_i(s,x) = E[V_i \mid S = s, X_i = X_{(1)}^{-i} = x]$. By the symmetry of u_i, v_i is identical for all i, that is, $v_i(s,x) = v(s,x)$. Because (s,x) are independent and hence affiliated, and because u is increasing in its arguments, $v(s,x)$ is also increasing in both variables (Milgrom and Weber [10, Theorem 5]). We add the non-degeneracy assumption so that v is strictly increasing in s and x.

Our specification of buyers' valuations is quite general. To give some examples, let ϕ, ψ and ζ be any positive increasing functions. Then the following valuation functions all fit in our framework:

(a) For any signal profile (s,z), $u_i(s,z) = \phi(s) + \psi(x_i)$ or $u_i(s,z) = \phi(s) \cdot \psi(x_i)$. In either of the two cases, buyers' valuations are independent of other buyers' signals. If their signals are independent and the seller's signal s is revealed to the buyers, then the ensuing auction has the features of independent private value auctions.

(b) For any signal profile (s,z), $u_i(s,z) = \phi(s) + \psi(x_i) + \zeta(x_{-i})$ or $u_i(s,z) = \phi(s) \cdot [\psi(x_i) + \zeta(x_{-i})]$, where, for example, $\psi(x_i) + \zeta(x_{-i}) = \sum_{j=1}^{n} x_j$, or $\psi(x_i) + \zeta(x_{-i}) = \frac{1}{n} \sum_{j=1}^{n} x_j$. In these cases, buyers' valuations depend on the seller's signal, their own signals as well as other buyers' signals. If their signals are independent and the seller's signal s is revealed to the buyers, then the auction has the features of independent-signal common value auctions.

(c) For any signal profile (s,z), $u_i(s,z)=\lambda\phi(s)+(1-\lambda)\max(x_1,x_2,...,x_n)$. This is another case of a common value auction in which the valuation common to all buyers is a linear combination of the seller's signal and the highest buyer signal.

Let $X_{(1)}$ and $X_{(2)}$ denote the highest and the second highest signal statistics among all the signals of the n buyers. For $k=1,2$, let $F_{(k)}(\cdot)$ and $f_{(k)}(\cdot)$ be the corresponding distribution and density functions, respectively. Our analysis will heavily rely on the following assumption:

Assumption (R): For any s, $J(s,x)=v(s,x)-\dfrac{\partial v(s,x)}{\partial x}\dfrac{F_{(2)}(x)-F_{(1)}(x)}{f_{(1)}(x)}$ is strictly increasing in x.

This assumption is a generalization of the standard assumption in the independent private value auction setting that the "virtual surplus" $x-(1-F(x))/f(x)$ is strictly increasing in x. The following lemma identifies sufficient conditions for Assumption (R) to hold when X_i's are independent.

Lemma 1: *When X_i's are independent, Assumption (R) is satisfied if*

(1) the hazard rate function of X is increasing; and (2) $\dfrac{\partial^2 v(s,x)}{\partial x^2}\le 0$.

Proof: See the appendix.

We study the following signaling game. The seller announces a reserve price r at the beginning of the auction. The buyers then submit sealed bids in the second price auction.[5] As is typical in the signaling literature, our game has many equilibria. By the standard equilibrium refinement concepts such as the Intuitive Criterion (Cho and Kreps [3]), pooling or partial pooling equilibria can be ruled out. In fact, by the results of Riley [12], there is a unique separating equilibrium in our game if the lowest type seller chooses the reserve price that is optimal under complete information.[6] Following this literature, we focus on such a unique separating equilibrium.

In the subgame following the seller's move, suppose the buyers, upon observing a reserve price r, believe that the value of the seller's signal is $\hat{s}$. By [10], it is a Bayesian Nash equilibrium for each buyer i to bid $v(\hat{s}, x_i) = E[V_i \mid S = \hat{s}, X_i = X_{(1)}^{-i} = x_i]$, that is, his expected valuation given that the seller's signal is $\hat{s}$ and that the highest signal among all other buyers equals his own signal x_i. Following the literature, we focus on the symmetric equilibrium in this paper.

A reserve price is a potential signal if a higher reserve price induces a higher probability of no sale. For then the marginal cost of increasing the reserve price is higher for a seller with a lower use value. Let $m(s)$ be the minimum buyer type that will enter the auction, given the belief that the seller's type is s. For there to be a lower probability

[5] For the independent valuation case our results generalize immediately to other auctions (see Corollary 1).

[6] See also Mailath [7] for his discussion of the existence and differentiability of such a separating equilibrium.

of sale when the seller's type is higher, it must be the case that the minimum type function $m(\cdot)$ is a strictly increasing function. In the previous paragraph we observed that the equilibrium bid by buyer *i*, with signal x_i is $v(s, x_i)$. Thus the bid by the minimum type who enters the auction is $v(s, m(s))$. In equilibrium the minimum bid must be equal to the reserve price, that is, $v(s, m(s)) - r = 0$. Since $v(\cdot)$ is an increasing function it follows that if $m(\cdot)$ is strictly increasing, then the equilibrium reserve price $r(s) = v(s, m(s))$ is strictly increasing.[7] Bidders can therefore infer the seller's type from the reserve price.

Formally, let the separating equilibrium be characterized by the minimum type schedule $m(\cdot):[\underline{s}, \overline{s}] \to \mathbb{R}_+$, a strictly increasing function mapping a seller's type to the minimum buyer type that will enter the auction. This induces a strictly increasing reserve price function

$$r(s) = v(s, m(s)) . \tag{1}$$

Given a reserve price $r(\hat{s})$, the auction has three possible outcomes. Contingent on these outcomes, the seller's payoffs are determined as follows:

(i) If $v(\hat{s}, X_{(1)}) \leq r(\hat{s}) = v(\hat{s}, m(\hat{s}))$, then the highest bid is below the reserve price and hence the good is not sold. In this case, the seller's payoff is $\xi(s)$.

(ii) If $v(\hat{s}, X_{(2)}) \leq r(\hat{s}) = v(\hat{s}, m(\hat{s})) < v(\hat{s}, X_{(1)})$, then only the highest bid is above the reserve price and hence the good is sold to the buyer with the highest bid at the

[7] In some cases, it is possible that there is an equilibrium seller type $\tilde{s}$ for which the minimum buyer type entering the auction given type $\tilde{s}$'s reserve price is $\overline{x}$. Then the probability of sale is zero for seller type $\tilde{s}$. The same will be true for all seller types higher than $\tilde{s}$. Thus, seller types in $[\tilde{s}, \overline{s}]$ do not actually participate in the auction market, and our analysis focuses on the signaling behavior of seller types lower than $\tilde{s}$.

price of $r(\hat{s})$. In this case, the seller's payoff is $r(\hat{s}) = v(\hat{s}, m(\hat{s}))$.

(iii) If $v(\hat{s}, X_{(2)}) > r(\hat{s}) = v(\hat{s}, m(\hat{s}))$, then at least two buyers submit bids greater than the reserve price and hence the good is sold to the buyer with the highest bid at the price of the second highest bid. In this case, the seller's payoff is $v(\hat{s}, X_{(2)})$.

Thus, if the type s seller is perceived to be type $\hat{s}$, she induces a minimum buyer type of m by announcing a reserve price $r = v(\hat{s}, m)$. Her expected payoff is then

$$U(s,\hat{s},m) = \xi(s)P_1 + v(\hat{s},m)P_2 + E_{X_{(2)}}\left[v(\hat{s}, X_{(2)})1_{\{X_{(2)}>m\}}\right] \qquad (2)$$

where P_1 and P_2 are the probabilities of the first two outcomes above and the last term is the expected second highest bid in the event that it is above the reserve price.

Since $v(\hat{s}, x)$ is strictly increasing in x, the first outcome occurs if $X_{(1)} \leq m$ hence $P_1 = F_{(1)}(m)$; the second outcome occurs if $X_{(2)} \leq m < X_{(1)}$ hence $P_2 = F_{(2)}(m) - F_{(1)}(m)$. Then the expected payoff for the seller with type s can be written as follows:

$$U(s,\hat{s},m) = \xi(s)F_{(1)}(m) + v(\hat{s},m)[F_{(2)}(m) - F_{(1)}(m)] + \int_m^{\bar{x}} v(\hat{s},x)dF_{(2)}(x) \qquad (3)$$

Differentiating (3), we have

$$\begin{aligned}\frac{\partial U}{\partial m} &= \xi(s)f_{(1)}(m) + v(\hat{s},m)[f_{(2)}(m) - f_{(1)}(m)] + \frac{\partial v}{\partial m}[F_{(2)}(m) - F_{(1)}(m)] - v(\hat{s},m)f_{(2)}(m) \\ &= (\xi(s) - v(\hat{s},m))f_{(1)}(m) + \frac{\partial v}{\partial m}[F_{(2)}(m) - F_{(1)}(m)] \qquad (4) \\ &= f_{(1)}(m)[\xi(s) - J(\hat{s},m)]\end{aligned}$$

$$\frac{\partial U}{\partial \hat{s}} = \frac{\partial v}{\partial \hat{s}}[F_{(2)}(m) - F_{(1)}(m)] + \int_{m}^{\bar{x}} \frac{\partial v}{\partial \hat{s}} dF_{(2)}(x) \tag{5}$$

Clearly, $\partial U / \partial m$ is increasing in s and $\partial U / \partial \hat{s}$ is independent of s. Thus, the slope of the indifference curve in the $m - \hat{s}$ plane is decreasing in s, i.e.,

$$\frac{d}{ds}\left[\left.\frac{d\hat{s}}{dm}\right|_{U=\bar{U}} \right] = \frac{d}{ds}\left[-\frac{\partial U / \partial m}{\partial U / \partial \hat{s}} \right] < 0 \tag{6}$$

Thus the single crossing condition holds, opening the possibility of signaling. To establish the existence of a signaling equilibrium we need to show that there is some increasing mapping $m(s)$ from seller's type to minimum buyer type and hence increasing reserve price function $r(s) = v(s, m(s))$, such that each type has an incentive to reveal its true type. That is $s = \arg \underset{\hat{s}}{Max}\, U(s, \hat{s}, m(\hat{s}))$, for all $s \in [\underline{s}, \bar{s}]$.

If s were directly observable to buyers, their perception $\hat{s} = s$, so the seller would choose the minimum buyer type to maximize $U(s, s, m)$. Let $m^*(s)$ be the optimal full information minimum type. Then, by Eq. (4) and Assumption (R), we have

$$m^*(s) = \begin{cases} \underline{x}, & \text{if } \xi(s) < J(s, \underline{x}); \\ J_s^{-1}\left(\xi(s)\right), & \text{if } J(s, \underline{x}) \le \xi(s) < J(s, \bar{x}); \\ \bar{x}, & \text{if } \xi(s) \ge J(s, \bar{x}); \end{cases} \tag{7}$$

where $J_s^{-1}(\cdot)$ is the inverse function of $J(s, \cdot)$.

3. Unique Separating Equilibrium

If $m(\cdot)$ characterizes a separating equilibrium, then (a) $U(s, s, m(s)) = \max_{\hat{s}} U(s, \hat{s}, m(\hat{s}))$, and (b) $m'(s) > 0$ for $s \in [\underline{s}, \bar{s}]$. Condition (a) implies

$$U_2(s,s,m(s)) + U_3(s,s,m(s)) \cdot m'(s) = 0 \tag{8}$$

To facilitate our later analysis, it is more convenient to work not with the minimum buyer type schedule $m(s)$, but with its inverse $s(m)$. We can then follow Riley [12] and rewrite (8) as follows:

$$\frac{ds}{dm} = -\frac{U_3(s,s,m)}{U_2(s,s,m)} \equiv b(m,s) \tag{9}$$

where by (4) and (5),

$$b(m,s) = \frac{f_{(1)}(m)\left[J(s,m) - \xi(s)\right]}{\frac{\partial v(s,m)}{\partial s}[F_{(2)}(m) - F_{(1)}(m)] + \int_m^{\bar{x}} \frac{\partial v(s,x)}{\partial s} dF_{(2)}(x)} \tag{10}$$

That is, given any separating equilibrium schedule, type s seller will optimally choose the minimum buyer type m according to the solution of (9).

Condition (9) says that the slope of the equilibrium schedule should equal the marginal rate of substitution between the minimum buyer type and the market perception about the type. To understand this, consider Figure 1.

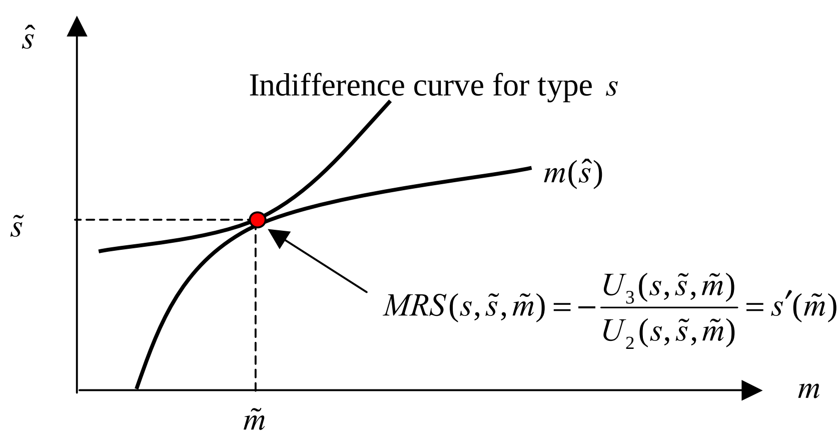

Figure 1: Separating Equilibrium

Buyers believe that the reserve price function $r(s)$ is strictly increasing. They therefore infer from a reserve price $\hat{r}$ that the seller's type is $\hat{s} = r^{-1}(\hat{r})$. Since $r(\hat{s}) = v(\hat{s}, m)$, this then determines the minimum buyer type that will enter $m(\hat{s})$.

Graphically, type s chooses a point $(\tilde{m}, \tilde{s})$ on the curve $m(\hat{s})$ to maximize her utility $U(s, \hat{s}, m)$. For incentive compatibility it must be the case that $\tilde{s} = s$. The indifference map for type s must therefore be tangential to the function $m(\hat{s})$ at $\tilde{s} = s$. Defining the inverse function $s(m) = m^{-1}(m)$ yields the first order condition

$$MRS(s, \tilde{s}, \tilde{m}) = -\frac{U_3(s, \tilde{s}, \tilde{m})}{U_2(s, \tilde{s}, \tilde{m})} = s'(\tilde{m}) \text{ at } \tilde{s} = s$$

We are now ready to state our characterization result.

Theorem 1: *The differential equation* (9) *with initial condition* $(\underline{s}, m^*(\underline{s}))$ *for the lowest type* $\underline{s}$ *characterizes the unique separating equilibrium.*

Proof: See the appendix.

Theorem 1 characterizes the unique separating equilibrium in terms of the minimum buyer type m for our general model.[8] The equilibrium reserve price function is then $r(s) = v(s, m(s))$.

Using the general characterization result in Theorem 1, the rest of the paper studies the properties of the equilibrium reserve schedule in more specific valuation

[8] Maskin and Tirole [9] study a very general model of informed principal problems from a mechanism design perspective. In our model, the seller can only choose within a restricted set of mechanisms (second price auctions with fixed reserve prices), thus their results do not apply here.

models. First we consider the case in which the bidders' signals X_i 's are independent. Under certain conditions regarding buyers' valuations, we can show that the equilibrium reserve price is increasing in the number of bidders n.

Theorem 2: *Suppose the bidders' signals X_i 's are independent. In the separating equilibrium, the minimum type that bids and hence the reserve price $r(s) = v(s, m(s))$ is higher for larger n for every $s > \underline{s}$ if (i) $v(s, x)$ is non-increasing in n, and (ii) $\partial v(s, x) / \partial s$ is non-decreasing in n.*

Proof: See the appendix.

Lemma 2 below shows that the conditions of Theorem 2 are easily satisfied by many specifications of buyers' valuations.

Lemma 2: *With independence of X_i 's, conditions (i) and (ii) in Theorem 2 are satisfied in the following cases:*

(a) $u_i(s, z) = \phi(s) + \psi(x_i)$ or $u_i(s, z) = \phi(s) \cdot \psi(x_i)$;

(b) $u_i(s, z) = \phi(s) + \frac{1}{n}[\alpha x_i + \beta \sum_{j \neq i} x_j]$, where $\alpha \geq \beta \geq 0$;

(c) $u_i(s, z) = \lambda \phi(s) + (1 - \lambda) \max(x_1, x_2, ..., x_n)$,

where ϕ and ψ are any positive and increasing functions and (s, z) is any signal profile.

Proof: See the appendix.

With affiliation, analysis of auctions such as the sealed high bid auction is much more complicated since a change in perceptions in general affects different types of buyer differently. However, under the assumption of independent private signals, the Revenue Equivalence Theorem applies. Hence we have the following Corollary.

Corollary 1: *If the buyers' private signals are independent, the optimal minimum type that bids* $m(s)$*, is the same in the sealed second price auction as in any other auctions that are revenue equivalent to the sealed second price auction.*

As is well known, in the literature of optimal reserve prices (Maskin and Laffont [8]) and optimal auctions with independent private valuations (Myerson [11], and Riley and Samuelson [13]), the optimal reserve price is invariant in the number of bidders. The seller's optimal reserve price is set to capture additional revenue when there is only one buyer who has a valuation much higher than her own, that is, when the item is sold at the reserve price. Thus, as the number of bidders becomes large, the probability that the item is sold at the reserve price converges to zero and hence the extra profit captured by setting a reserve price converges to zero as well.

Theorem 2 shows that when the signaling role is taken into account, the optimal reserve price increases in the number of bidders in many valuation models, including independent private valuations. Consequently, a reserve price plays a more central role than perceived by the traditional literature. One question naturally arises: can the probability that the item is sold at the reserve price increase as the number of bidders becomes larger?

Since the minimum type that bids is *m*, the probability of the highest type being below *m* is $F_{(1)}(m)$ and the probability of the second highest type being below *m* is $F_{(2)}(m)$. Thus, given independence, the probability that the item will be sold at the reserve price is

$$P_2 = F_{(2)}(m) - F_{(1)}(m) = nF^{n-1}(m)(1 - F(m)) \tag{11}$$

Note that at $s = \underline{s}$, the seller uses her optimal reserve price under complete information, $m^*(\underline{s})$, which is independent of n. In this case,

$$dP_2/dn = [1 + n\ln F(m^*(\underline{s}))]F^{n-1}(m^*(\underline{s}))[1 - F(m^*(\underline{s}))]$$

Clearly, since $F(m^*(\underline{s})) < 1$, P_2 is decreasing in n unless n is very small. Furthermore, $P_2 \to 0$ as $n \to \infty$. Of course, this is just the standard result of the optimal reserve price literature mentioned above.

For $s > \underline{s}$, under the conditions of Theorem 2, the minimum bidder type, $m(s)$, is increasing in n. Thus,

$$\begin{aligned} \frac{dP_2}{dn} &= \frac{\partial P_2}{\partial n} + \frac{\partial P_2}{\partial m}\frac{\partial m}{\partial n} \\ &= [1 + n\ln F(m)]F^{n-1}(m)[1 - F(m)] + n^2 f(m)F^{n-2}(m)[\frac{n-1}{n} - F(m)]\frac{\partial m}{\partial n} \end{aligned} \tag{12}$$

Note that $e^{-1/n} < (n-1)/n$ for $n \geq 3$. Clearly, the probability of a seller of type s having a binding reserve prices increases with the number of bidders if $e^{-1/n} \leq F(m(s)) \leq (n-1)/n$.

If these sufficient conditions do not hold, then the two terms on the RHS of (12) have the opposite signs, and hence whether P_2 is increasing or decreasing in n depends on their relative magnitudes. The net effect cannot be determined in this general formulation. In the next section we investigate this issue using a numerical solution of a linear valuation model.

4. A Linear Valuation Model

In this section we will work with a linear valuation model and solve for the equilibrium reserve price schedule analytically. Specifically, buyer i's valuation for the object is $u_i = s + x_i$, where s is a common value component only observed by the seller, and x_i is a private value component only observed by buyer i. We suppose that s and the x_i's are all independent. The seller's own valuation $u_0 = \gamma s$, $\gamma > 0$. This corresponds to the case in our general framework in which $v(s, x_i) = s + x_i$, and $\xi(s) = \gamma s$. If the equilibrium mapping from the seller's signal, s, to the minimum type that enters is $m(s)$, then the mapping from the seller's signal to the reserve price is $r(s) = s + m(s)$.

Assumption (R) is now equivalent to the following:

Assumption (R'**):** $J(x) = x - (1 - F(x)) / f(x)$ is strictly increasing in x.

This assumption holds as long as the hazard rate of X is increasing, which is satisfied for many common distributions including uniform, normal, and exponential.[9]

Following exactly the same analysis as in the previous section, we can show that for $0 < \gamma \le 1$, the inverse minimum bidder type schedule below completely characterizes the separating equilibrium.

$$s(m) = (1 - F_{(1)}(m))^{\gamma-1}\left[\int_{\underline{m}}^{m} f_{(1)}(t)(1 - F_{(1)}(t))^{-\gamma} J(t)dt + (1 - F_{(1)}(\underline{m}))^{1-\gamma}\underline{s}\right] \tag{13}$$

where $\underline{m}$ is the full information optimal minimum bidder type for the lowest seller type.

As an example, when (i) X is uniform with support [0,1]; (ii) $n = 2$; (iii) $\gamma = 1$; and (iv) $\underline{s} = 0$; we can integrate (13) analytically to obtain

$$s(m) = -4(m - \underline{m}) + 3\log\left(\frac{1+m}{1+\underline{m}}\right) - \log\left(\frac{1-m}{1-\underline{m}}\right)$$

where $\underline{m} = 1/2$.

When $\gamma > 1$, the equilibrium reserve price schedule may be truncated at some critical type, because the seller can be better off holding the item unsold as her own valuation becomes sufficiently large to exceed the equilibrium reserve price. With full information the seller is better off not selling if and only if his valuation γs exceeds the maximum buyer valuation $s + \overline{x}$. Thus the item is sold if and only if $s \le s^c$, where

$$s^c + \overline{x} = \gamma s^c . \tag{14}$$

We will show that with asymmetric information, the critical type is again s^c.

[9] It can be shown that if a distribution has an increasing hazard rate, then its truncated distribution also does.

Theorem 3: *Under Assumption* (R'), *Eq.* (13) *completely characterizes the solution for the unique separating equilibrium when* $0 < \gamma \le 1$, *or when* $\gamma > 1$ *and* $\bar{x} > (\gamma - 1)\bar{s}$. *When* $\gamma > 1$ *and* $\bar{x} \le (\gamma - 1)\bar{s}$, *the equilibrium schedule determined by* (13) *is truncated at* $(m^c = \bar{x}, s^c = \bar{x}/(\gamma - 1))$; *those types of* $s \in [\bar{x}/(\gamma - 1), \bar{s}]$ *will withdraw from the market.*

Proof: See the appendix.

Theorem 3 thus gives a complete characterization of the unique separating equilibrium for the linear valuation model. Jullien and Mariotti [6] study a reserve price signaling model in which the seller's valuation is $\tilde{\theta}$ and the buyers' valuations are $\lambda\tilde{\theta} + (1-\lambda)\tilde{\varepsilon}_i$, where $\lambda \in [0,1]$. Their setup thus corresponds to the case of $\gamma \ge 1$ in our linear valuation model.[10]

Using the equilibrium characterization of Theorem 3, we now return to the issue of whether the probability of selling at the reserve price can be increasing in the number of bidders. For computational simplicity, we consider the case $X \sim U[0,1]$, $\gamma = 1$, and $\underline{s} = 0$. For any given s and n, we first numerically solve for the reserve markup based on (13),[11] then compute the probability that the item will be sold at the reserve price according to (11). The results are reported in Table I.

[10] The equilibrium characterization of Theorem 3 is very useful for studying applications of the linear valuation model. In Cai, Riley and Ye [1], we illustrate this with three simple applications, the first to outside certification, the second to an analysis of relative importance of private values, and the last to a Lemons market analysis.

[11] The algorithm is straightforward as $m(s)$ is strictly increasing in s.

Table I: Probabilities of the event that the item is sold at the reserve price

n	s=0	s=0.1	s=0.2	s=0.3	s=0.4	s=0.5	s=0.6	s=0.7	s=0.8	s=0.9	s=1.0
2	**.5000**	.4103	.3506	.3041	.2661	.2344	.2073	.1840	.1638	.1461	.1305
3	**.3750**	**.4308**	.3912	.3514	.3150	.2823	.2531	.2271	.2040	.1833	.1648
4	**.2500**	**.4219**	.4042	.3740	.3419	.3108	.2818	.2552	.2308	.2087	.1886
5	**.1563**	**.4042**	**.4053**	.3843	.3571	.3286	.3006	.2741	.2494	.2266	.2057
6	**.0938**	**.3855**	**.4016**	.3886	.3659	.3399	.3133	.2874	.2628	.2397	.2182
7	**.0547**	**.3685**	**.3962**	**.3898**	.3711	.3475	.3223	.2970	.2726	.2495	.2278
8	**.0313**	**.3538**	**.3904**	**.3895**	.3742	.3527	.3288	.3042	.2801	.2570	.2353
9	**.0176**	**.3413**	**.3849**	**.3885**	.3760	.3564	.3336	.3097	.2860	.2630	.2412
10	**.0098**	**.3307**	**.3798**	**.3871**	.3771	.3591	.3373	.3141	.2907	.2678	.2460
13	**.0016**	**.3073**	**.3673**	**.3824**	**.3780**	.3636	.3444	.3227	.3002	.2779	.2561
53	**.0000**	**.2442**	**.3256**	**.3601**	**.3710**	**.3682**	.3571	.3415	.3228	.3032	.2832
10,000	**.0000**	**.2177**	**.3106**	**.3453**	**.3662**	**.3679**	.3595	.3476	.3293	.3033	.3033

Note: The formula used in calculating the probabilities are $s(m) = n\int_{0.5}^{m} \frac{(2t-1)t^n}{1-t^n}dt$, and $P_2(m) = F_{(2)}(m) - F_{(1)}(m) = n(1-m)m^{n-1}$.

In each column, the probability of a sale at the reserve price is shown in boldface if the term below is smaller. Consider the first column where the seller's signal is the lowest ($s = \underline{s} = 0$). The probability that the item is sold at the reserve price decreases in the number of bidders and converges (quite rapidly) to zero as n becomes large. This is the standard result for optimal reserve prices since the lowest type has no incentive to signal. However, for any $s \in \{0.1, 0.2, ..., 1\}$, there exists a threshold of n^* such that for $n \leq n^*$, the probability of a sale at the reserve price is *increasing* in n.[12] Moreover, such a threshold n^* appears to be increasing in s : n^* is 2, 5,7,13, 53 for $s =$0.1, 0.2, 0.3, 0.4, 0.5 respectively (results for $n > 10$ are not reported to save space, except for the

[12] For $n > n^*$, our numerical results suggest that the probability of binding reserve price tends to decrease, but not always monotonically.

threshold values 13 and 53). For s greater than 0.5, our program does not find a threshold within $n \leq 100$. Therefore, these results indicate that for relatively large s, the probability of a sale at the reserve price is increasing in the number of bidders in a broad range of auction sizes.[13] Also note that as the separating equilibrium is independent of the seller's type distribution, this result also holds in an ex-ante sense if the distribution is sufficiently concentrated at the top.

Table I reveals another interesting result. For very large n (as an example, we report $n = 10,000$ in the last row of Table I), the probability of a sale at the reserve price is substantially above zero and quite large in magnitude for all s except $\underline{s} = 0$. This is in sharp contrast with the traditional case when reserve price does not serve a signaling role. This contrast is stark even for relatively small n. For example, consider $n = 10$ in Table I, item is sold at the reserve price with a probability of less than 1% when it does not have a signaling role ($\underline{s} = 0$), but is above 25% for all other s when it is also used to signal the seller's type.

In summary, these results indicate that the reserve price plays a much more central role in auctions when its signaling role is taken into account than in the traditional auction literature.

5. Conclusion

In this paper we study an auction model in which the seller has private information about the object's characteristics that are valued by both the seller and the buyers. In a fairly general framework in which buyers' signals are affiliated, we identify conditions

[13] In practice auctions with more than 50 bidders are usually considered large (and rare).

under which reserve prices can serve as an effective signaling instrument to reveal the seller's private information. Under such conditions we characterize the unique separating equilibrium with reserve price signaling. When the buyers' signals are independent, it is shown that the optimal reserve price is increasing in the number of bidders under certain conditions about the valuations. This is in contrast with the optimal auction literature with independent private values. Our analysis thus suggests a more central role for reserve prices than perceived by the standard auction model.

In our analysis, we assume that the seller's own value (u_0) and the "common value component" which affects all the buyers' valuations (s) are perfectly correlated, that is, $u_0 = \xi(s)$. Our model can be extended to a more general setting that allows for positive but not necessarily perfect correlation between the seller's expected valuation and a common value component for the buyers' valuations. Such a setting is very natural when the object's characteristics are multi-dimensional and the seller and the bidders may place different weights on the relative importance of different dimensions. For example, a seller of an artwork may be mostly concerned with the artwork's secondary market value, while potential buyers (who buy for self consumption) may care more about its impact in the setting for which it is intended. In Cai, Riley and Ye [1], we show that our analysis of the basic model carries through to this more general setting almost unchanged, thus giving our results a greater applicability.[14]

[14] Another possible extension is to allow for positive correlation between the seller's private information and the buyers' private signals. However, it is unclear under what conditions the single crossing property will hold. Thus, this extension presents technical difficulties not easy to overcome. See Cai, Riley and Ye [1] for more detailed discussion.

Following the signaling literature, we appeal to standard refinements and focus on the unique separating equilibrium of the signaling game in which the lowest type seller sets the reserve price that is optimal under complete information (the “Riley outcome”). When the object’s characteristics are multi-dimensional, it is very natural that the seller’s valuation and the buyers’ common value component are positively, but not perfectly correlated. In that case, standard equilibrium refinements such as the Cho-Kreps Intuitive Criterion (Cho and Kreps [3]) no longer reduce the number of equilibria. In a separate paper (Cai, Riley and Ye [2]), we argue that in a general signaling model for which our reserve price signaling model is a special case, it is sensible to consider a refinement criterion which we call the “Local Credibility Test .” This is somewhat stronger than Cho and Kreps Intuitive criterion but weaker than the “strong Intuitive Criterion” of Grossman and Perry [4,5]. There we explore necessary and sufficient conditions under which the separating equilibrium survives the Local Credibility Test .

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Appendix

Proof of Lemma 1: Let $f(\cdot)$ be the density function of x_i for all i. Let $h(x) = f(x)/(1-F(x))$ denote the hazard rate function of X. Suppose it is increasing (the regular distribution case). Then, by independence,

$$\frac{F_{(2)}(x) - F_{(1)}(x)}{f_{(1)}(x)} = \frac{1-F(x)}{f(x)}$$

Differentiating $J(s,x)$ with respect to x we have

$$\frac{\partial J}{\partial x} = \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2}\frac{1}{h(x)} - \frac{\partial v}{\partial x}\frac{\partial}{\partial x}\left(\frac{1}{h(x)}\right)$$

The first and third terms above are positive by assumptions. The middle term is also positive as long as $\dfrac{\partial^2 v}{\partial x^2}$ is non-positive. ***Q.E.D.***

Proof of Theorem 1: Consider the function $b(m,s)$ (See Eq. (10)). By Assumption (R), for each s either $b(m,s) < 0$ or $b(m,s) = 0$ at a unique point in $[\underline{x}, \overline{x}]$ and $b(m,s) > 0$ only for larger m. Therefore, following arguments paralleling those in Riley [12], there is a unique solution going through $(\underline{s}, m^*(\underline{s}))$, the full information optimum for the lowest seller type, which is strictly increasing. We next show that this unique solution is incentive compatible, that is, no type s wants to deviate from $m(s)$ on the signaling schedule, hence constitutes the unique separating equilibrium.

Suppose that the buyers' perception is given by $\hat{s} = s(m)$, which is the solution to (9). A seller of type s thus chooses m to maximize $U(s, s(m), m)$. Differentiating with respect to *m,*

$$\frac{d}{dm}U(s,s(m),m) = U_2(s,s(m),m)s'(m) + U_3(s,s(m),m)$$
$$= U_2(s,s(m),m)\left[s'(m) + \frac{U_3(s,s(m),m)}{U_2(s,s(m),m)}\right]$$
$$= U_2(s,s(m),m)\left[-\frac{U_3(s(m),s(m),m)}{U_2(s(m),s(m),m)} + \frac{U_3(s,s(m),m)}{U_2(s,s(m),m)}\right]$$

By the single crossing condition (6), the terms in the bracket above only changes signs once and $U(s,s(m),m)$ takes on its maximum at m where $s(m)=s$. Therefore, incentive compatibility is satisfied. ***Q.E.D.***

Proof of Theorem 2: Since X_i 's are independent, $\frac{F_{(2)}(x)-F_{(1)}(x)}{f_{(1)}(x)} = \frac{1-F(x)}{f(x)}$.

Therefore, $J(s,x)$ is non-increasing in n as $v(s,x)$ is non-increasing in n and $\partial v(s,x)/\partial s$ is non-decreasing in n.

Define

$$\kappa(m,s;n) =: \frac{\partial m}{\partial s} = \frac{\frac{\partial v(s,m)}{\partial s}[F_{(2)}(m)-F_{(1)}(m)] + \int_m^{\bar{x}} \frac{\partial v(s,x)}{\partial s}dF_{(2)}(x)}{f_{(1)}(m)\;[J(s,m)-\xi(s)]}$$
$$= \frac{\frac{\partial v(s,m)}{\partial s}}{[J(s,m)-\xi(s)]}\frac{1-F(m)}{f(m)} + \frac{\int_m^{\bar{x}} \frac{\partial v(s,x)}{\partial s}dF_{(2)}(x)}{f_{(1)}(m)\;[J(s,m)-\xi(s)]}$$

We want to show that $\kappa(m,s;n)$ is strictly increasing in n. Since $J(s,m)$ is non-increasing in n and $\partial v(s,x)/\partial s$ is non-decreasing in n, the first term above is non-decreasing in n. So it suffices to show that

$$\rho(s,m;n) = \frac{\int_m^{\bar{x}} \frac{\partial v(s,x)}{\partial s}dF_{(2)}(x)}{f_{(1)}(m)} = \frac{(n-1)\int_m^{\bar{x}} \frac{\partial v(s,x)}{\partial s}F^{n-2}(x)(1-F(x))dF(x)}{F^{n-1}(m)f(m)}$$

is strictly increasing in n. Taking logarithms and then differentiating with respect to n, we have

$$\frac{\partial \log \rho}{\partial n} = \frac{1}{n-1} - \log F(m) + \frac{\int_m^{\bar{x}} \frac{\partial v}{\partial s} F^{n-2}(x)(1-F(x)) \log F(x) dF(x) + \int_m^{\bar{x}} \frac{\partial^2 v}{\partial s \partial n} F^{n-2}(x)(1-F(x)) dF(x)}{\int_m^{\bar{x}} \frac{\partial v}{\partial s} F^{n-2}(x)(1-F(x)) dF(x)}$$

$$\geq \frac{1}{n-1} - \log F(m) + \frac{\int_m^{\bar{x}} \frac{\partial v}{\partial s} F^{n-2}(x)(1-F(x)) \log F(x) dF(x)}{\int_m^{\bar{x}} \frac{\partial v}{\partial s} F^{n-2}(x)(1-F(x)) dF(x)}$$

$$\geq \frac{1}{n-1} - \log F(m) + \frac{\int_m^{\bar{x}} \frac{\partial v}{\partial s} F^{n-2}(x)(1-F(x)) \log F(m) dF(x)}{\int_m^{\bar{x}} \frac{\partial v}{\partial s} F^{n-2}(x)(1-F(x)) dF(x)}$$

$$= \frac{1}{n-1} - \log F(m) + \log F(m)$$

$$= \frac{1}{n-1} > 0$$

Therefore, $\partial \kappa(s, m; n)/\partial n > 0$, which implies that

$$\frac{\partial}{\partial n}\left(\frac{\partial m}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial m}{\partial n}\right) > 0 \tag{15}$$

Now we consider the sign of $\partial m^*(\underline{s})/\partial n$. By (7), if $m^*(\underline{s})$ takes a corner solution, it is independent of n; when it takes an interior solution, we have $\xi(\underline{s}) = J(\underline{s}, m^*(\underline{s}))$.

Differentiating with respect to n on both sides, we have

$$0 = \frac{\partial J(\underline{s}, m^*(\underline{s}))}{\partial n} + \frac{\partial J(\underline{s}, m^*(\underline{s}))}{\partial m} \frac{\partial m^*(\underline{s})}{\partial n}$$

Since $J(s,m)$ is non-increasing in n and strictly increasing in m, $m(\underline{s}) = m^*(\underline{s})$ must be non-decreasing in n. Therefore, by (15) we have $\frac{\partial m(s)}{\partial n} > \frac{\partial m(\underline{s})}{\partial n} \geq 0$ for all $s > \underline{s}$.

Q.E.D.

Proof of Lemma 2: Recall that $v_i(s,x) = E[V_i \mid S = s, X_i = X_{(1)}^{-i} = x]$. It is easily verified, corresponding to the three cases, that

(a) $v_i(s,x) = \phi(s) + \psi(x)$ or $v_i(s,z) = \phi(s) \cdot \psi(x)$;

(b) $v_i(s,x) = \phi(s) + \frac{\alpha+\beta}{n} x + \frac{n-2}{n} \beta E(X \mid X \leq x)$;

(c) $v_i(s,x) = \lambda\phi(s) + (1-\lambda)x$.

Clearly, $\partial v(s,x)/\partial s$ is independent of n in all three cases. Also, $v(s,x)$ is independent of n for cases (a) and (c). For case (b), note that

$$v_i(s,x) = \phi(s) + E(X \mid X \leq x) + \frac{1}{n}[(\alpha+\beta)x - 2\beta E(X \mid X \leq x)]$$

Since $\alpha \geq \beta \geq 0$ and $x \geq E(X \mid X \leq x)$, $v_i(s,x)$ is decreasing in n. ***Q.E.D.***

Proof of Theorem 3:

Since $v(\hat{s},m) = \hat{s} + m$, following exactly the same analysis as in the previous section, we have

$$\begin{aligned} U(s,\hat{s},m) &= \xi(s)F_{(1)}(m) + (\hat{s}+m)[F_{(2)}(m) - F_{(1)}(m)] + \int_m^{\bar{x}} (\hat{s}+x)\,dF_{(2)}(x) \\ U_2(s,\hat{s},m) &= 1 - F_{(1)}(m) \\ U_3(s,\hat{s},m) &= (\gamma s - \hat{s} - J(m)) f_{(1)}(m) \end{aligned} \qquad (16)$$

Given Assumption (R′), the single crossing condition is obviously satisfied.

If s is directly observable to buyers, their perception $\hat{s} = s$, then the seller will choose the minimum bidder type m to maximize $U(s,s,m)$. Let $m^*(s)$ be the optimal full information minimum bidder type, then by Assumption (R′) and Eq. (7), we have

$$m^*(s) = \begin{cases} \underline{x}, & \text{if } (\gamma-1)s < J(\underline{x}), \\ J^{-1}\left((\gamma-1)s\right), & \text{if } J(\underline{x}) \le (\gamma-1)s < J(\overline{x}) = \overline{x}, \\ \overline{x}, & \text{if } (\gamma-1)s \ge \overline{x}. \end{cases} \tag{17}$$

As argued in the previous section, the inverse minimum bidder type schedule $s(m)$ in the separating equilibrium must satisfy the following differential equation:

$$s'(m) = -\frac{U_3(s,s,m)}{U_2(s,s,m)} = \frac{(J(m)-(\gamma-1)s)f_{(1)}(m)}{1-F_{(1)}(m)} \tag{18}$$

Assuming an interior solution for the full information optimum, (18) can be rewritten as

$$s'(m) = \frac{(J(m)-J(m^*(s))f_{(1)}(m)}{1-F_{(1)}(m)}$$

This implies that $m(s) > m^*(s)$ for all $s > \underline{s}$.

Eq. (18) can be rewritten as:

$$(1-F_{(1)}(m))\frac{ds}{dm} + (\gamma-1)f_{(1)}(m)s = f_{(1)}(m)J(m)$$

Multiplying both sides by $(1-F_{(1)}(m))^{-\gamma}$, we have

$$\frac{d}{dm}[(1-F_{(1)}(m))^{1-\gamma}s(m)] = f_{(1)}(m)(1-F_{(1)}(m))^{-\gamma}J(m)$$

Integrating we obtain:

$$(1-F_{(1)}(m))^{1-\gamma}s(m)-(1-F_{(1)}(\underline{m}))^{1-\gamma}\underline{s}=\int_{\underline{m}}^{m}f_{(1)}(t)(1-F_{(1)}(t))^{-\gamma}J(t)dt$$

which gives rise to (13).

We thus have proved the case for $0<\gamma\leq 1$ since there is no truncation. When $\gamma>1$ and $\overline{x}>(\gamma-1)\overline{s}$, clearly the constraint of (14) does not bind since $(\gamma-1)s\leq(\gamma-1)\overline{s}<\overline{x}$. It thus remains to show the case in which $\gamma>1$ and $\overline{x}\leq(\gamma-1)\overline{s}$.

In this case, the constraint of (14) implies that there exists a truncated type $s^c=\overline{x}/(\gamma-1)\in[\underline{s},\overline{s}]$ so that $m^c=m(s^c)=\overline{x}$.

It remains to verify that this new endpoint condition is also implied in (13). First, as $m\to\overline{x}$,

$$(1-F_{(1)}(m))^{\gamma-1}\left[(1-F_{(1)}(\underline{m}))^{1-\gamma}\underline{s}\right]\to 0$$

Second, applying L'Hopital's rule, we have

$$\begin{aligned}
&\lim_{m\to\overline{x}}(1-F_{(1)}(m))^{\gamma-1}\int_{\underline{m}}^{m}f_{(1)}(t)(1-F_{(1)}(t))^{-\gamma}J(t)dt\\
&=\lim_{m\to\overline{x}}\frac{\int_{\underline{m}}^{m}f_{(1)}(t)(1-F_{(1)}(t))^{-\gamma}J(t)dt}{(1-F_{(1)}(m))^{1-\gamma}}\\
&=\lim_{m\to\overline{x}}\frac{f_{(1)}(m)(1-F_{(1)}(m))^{-\gamma}J(m)}{(1-\gamma)(1-F_{(1)}(m))^{-\gamma}(-f_{(1)}(m))}\\
&=\lim_{m\to\overline{x}}\frac{J(m)}{\gamma-1}\\
&=\frac{\overline{x}}{\gamma-1}
\end{aligned}$$

Therefore, taking limits on both sides of Eq. (13), we have $s(\overline{x})=\lim_{m\to\overline{x}}s(m)=s^c$

$= \bar{x}/(\gamma - 1)$, which confirms that for $\gamma > 1$, the separating equilibrium is given by (13) with a truncation at the second endpoint $(m^c = \bar{x}, s^c = \bar{x}/(\gamma - 1))$.

Q.E.D.