# Competitive Nonlinear Income Taxation Revisited*

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**Abstract**

In the current literature on competitive nonlinear income taxation, competition is usually modeled as a game in which different tax authorities compete in tax schedules. An undesirable feature of this traditional approach is that the resource constraint is required only in equilibrium: following a deviation by one state, the resource constraints in other competing states are typically unbalanced. We propose a new approach in which the tax authorities compete in marginal tax rates, with the poll subsidies adjusting to satisfy budget balance. We show that our new approach in general leads to an equilibrium outcome different from the traditional approach. Under certain regular conditions we demonstrate that the new approach leads to increased competition, reducing the amount of income redistribution from high-income to low-income workers.

## 1 Introduction

One of the important extensions of the original Mirrlees model of optimal income taxation has been to include migration responses to changes in the tax schedule. Indeed, Mirrlees provided one of the early contributions.[1] This literature initially focused primarily on the optimal income tax for a single country, treating as exogenous the tax schedules in other countries. More recent literature has begun to examine the tax competition problem between governments, in which regions (countries or states) independently choose their income tax schedules, taking into account the resulting migration of workers. Regions effectively compete for high-skilled residents, who provide the tax revenue that can be redistributed to

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1 See Mirrlees (1982). Wilson (2009) reviews the literature on optimal income taxation and migration, and Wilson (2013) discusses more recent contributions.

lower-skilled residents. In the absence of moving costs, this competition becomes severe enough to eliminate tax payments by the highest-skilled individuals. In fact, Bierbrauer, Brett, and Weymark (2013) show that, without moving costs, competition for the most highly-skilled individuals becomes so intense that no region taxes them in equilibrium. There is also a race-to-the-bottom result for this model, where competition to prevent low-skilled workers from moving to the region results in the subsidies they receive going to zero. But with moving costs, redistribution becomes possible, and the equilibrium nonlinear income taxes in this case have now been extensively studied. See, in particular, Lipatov and Weichenrieder (2012) for competition between two regions for two types of workers, distinguished by productivity, and Morelli, Yang, and Ye (2012) for competition for three types and continuous types.[2] A main message from these studies is that competition leads to less income redistribution and there is also too little redistribution from the viewpoint of the system of regions as a whole. On the other hand, Gordon and Cullen (2012) show that if income taxes are chosen by both the central government and lower-level governments in a federal system, then the lower-level governments may engage in excess income redistribution, if the federal government is not optimizing its tax system. The problem stems from the vertical externalities created by the use of an income tax at the federal level. A region essentially ignores the fact its redistribution activities will impact other regions through required changes in the federal tax schedule. More recently, Lehmann, Simula, and Trannoy (2014) work with a continuous type model allowing for correlations between worker productivity types and moving costs, and extend the well known Diamond-Saez formula to the competitive income taxation setting. They also demonstrate that the shape of the optimal marginal income tax schedule depends on the semi-elasticity of migration, defined as the percentage change in the number of residents of a particular type caused by a dollar increase in their after-tax income. In particular, the shape of the schedule is sensitive to how this semi-elasticity varies across individuals with different incomes.

Although there has been considerable progress in our understanding of competitive income taxation, this literature suffers from a fundamental shortcoming: a simultaneous-move Nash game for the competing regions is not properly modeled. More specifically, it is typically assumed that each region maximizes its welfare, given the other region's entire tax policy and exogenous public expenditure requirements. In a rare discussion of the shortcoming inherent in this approach, Piaser (2007) explains, "... a government does not anticipate that after a deviation from equilibrium the policy of the other government could not be sustainable. The budget constraint depends on the proportions of both kinds of workers: after a migration from one country to the other induced by a change in the fiscal policy of one country, the other country's budget constraint is not balanced anymore." Stated differently, it cannot be true that when one region changes its tax policy, there will be no change in the other regions' tax policies or public expenditures, because the migration resulting from this change will throw the other regions' government budgets out of balance. A region should recognize the balanced-budget requirement for other regions and take

[2] The analysis of continuous types was published as an online appendix.

into account how other regions' policies adjust.

Piaser (2007) gives two justifications for taking the usual approach: "First, there is no consensus on an alternative definition of equilibrium. Second, we want to keep the Nash equilibrium concept to be consistent with the usual assumption made in the economic literature." But in traditional models of competition for capital, there is a usual way of modeling tax competition as a simultaneous-move Nash game: tax rates are the strategies and public good levels adjust to keep the government budgets satisfied. For competition in income tax schedules, we could introduce public expenditures into the model. Governments would then play a Nash game with tax schedules as strategies, or, as often assumed in this literature, they would play a Nash game in direct mechanisms, which specify consumption and before-tax income for each worker, subject to the constraint that there exists a tax schedule that supports this choice. If one government deviated from its equilibrium strategy, it would recognize that changing its tax schedule causes public good levels to change in other regions, given their tax policies, and these changes would affect migration responses. We discuss this approach in our concluding remarks, but our current paper follows much of the optimal income tax literature by assuming that governments choose their optimal income tax policy, given exogenous public expenditures.

The problem resembles the well-known debate over Recardian approach vs. Non-Recardian approach among macroeconomists. In part due to central banks' tendency to choose an interest rate as the instrument of monetary policy, fiscal policy is thought to play a more fundamental role in price determination and control, which gives rise to the influential fiscal theory of the price level (FTPL, Leeper, 1991; Woodford, 1994, 1995; Sims, 1994; and Cochrane, 1998). While the FTPL offers a solution to the well-known price determinacy puzzles, it is non-Ridardian as it only requires that the consolidated government present value budget constraint hold in equilibrium. In part because of this, it has been controversial and widely criticized (e.g., Buiter, 2002; Bassetto, 2002, 2005; and Niepelt, 2004). For example, Buiter (2002) argues that the government present value budget constraint is a real constraint on government behavior, both in equilibrium and along off equilibrium paths. The government must obey its budget constraint just like households, and equilibria that suggest otherwise are invalid.

Our approach consists of dividing the tax schedule for each of two identical "states", $i = 1, 2$, into a poll subsidy, $A_i$, which every resident in state $i$ receives, and a tax function, $T_i(Q)$, where $Q$ is before-tax income and $T_i(0) = 0$. The tax function defines the schedule of marginal tax rates, $T_i'(Q)$ at income $Q$, and the state's income tax policy is completely described by the poll subsidy and the marginal tax function. We utilize the common assumption of quasilinear utility functions, in which case labor supplies for a state's residents are independent of the poll subsidy. We may then model competition within the federation as a Nash game between states, in which each state optimizes by choosing its marginal tax function, given the other state's marginal tax function, and the poll subsidies adjustment to satisfy each state's government budget constraint, taking into account the tax-induced migration that occurs between states. The pattern of migration across individuals with different labor productivities depends on how the distribution of moving costs varies across worker abilities. The distribution of these costs determines how a tax change in one state affects the other state's budget-balancing poll subsidy. In contrast, the traditional approach to modeling competitive taxation would be to treat both the marginal tax function and poll subsidies combined as strategies in a Nash game, thereby ignoring budget balance.

We identify conditions under which the traditional approach and our new approach yield identical symmetric equilibria, but these assumptions are quite restrictive. For example, if the income tax is utilized only to redistribute income, then there must be a zero correlation across workers in their tax payments and the semi-elasticity of migration at equilibrium. The basic argument behind this result is that a small rise in state $j$'s poll subsidy from its equilibrium level has no impact on $i$'s tax revenue under these assumptions because the resulting migration from state $i$ to state $j$ consists of individuals with the same distribution of skills as those individuals currently residing in each state; the net tax revenue obtained from these individuals equals zero. Thus, whether or not state $i$ treats as fixed state $j$'s poll subsidy is irrelevant for the choice of its tax policy; any migration resulting from a change in $j$'s poll subsidy has no effect on $i$'s budget constraint.

It is usually the case, however, that our new approach yields equilibria that differ from those under the traditional approach. Suppose again that the semi-elasticities of migration (or roughly speaking, the migration propensities) are identical across individuals with different labor productivities, but each government must finance a given public expenditure level, i.e., net tax revenue is positive. Starting from a symmetric equilibrium, suppose now that state $i$ raises the marginal tax rate by a small amount at some high income level, $Q'$, where all individuals with equal or higher incomes are net taxpayers; that is, their tax payments are positive, calculated net of the poll subsidy they receive. With tax payments now increased for all individuals at $Q'$ and above, some of these individuals now migrate from state $i$ to state $j$, causing a rise in $j$'s net tax revenue. As a result, $j$'s budget-balancing poll subsidy rises, which makes $j$ even more attractive to current residents of $i$, inducing yet more migration to $j$. This latter migration occurs across the entire range of incomes, since all residents receive the same poll subsidy, and it causes a further decline in $i$'s tax revenue under our assumptions. This revenue loss reduces $i$'s potential welfare gain from a rise in its tax rate, suggesting that the equilibrium tax systems will be less progressive under our new approach than under the traditional approach, where government balance is ignored. In fact, we show that this is the case, and we generalize our results to environments that imply a reasonable equilibrium property, which is a counterpart of increasing average tax rates in the close economy. More specifically, we show that there exists a cutoff income $Q^*$ (or cutoff productivity type $\theta^*$) so that workers with income above $Q^*$ pay less taxes while workers with incomes below $Q^*$ pay more taxes (or receive less subsidies) under the new approach. Thus, our basic message is that tax competition leads to less redistribution under our new approach than under the traditional approach.

The plan of the paper is as follows. The model is described in the next section, and then Section 3 presents the standard Diamond-Saez formula for optimal marginal tax rates in a closed economy. Section 4 then derives the optimal tax formula when there is competition for residents between states, modeled using the traditional approach. Section 5 derives the formula under the new approach. In Section 6, we compare the solutions under the traditional and new approaches, demonstrating that the new approach leads to less redistribution under reasonable assumptions. Section 7 concludes.

## 2 The Model

We consider two states in a potential federation, indexed by $i = 1,2$. This is the minimal situation in which we can compare the progressivity of competitive state taxation versus that of a unified federal tax. The federation has a total measure (population) of 2 citizens (or workers/consumers), which are evenly split between two states (so each state has a total measure of 1 original citizens attached to it). The state that a citizen is initially attached to is called her home state. By incurring a moving/migration cost, each citizen may move from her home state to the other state. Each citizen is characterized by three characteristics: her native state $i \in \{1,2\}$, her productivity (or skill) $\theta \in [0,+\infty)$, and the moving cost $z \in [0,+\infty)$ that she has to incur if she moves from her home state to the other state.[3] The moving cost $z$ captures various material and psychic costs of moving, such as the costs in adapting to different culture, landscape, food, political system, weather condition, etc.

We assume that consumers' productivity types ($\theta$'s) (in both states) are independently distributed according to a distribution function $F(\cdot)$ over $[0,\infty)$ (with strictly positive density function $f(\cdot)$ over its support). As in Lehmann et al. (2014), we allow correlations between skills and moving costs: for each skill $\theta$, the moving costs ($z$'s) are independently distributed according to a distribution function $H(\cdot|\theta)$ on the interval $[0,+\infty)$ (with density function $h(\cdot|\theta)$). The initial joint density of $(\theta, z)$ is thus given by $h(z|\theta)f(\theta)$.

Neither the ability $\theta$ nor the moving cost $z$ is observable to the tax authority. The tax authority (the federation or each state in our model) is also constrained to treat native and immigrant workers in the same way. Therefore, the tax authority can only condition transfers on pre-tax income $Q$ through a tax schedule $T(\cdot)$ – it cannot base the tax on an individual's skill type $\theta$, moving cost $z$, or native state.

Given consumption (or after-tax income) $c$ and labor supply $l$, following recent literature (e.g., Gordon and Cullen, 2012, and Lehmann et al., 2014) we assume that a consumer's preference can be represented by the following quasi-linear utility function:[4]

$$U(c,l) = c - v(l). \tag{1}$$

where $v$ is increasing and convex. We assume that an individual of ability $\theta$ has a constant-returns-to-

[3] Those who are immobile are captured by $z = +\infty$.

[4] The quasilinear preference is a good approximation given some recent empirical findings (e.g., Gruber and Saez, 2002).

scale production function so that

$$Q = \theta l.$$

The productivity of this individual, $\theta$, equals her hourly wage in a competitive labor market. In an autarkic economy where every agent consumes the product of her own labor, her labor supply is given by

$$v'(l) = \theta.$$

In an economy with taxation, $c = Q - T(Q)$, where $T(\cdot)$ is the tax schedule set by the tax authority.

Suppose in the equilibrium induced by a tax policy $T(\cdot)$, a type-$\theta$ citizen chooses labor supply $l(\theta)$, and hence pre-tax income $Q(\theta) = \theta l(\theta)$ and consumption $c(\theta) = \theta l(\theta) - T(\theta l(\theta))$. We will consider the "weighted utilitarian" criterion so that the tax authority (the federation or each state) maximizes the following weighted social welfare:

$$\int_0^\infty [c(\theta) - v(l(\theta))] g(\theta) d\theta, \tag{2}$$

where $g$ is a social weighting function, which is also a probability density, that typically differs from $f$ in so far as the tax authority has redistributive objectives. The tax authority puts a higher weight on lower $\theta$'s, so that $G \geq F$, the cumulative distribution function of $g$ first-order stochastically dominates that of $f$. Note that when all the weights are given to the lowest type, i.e., when $G(0) = 1$, our objective is reduced to maximin or Rawlsian, which is the criterion adopted by Lehmann et al. (2014). Our model hence nests theirs as an important special case.

We will consider two taxation regimes. The first is unified taxation, in which the tax policies in both states are chosen by the federation. The second is the independent (or competitive) taxation, in which each state decides on its own tax policy. In the unified taxation regime, the federation aims to maximize social welfare given by (2) among all the citizens residing in the federation. In the competitive taxation setting, when making policy choices, each state takes as given the policies chosen by the other state and maximizes (2) for the citizens who reside in its own state. As described in the introduction, there is an issue regarding the solution concept. In the traditional approach, states compete in tax schedules, but budget balance is required only in equilibrium. Under our new approach, states compete in marginal tax rates, $T_i'(\cdot)$, $i = 1, 2$. A pair of these schedules, $\{T_1'(\cdot), T_2'(\cdot)\}$, constitutes a Nash equilibrium in the competitive taxation setting if given $T_{-i}'(\cdot)$, $T_i'(\cdot)$ maximizes the weighted welfare for the citizens residing in state $i$, with resource constraints (in both states) being maintained all the time (both on equilibrium and off equilibrium paths) via poll subsidy adjustment processes, which will be made clear below.

Denoting net tax payments as a function of earnings as $T(Q(\theta)) - A$ (where $T(0) = 0$ and $A$ is a poll subsidy that everyone receives). Note that given $T(0) = 0$, $T(Q(\theta))$ and $T'(Q(\theta))$ are uniquely determined by each other. So while the traditional approach considers the tax schedule $T(Q(\theta)) - A$ as the strategic choice variable, the new approach considers the tax function $T(Q(\theta))$ or the marginal tax rate $T'(Q(\theta))$ as strategic choice variable.

## 3 Unified Taxation

Since the two states are identical in terms of the original composition of the population, we focus on the symmetric solution in which each state offers the same tax policies and the resulting "market shares" are symmetric.[5] Given that both states offer the same tax schedules (the marginal tax rates and poll subsidies), there is no migration, so all citizens stay in their home states. An aggregate budget constraint on this maximization problem can be stated as

$$\int_0^\infty T(\theta l(\theta)) f(\theta) d\theta \geq A + R, \tag{3}$$

where $R$ is the required per-capital government net revenue (so the total revenue required is $2R$ for the federation).

Consumption equals the difference between earnings and taxes, $c(\theta) = \theta l(\theta) - T(\theta l(\theta)) + A$. A worker with skill $\theta$ maximizes her utility:

$$u(\theta) + A = \max_{l'} \{\theta l' - T\left(\theta l'\right) + A\} - v\left(l'\right). \tag{4}$$

By the first-order condition and the envelope theorem, we have

$$u'(\theta) = (1 - T'(\theta l(\theta))) \cdot l(\theta) = v'(l(\theta)) \frac{l(\theta)}{\theta}. \tag{5}$$

More generally, the following lemma is standard:

**Lemma 1.** The tax function $T(\cdot)$ or marginal tax schedule $T'(\cdot)$ is incentive compatible if and only if the following conditions hold:

1. $Q(\theta)$ or $\theta l(\theta)$ increases in $\theta$;
2. The envelope formula (5) holds.

Note that choosing $T'$ is equivalent to choosing $u$: Given $u(\theta)$, $l(\theta)$ can be recovered from $u'(\theta) = v'(l(\theta)) \frac{l(\theta)}{\theta}$; then $T'(\theta l(\theta))$ can be further recovered from $(1 - T'(\theta l(\theta))) = v'(l(\theta))/\theta$. Given $T'(\theta l(\theta))$, $l(\theta)$ can be recovered from $(1 - T'(\theta l(\theta))) = v'(l(\theta))/\theta$, and $T(\theta l(\theta))$ can be recovered by simply integrating $T'(\theta l(\theta))$ (as $T(0) = 0$). Hence $u(\theta)$ can be recovered from (4): $u(\theta) = \theta l(\theta) - T(\theta l(\theta)) - v(l(\theta))$. For this reason we can work with $u(\theta)$ instead of $T'$ (or $T$) in our analysis throughout.

[5] We focus on the symmetric solution here for ease of comparison with the independent case, where we will focus on symmetric equilibrium.

Budget constraint (3) can be rewritten as

$$\int_0^\infty [\theta l(\theta) - u(\theta) - v(l(\theta))] f(\theta) d\theta \geq A + R.$$

The federation's problem can now be formulated as follows:

$$\max_{l(\theta), u(\theta), A} \int_0^\infty (u(\theta) + A) g(\theta) d\theta$$
$$\text{s.t.} \int_0^\infty [\theta l(\theta) - u(\theta) - v(l(\theta))] f(\theta) d\theta \geq A + R,$$
$$u'(\theta) = v'(l(\theta)) \frac{l(\theta)}{\theta}$$
$$\frac{d(\theta l(\theta))}{d\theta} \geq 0.$$

Let the subscript "$f$" denote the (federation) solution. Then $A_f$ is the equilibrium poll subsidy, and the equilibrium tax payment at $\theta$ is

$$T_f(\theta) = \theta l_f(\theta) - u_f(\theta) - v(l_f(\theta)).$$

With this notation, we have

**Proposition 1.** The optimal marginal tax rate under unified taxation is characterized by the following condition:

$$\frac{T'_f(\theta)}{1 - T'_f(\theta)} = \frac{1 + e_f^{-1}(\theta)}{\theta} \frac{G(\theta) - F(\theta)}{f(\theta)} = \left(1 + \frac{1}{e_f(\theta)}\right) \frac{1 - F(\theta)}{\theta f(\theta)} \frac{G(\theta) - F(\theta)}{1 - F(\theta)}, \tag{6}$$

where $T'_f(\theta) = T'(\theta l_f(\theta))$ and $e_f(\theta) = \frac{v'(l_f(\theta))}{v''(l_f(\theta)) l_f(\theta)}$ is the elasticity of labor supply.

This proposition can be demonstrated following the standard point-wise maximization technique (e.g., pages 47-51, Salanie, 2005). Equation (6) suggests that the marginal tax rate depends on the elasticity of labor supply, on the shape of the distribution of productivities, and on the government's redistributive objectives. Since these three terms are all positive, we have that under unified taxation, the marginal tax rate $T'_f(\theta) \geq 0$.

## 4 Independent Taxation: The Traditional Approach

We are now ready to consider independent taxation, where the federation lets two states make their tax policy decisions independently. We first consider the traditional approach to modeling competition between the two states, followed by our new approach.

With the traditional approach, states compete in taxation schedules simultaneously and independently, with the requirement of budget balance being only satisfied in equilibrium. Following the same notation as in the previous section, Lemma 1 now holds for each state. The only difference is that each state's optimization problem now reflects the variable population. Given one state's tax policy, the other state will choose a tax policy, $(T(\cdot), A)$ to maximize the weighted average utility of the citizens residing in its own state.[6] We will focus on symmetric equilibria, in which the two states choose the same tax policy.

Suppose that state 2's rent provision contract is given by $\overline{u}(\theta)+\overline{A}$. Then if state 1 offers the rent provision contract, $u(\theta)+A$, the type-$\theta$ "market share," or population, for state 1 is given by

$$\eta_1(\theta,\Delta) = \begin{cases} f(\theta)+f(\theta)H(\Delta|\theta) & \text{if} \quad \Delta > 0 \\ f(\theta)(1-H(-\Delta|\theta)) & \text{if} \quad \Delta \leq 0 \end{cases} \tag{7}$$

where $\Delta = u(\theta)+A-\overline{u}(\theta)-\overline{A}$. The type-$\theta$ "market share" or population for state 2, $\eta_2(\theta,\Delta)$, can be analogously defined.

As in Lehmann et al., we define the semi-elasticity of migration as follows:

$$\varepsilon(\theta,\Delta) = \frac{\partial \log \eta_1(\theta,\Delta)}{\partial \Delta} = \begin{cases} \frac{h(\Delta|\theta)}{1+H(\Delta|\theta)} & \text{if} \quad \Delta > 0 \\ \frac{h(-\Delta|\theta)}{1-H(-\Delta|\theta)} & \text{if} \quad \Delta \leq 0 \end{cases}$$

In particular, we define $\varepsilon(\theta) \equiv \varepsilon(\theta,0) = h(0|\theta)$, which is the semi-elasticity of migration for type-$\theta$ worker, evaluated at the symmetry equilibrium ($\Delta = 0$).

The budget constraint for state 1 can be stated as $\int_0^\infty (T(\theta l(\theta))-A)\eta_1(\theta,\Delta)d\theta \geq R$, where $R$ is the total revenue required (or the per-capital net revenue requirement). Using $l(\theta)$ as the control variable and $u(\theta)$ as the state variable, state 1's maximization problem can be formulated as the following optimal control program:

$$\begin{aligned}
&\max_{l(\theta),u(\theta),A} \int_0^\infty (u(\theta)+A)g(\theta)d\theta \\
&\text{s.t.} \int_0^\infty [\theta l(\theta)-u(\theta)-v(l(\theta))-A]\eta_1(\theta,\Delta)d\theta \geq R, \\
&u'(\theta) = v'(l(\theta))\frac{l(\theta)}{\theta} \\
&\frac{d(\theta l(\theta))}{d\theta} \geq 0
\end{aligned}$$

Let the subscript "$o$" denote the ("old," or traditional). Then $A_o$ is the equilibrium poll subsidy, and the equilibrium tax payment at $\theta$ is

$$T_o(\theta) = \theta l_o(\theta) - u_o(\theta) - v(l_o(\theta)).$$

With this notation, we have –

[6]It is well known that in the competitive mechanism design setting, it is no longer without loss of generality to apply the revelation principle. To sidestep this issue, we restrict attention to deterministic tax schedules.

**Proposition 2.** The symmetric equilibrium marginal tax rate under independent taxation is given by

$$\frac{T_o'(\theta)}{1-T_o'(\theta)} = \left(1+\frac{1}{e_o(\theta)}\right)\frac{1-F(\theta)}{\theta f(\theta)}\left(\frac{(1-F(\theta))-\frac{1}{p_o}(1-G(\theta))}{1-F(\theta)} - \frac{\int_\theta^\infty (T_o(t)-A_o)\varepsilon(t)f(t)dt}{(1-F(\theta))}\right), \tag{8}$$

where $T_o'(\theta) = T_o'(\theta l_o(\theta))$, $p_o = \frac{1}{1-\int_0^\infty (T_o(t)-A_o)\varepsilon(t)f(t)dt}$, $A_o = \int_0^\infty T_o(t)f(t)dt - R$, and $e_o(\theta) = \frac{v'(l_o(\theta))}{v''(l_o(\theta))l_o(\theta)}$.

*Proof.* See Appendix. □

Note that when $G(\theta) \equiv 1$ (the Rawlsian case), (8) becomes

$$\frac{T_o'(\theta)}{1-T_o'(\theta)} = \left(1+\frac{1}{e_o(\theta)}\right)\frac{\int_\theta^\infty [1-(T_o(t)-A_o)\varepsilon(t)]f(t)dt}{\theta f(\theta)},$$

which is exactly the same equilibrium marginal tax rate derived by Lehmann et al. (equation (13) of their Proposition 1).

The variable $p_o$ is the Lagrangian parameter for the budget constraint, which can be interpreted as the welfare value of additional government revenue. In the unified taxation, $p_f = 1$. Compared to (6), the effect of competition is reflected by the term

$$\int_0^\infty (T_o(t)-A_o)\varepsilon(t)f(t)dt,$$

where $\varepsilon(t) = h(0|t)$ is the measure of type-$t$ consumers who are perfectly movable (with zero moving cost), which is also the semi-elasticity of migration (evaluated at symmetric equilibrium $\Delta = 0$). A sufficiently small deviation from the proposed symmetric equilibrium ($u = \overline{u} = u^*$) by a state only has a marginal effect on the migration for the consumers with moving cost $z = 0$. This portion of consumers has a measure/mass of $\varepsilon(t)$ for each type $t$. For this reason we also refer to $\varepsilon(t)$ as the mass of type-$t$ marginal consumers. As we will see, $\varepsilon(t)$ plays an crucial role in our equilibrium characterization.

While the formal proof of Proposition 2 is a bit tedious, the driving force can be understood intuitively. When the state lowers $A_o$ to $A_o - \delta A$, there are three effects: the first is the loss in utility for all workers, $-\delta A \cdot \int_0^\infty dG(\theta)$; the second is the direct gain in tax revenue, $\delta A \cdot p_o \int_0^\infty dF(\theta)$; and the third is the indirect loss in tax revenue due to excluding more workers, $-\delta A \cdot p_o \int_0^\infty (T_o(t)-A_o)\varepsilon(t)f(t)dt$. $A_o$ is optimal if these three welfare effects sum to zero, implying that

$$p_o = \frac{1}{1-\int_0^\infty (T_o(t)-A_o)\varepsilon(t)f(t)dt}.$$

Since $p_o$ is the welfare value of additional government revenue (by the envelope theorem), we must have $p_o \geq 0$. Thus in equilibrium we must have

$$\int_0^\infty (T_o(t) - A_o)\varepsilon(t) f(t)\, dt < 1. \tag{9}$$

To see that (9) is a necessary condition for equilibrium, suppose in negation, the inequality in (9) is reversed. In that case it can be verified that a (marginal) rise in the poll subsidy ($A$) increases tax revenue, calculated net of the tax rate. That is, a higher poll subsidy creates a budget surplus, which implies that increasing the poll subsidy is feasible. Since raising a poll subsidy increases all resident utilities, the original tax policy could not have been in equilibrium. In the case with perfect mobility, Bierbrauer et al. (2013) show that there does not exist any equilibrium in which the highest type pays a positive amount of tax or the lowest type receives a positive amount of subsidy. This is consistent with our result, as the perfect mobility case ($h = +\infty$) clearly violates (9).

In general, $p_o$ may exceed or fall short of one, depending on the shape of $\varepsilon(t)$. If consumers with higher income are generally more movable, then $\varepsilon(t)$ would be relatively higher for higher types, which would tend to put more weight on positive values of $T_o(t) - A_o$, leading to $\int_0^\infty (T_o(t) - A_o)\varepsilon(t) f(t)\, dt > 0$, and hence $p_o > 1$. But if the most movable workers are predominantly low-income workers who are net recipients of government revenue, then $p_o$ can be less than one. In this latter case, the marginal social value of government revenue is less than it would be under unified taxation, because another dollar of revenue increases the poll subsidy by less than a dollar, due to the influx of these tax recipients.

In the traditional approach, consider raising the marginal tax rate by $\delta T$ over the interval $\theta$ to $\theta + \delta\theta$. This tax change is equivalent to a lump sum tax of $\delta T \delta\theta$ on the $1 - F(\theta)$ individuals whose skill is larger than $\theta$, leading to a loss $(-(1 - G(\theta)) \cdot \delta T \delta\theta)$ in utility for those individuals, and a loss $(-p_o \int_\theta^\infty (T_o(t) - A_o)\varepsilon(t) f(t)\, dt \cdot \delta T \delta\theta)$ in revenue for the state government, due to emigration from those individuals, but a gain in revenue for the state government $(p_o(1 - F(\theta)) \cdot \delta T \delta\theta)$ . However, the $f(\theta)\delta\theta$ individuals with skill between $\theta$ and $\theta + \delta\theta$ now face a higher marginal tax rate, inducing a drop in their labor supply and a resulting loss $(-p_o \frac{T_o'(\theta)}{1 - T_o'(\theta)} \frac{e_o(\theta)}{1 + e_o(\theta)} \theta f(\theta) \cdot \delta T \delta\theta)$ in tax revenue. The loss in tax revenue varies directly with the elasticity of labor supply with respect to the after-tax wage rate for individuals with labor skill of $\theta$, denoted $e_o(\theta)$.

The initial tax schedule is optimal if these four welfare effects sum to zero, implying that

$$\left\{ -(1 - G(\theta)) + p_o(1 - F(\theta)) - p_o \frac{T_o'(\theta)}{1 - T_o'(\theta)} \frac{e_o(\theta)}{1 + e_o(\theta)} \theta f(\theta) - p_o \int_\theta^\infty (T_o(t) - A_o)\varepsilon(t) f(t)\, dt \right\} \delta T \delta\theta = 0.$$

Simplifying, we obtain condition (8), which can be rewritten as

$$\frac{T_o'(\theta)}{1 - T_o'(\theta)} = \left(1 + \frac{1}{e_o(\theta)}\right) \frac{1 - F(\theta)}{\theta f(\theta)} \left( \frac{(1 - F(\theta)) - \frac{1}{p_o}(1 - G(\theta))}{1 - F(\theta)} - \frac{\int_\theta^\infty (T_o(t) - A_o)\varepsilon(t) f(t)\, dt}{(1 - F(\theta))} \right), \tag{10}$$

or alternatively,

$$\begin{aligned}\frac{T_o'(\theta)}{1-T_o'(\theta)} &= \left(1+\frac{1}{e_o(\theta)}\right)\frac{1-F(\theta)}{\theta f(\theta)}\\ &\cdot\left[\frac{G(\theta)-F(\theta)}{1-F(\theta)}+\left(1-\frac{1}{p_o}\right)\left(\frac{1-G(\theta)}{1-F(\theta)}-\frac{\int_\theta^\infty (T_o(t)-A_o)\varepsilon(t)f(t)dt/(1-F(\theta))}{\int_0^\infty (T_o(t)-A_o)\varepsilon(t)f(t)dt}\right)\right].\end{aligned} \tag{11}$$

## 5 Independent Taxation: The New Approach

The difference between the new and traditional approaches is that when a state chooses its tax policy, it recognizes that the other state's poll subsidy will adjust to prevent any migration from unbalancing its government budget. In other words, each state is now choosing its marginal tax function, $T'(\cdot)$, simultaneously and independently, with the poll subsidies being adjusted automatically to satisfy the government budget constraints, requiring that each state generate net tax revenue, $R$.

Again we focus on symmetric equilibria, in which the two states choose the same marginal tax rate function, or equivalently, the same rent provision function, $u$. After the poll subsidies are determined, suppose that state 2's rent provision contract is given by $\overline{u}(\theta)+\overline{A}$. Then if state 1 offers rent provision contract, $u(\theta)+A$, the measures (or "market shares" ) of type-$\theta$ workers residing in states 1 and 2 are is also given by (7) and (**??**), respectively.

From $u'(\theta)=v'(l(\theta))\frac{l(\theta)}{\theta}$ (Lemma 1), we can obtain $l(\theta)=\xi\left(u'(\theta),\theta\right)$ for some function $\xi$, and

$$\xi_{u'}\left(u'(\theta),\theta\right)=\frac{dl(\theta)}{du'(\theta)}=\frac{\theta}{v'(l(\theta))+v''(l(\theta))l(\theta)}=\frac{\theta}{v'(\xi(u'(\theta),\theta))+v''(\xi(u'(\theta),\theta))\xi(u'(\theta),\theta)}. \tag{12}$$

Define

$$M\left(\theta,u(\theta),u'(\theta)\right)\equiv\theta\xi(u'(\theta),\theta)-u(\theta)-v(\xi(u'(\theta),\theta)). \tag{13}$$

$M\left(\theta,u(\theta),u'(\theta)\right)$ is the tax revenue from type-$\theta$ workers. Using this notation, the budget constraint for state 1 can be stated as

$$\int_0^\infty\left[M\left(\theta,u(\theta),u'(\theta)\right)-A\right]\eta_1\left(\theta,u(\theta)+A-\overline{u}(\theta)-\overline{A}\right)d\theta\geq R \tag{14}$$

Similarly, state 2's budget constraint is given by

$$\int_0^\infty\left[M\left(\theta,\overline{u}(\theta),\overline{u}'(\theta)\right)-\overline{A}\right]\eta_2\left(\theta,u(\theta)+A-\overline{u}(\theta)-\overline{A}\right)d\theta\geq R \tag{15}$$

We now introduce state 2's poll subsidy adjustment process following any deviation by state 1. The timeline is as follows:

1. State 1 announces tax policy $\{u,A\}$;

2. Given $\{u, A\}$ and $\overline{u}$, state 2 will adjust its poll subsidy $\overline{A}$ so that (15) is satisfied, which gives rise to $\overline{A} = \overline{A}(u, A)$ (the dependence on $\overline{u}$ and $\overline{u}'$ are suppressed).

Anticipating state 2's poll subsidy adjustment process, state 1 chooses $\{u, A\}$ jointly to satisfy its own resource constraint (14). A pair $\{u, A\}$ is a *feasible* deviation for state 1 in response to state 2's strategy $\overline{u}$ if (1) $\overline{A}$ is chosen so that (15) is satisfied; (2) given $\overline{A} = \overline{A}(u, A)$, $\{u, A\}$ satisfies (14). Let

$$\begin{aligned} & N\left(u, A, \overline{A}\right) \\ = & \int_0^{\infty} \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}\right] \eta_2\left(\theta, u(\theta) + A - \overline{u}(\theta) - \overline{A}\right) d\theta. \end{aligned}$$

Then we have the partial derivatives

$$\begin{aligned} N'_A\left(u, A, \overline{A}\right) &= \int_0^{\infty} \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}\right] \frac{\partial \eta_2\left(\theta, u(\theta) + A - \overline{u}(\theta) - \overline{A}\right)}{\partial \Delta} d\theta, \\ N'_{\overline{A}}\left(u, A, \overline{A}\right) &= -\int_0^{\infty} \eta_2\left(\theta, u(\theta) + A - \overline{u}(\theta) - \overline{A}\right) d\theta \\ & \quad -\int_0^{\infty} \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}\right] \frac{\partial \eta_2\left(\theta, u(\theta) + A - \overline{u}(\theta) - \overline{A}\right)}{\partial \Delta} d\theta, \\ N'_u\left(u, A, \overline{A}\right) &= \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}\right] \frac{\partial \eta_2\left(\theta, u(\theta) + A - \overline{u}(\theta) - \overline{A}\right)}{\partial \Delta}, \end{aligned}$$

where $N'_u\left(u, A, \overline{A}\right)$ above is the Fréchet derivative (Luenberger, 1969) on $[0, \infty)$. By the implicit function theorem,

$$\begin{aligned} \overline{A}'_A(u, A) &= -\frac{N'_A\left(u, A, \overline{A}\right)}{N'_{\overline{A}}\left(u, A, \overline{A}\right)} \\ &= \frac{\int_0^{\infty} \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}\right] \frac{\partial \eta_2\left(\theta, u(\theta) + A - \overline{u}(\theta) - \overline{A}\right)}{\partial \Delta} d\theta}{\int_0^{\infty} \eta_2\left(\theta, u(\theta) + A - \overline{u}(\theta) - \overline{A}\right) d\theta + \int_0^{\infty} \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}\right] \frac{\partial \eta_2\left(\theta, u(\theta) + A - \overline{u}(\theta) - \overline{A}\right)}{\partial \Delta} d\theta}; \\ \overline{A}'_u(u, A) &= -\frac{N'_u\left(u, A, \overline{A}\right)}{N'_{\overline{A}}\left(u, A, \overline{A}\right)} \\ &= \frac{\left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}\right] \frac{\partial \eta_2\left(\theta, u(\theta) + A - \overline{u}(\theta) - \overline{A}\right)}{\partial \Delta}}{\int_0^{\infty} \eta_2\left(\theta, u(\theta) + A - \overline{u}(\theta) - \overline{A}\right) d\theta + \int_0^{\infty} \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}\right] \frac{\partial \eta_2\left(\theta, u(\theta) + A - \overline{u}(\theta) - \overline{A}\right)}{\partial \Delta} d\theta}. \end{aligned}$$

$\overline{A}'_u(u, A)$ above is the Fréchet derivative on $[0, \infty)$. Given state 2's strategy $\overline{u}$ and its poll subsidy adjustment rule $\overline{A}(u,A)$, state 1's maximization problem can be formulated as follows:

$$
\begin{aligned}
&\max_{u(\theta),A} \int_0^\infty (u(\theta)+A)\, g(\theta)\, d\theta \\
&\text{s.t.} \int_0^\infty \left[M\left(\theta,u(\theta),u'(\theta)\right)-A\right]\eta_1\left(\theta,u(\theta)-\overline{u}(\theta)+A-\overline{A}(u,A)\right)d\theta \geq R; \qquad (16)\\
&\frac{d\left(\theta\xi(u'(\theta),\theta)\right)}{d\theta} \geq 0.
\end{aligned}
$$

It is worth noting that the integrand in the resource constraint is a function of not just $u(\theta)$, but also the entire path $u$, exactly due to the poll subsidy adjustment rule $\overline{A}(u,A)$. It is not clear what can be used as appropriate state variables for us to formulate our program (16) as an optimal control problem. As such, we will turn to the (presumably more basic) method of calculus variations to characterize the equilibrium under our new approach.

Let the subscript "$n$" denote the (new) solution. Then $A_n$ is the equilibrium poll subsidy, and the equilibrium tax payment at $\theta$ is

$$T_n(\theta) = M\left(\theta, u_n(\theta), u_n'(\theta)\right).$$

With this notation, we have –

**Proposition 3.** The symmetric equilibrium marginal tax rate under the new approach is given by

$$
\begin{aligned}
\frac{T_n'(\theta)}{1-T_n'(\theta)} &= \left(1+\frac{1}{e_n(\theta)}\right)\frac{1-F(\theta)}{\theta f(\theta)} \\
&\quad \cdot\left(\frac{(1-F(\theta))-\frac{1}{p_n}(1-G(\theta))}{1-F(\theta)} - \frac{\int_\theta^\infty [T_n(t)-A_n]\varepsilon(t)f(t)dt}{\left(1-\int_0^\infty [T_n(t)-A_n]\varepsilon(t)f(t)dt\right)(1-F(\theta))}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
p_n &= \frac{1}{1-\frac{\int_0^\infty [T_n(t)-A_n]\varepsilon(t)f(t)dt}{1-\int_0^\infty [T_n(t)-A_n]\varepsilon(t)f(t)dt}};\; A_n = \int_0^\infty T_n(t)f(t)dt - R; \\
T_n'(\theta) &= T'(\theta l_n(\theta));\; e_n(\theta) = \frac{v'(l_n(\theta))}{v''(l_n(\theta))l_n(\theta)}.
\end{aligned}
$$

*Proof.* See Appendix. □

The proof is based on calculus of variation. It also makes use of the Fréchet derivative, which turns out to be the right tool in capturing the reaction of one state's common subsidy to the other state's tax policy changes. Proposition 3 can be also understood intuitively, as follows.

When the state lowers poll subsidy $A_n$ to $A_n - \delta A$, there are four effects: the first is the loss in utility for all workers, $-\delta A \cdot \int_0^\infty dG(\theta)$; the second is the direct gain in tax revenue, $\delta A \cdot p_n \int_0^\infty dF(\theta)$; the third is the indirect loss in tax revenue due to excluding more workers, $-\delta A \cdot p_n \int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt$; and the fourth is the indirect loss in tax revenue due to the other state attracting more workers by raising $\overline{A}$ to react to the change of $A_n$, $\delta A \cdot p_n \int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt \cdot \overline{A}_A$. Note that when $A_n$ reduces, the other state will end up with revenue surplus. If $\int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt > 1$, the other state can raise $\overline{A}$ to increase revenue (leading to more surplus). In this case $\overline{A}_A < 0$. If $\int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt \leq 1$, on the other hand, the other state will raise $\overline{A}$ (to reduce the surplus until the budget balances). In this case, we have

$$\overline{A}_A = \frac{\int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt}{\int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt - 1}.$$

$A_n$ is optimal if the above four welfare effects sum to zero. Note that this is impossible in the case with $\int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt > 1$ (as the first effect is negative, the combined second and third effect is negative, and the last effect is also negative due to $\overline{A}_A < 0$). So in equilibrium, we must have $\int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt \leq 1$, and the optimality leads to

$$p_n = \frac{1}{1 - \frac{\int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt}{1 - \int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt}}$$

By a similar argument in the previous section, $p_n \geq 0$, which implies that in equilibrium,

$$\int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt < \frac{1}{2} \tag{17}$$

Now consider raising the marginal tax rate by $\delta T$ over the interval $\theta$ to $\theta + \delta\theta$. This tax change is equivalent to a lump sum tax of $\delta T \delta\theta$ on the $1 - F(\theta)$ individuals whose skill is larger than $\theta$, leading to a loss in utility, $-(1 - G(\theta)) \cdot \delta T \delta\theta$, for those individuals; and a loss in revenue, $-p_n \int_\theta^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt \cdot \delta T \delta\theta$, for the state government, due to emigration of those individuals; a loss (given below) in revenue for the state government, due to emigration of those individuals (caused by the other state increasing its poll subsidy),

$$\begin{aligned}
& p_n \int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt \cdot \int_\theta^\infty \overline{A}_u(u, A)dt \cdot \delta T \delta\theta \\
= \quad & p_n \left( \int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt \right) \cdot \left( \frac{\int_\theta^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt}{\int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt - 1} \right) \cdot \delta T \delta\theta \\
= \quad & p_n \left( \int_\theta^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt \right) \cdot \left( \frac{\int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt}{\int_0^\infty [T_n(t) - A_n]\varepsilon(t) f(t)dt - 1} \right) \cdot \delta T \delta\theta;
\end{aligned}$$

and a gain in revenue for the government, $p_n(1 - F(\theta)) \cdot \delta T \delta\theta$ . However, the $f(\theta)\delta\theta$ individuals with skill between $\theta$ and $\theta + \delta\theta$ now face a higher marginal tax rate, inducing a drop in their labor supply and a resulting loss in tax revenue, $-p_n \frac{T'_n(\theta)}{1-T'_n(\theta)} \frac{e_n(\theta)}{1+e_n(\theta)} \theta f(\theta) \cdot \delta T \delta\theta$. The loss in tax revenue varies directly with the elasticity of labor supply with respect to the after-tax wage rate for individuals with labor skill of $\theta$, denoted $e_n(\theta)$.

The initial tax schedule is optimal if these five welfare effects sum to zero, implying that

$$\left\{ \begin{array}{c} -(1-G(\theta)) + p_n(1-F(\theta)) - p_n \frac{T'_n(\theta)}{1-T'_n(\theta)} \frac{e_n(\theta)}{1+e_n(\theta)} \theta f(\theta) \\ -p_n \left(1 - \frac{\int_0^\infty [T_n(t)-A_n]\varepsilon(t) f(t) dt}{\int_0^\infty [T_n(t)-A_n]\varepsilon(t) f(t) dt - 1}\right) \int_\theta^\infty [T_n(t)-A_n]\varepsilon(t) f(t) dt \end{array} \right\} \delta T \delta\theta = 0.$$

Simplifying, we find the following expression determining the optimal marginal tax rate at any skill $\theta$:

$$\frac{T'_n(\theta)}{1-T'_n(\theta)} = \left(1 + \frac{1}{e_n(\theta)}\right) \frac{1-F(\theta)}{\theta f(\theta)} \left( \frac{(1-F(\theta)) - \frac{1}{p_n}(1-G(\theta))}{1-F(\theta)} - \frac{\int_\theta^\infty [T_n(t)-A_n]\varepsilon(t) f(t) dt}{\left(1 - \int_0^\infty [T_n(t)-A_n]\varepsilon(t) f(t) dt\right)(1-F(\theta))} \right),$$

or alternatively,

$$\begin{aligned} \frac{T'_n(\theta)}{1-T'_n(\theta)} &= \left(1 + \frac{1}{e_n(\theta)}\right) \frac{1-F(\theta)}{\theta f(\theta)} \\ &\cdot \left[ \frac{G(\theta)-F(\theta)}{1-F(\theta)} + \left(1 - \frac{1}{p_n}\right) \left( \frac{1-G(\theta)}{1-F(\theta)} - \frac{\int_\theta^\infty [T_n(t)-A_n]\varepsilon(t) f(t) dt / (1-F(\theta))}{\int_0^\infty [T_n(t)-A_n]\varepsilon(t) f(t) dt} \right) \right]. \end{aligned} \tag{18}$$

## 6 Comparison: Traditional vs. New Approaches

The expressions for the equilibrium marginal tax rates (11) and (18) can be unified as follows:

$$\begin{aligned} \frac{T'_i(\theta)}{1-T'_i(\theta)} &= \left(1 + \frac{1}{e_i(\theta)}\right) \frac{1-F(\theta)}{\theta f(\theta)} \\ &\cdot \left[ \frac{G(\theta)-F(\theta)}{1-F(\theta)} + \left(1 - \frac{1}{p_i}\right) \left( \frac{1-G(\theta)}{1-F(\theta)} - \frac{\int_\theta^\infty [T_i(t)-A_i]\varepsilon(t) f(t) dt / (1-F(\theta))}{\int_0^\infty [T_i(t)-A_i]\varepsilon(t) f(t) dt} \right) \right], \ i \in \{o,n\} \end{aligned} \tag{19}$$

where

$$\begin{aligned} T'_i(\theta) &= T'(\theta l_i(\theta)); \ e_i(\theta) = \frac{v'(l_i(\theta))}{v''(l_i(\theta)) l_i(\theta)}; \ A_i = \int_{\underline{\theta}}^{\overline{\theta}} T_i(t) f(t) dt - R; \\ p_n &= \frac{1}{1 - \frac{\int_0^\infty [T_n(t)-A_n]\varepsilon(t) f(t) dt}{1 - \int_0^\infty [T_n(t)-A_n]\varepsilon(t) f(t) dt}}; \ p_o = \frac{1}{1 - \int_0^\infty [T_o(t)-A_o]\varepsilon(t) f(t) dt}. \end{aligned} \tag{20}$$

The marginal tax differs between the two approaches only through the Lagrange multiplier on the state government's budget constraint. This difference is due to an additional effect in our new approach, compared to the traditional approach, when identifying the welfare value of additional state revenue (the Lagrange multiplier). When state 1 uses revenue to raise its poll subsidy, it attracts residents from state 2, and state 2's poll subsidy then adjusts to keep its government budget balanced, which creates more migration. For this reason, the new approach differs from the old approach via the responsiveness of migration to a change in the poll subsidy. (19) can also be rewritten as follows:

$$\frac{T_i'(\theta)}{1-T_i'(\theta)} = \left(1+\frac{1}{e_i(\theta)}\right)\left(\frac{G(\theta)-F(\theta)}{\theta f(\theta)} + \frac{(1-G(\theta))\int_0^\infty \frac{T_i(t)-A_i}{\alpha_i}\varepsilon(t)f(t)dt - \int_\theta^\infty \frac{T_i(t)-A_i}{\alpha_i}\varepsilon(t)f(t)dt}{\theta f(\theta)}\right), \tag{21}$$

where $\alpha_o = 1$ and $\alpha_n = 1-\int_0^\infty [T_n(t)-A_n]\varepsilon(t)f(t)dt$.

Inspecting the expression of (21), it is clear that replacing the traditional approach with the new approach has the same effect on the marginal tax function $T_n'(\theta)$ as a change in the semi-elasticity of migration from $\varepsilon(\theta)$ to $\varepsilon(\theta)/\left(1-\int_0^\infty [T_n(\theta)-A_n]\varepsilon(\theta)dF(\theta)\right)$. In particular, replacing the traditional approach with the new approach is equivalent to reducing (increasing) $\varepsilon(\theta)$ by the same percentage amount at each $\theta$, if $\int_0^\infty [T_n(\theta)-A_n]\varepsilon(\theta)dF(\theta)$ is negative (positive). In particular, there is no difference between the two approaches if $\int_0^\infty [T_n(\theta)-A_n]\varepsilon(\theta)dF(\theta)=0$. Note that $\int_0^\infty [T_n(\theta)-A_n]\varepsilon(\theta)dF(\theta) = R\cdot E\varepsilon(\theta)+cov(T_n(\theta)-A_n, \varepsilon(\theta))$. As such, we have

**Proposition 4.** There is no difference between the traditional and new approaches if the income tax is purely redistributive ($R=0$) and there is no correlation between the tax payment and semi-elasticity of migration in equilibrium, e.g., when $\varepsilon(\theta)=h$, a constant.

The more relevant case, however, is where $R>0$ or $\varepsilon(\theta)$ varies with $\theta$. As such, our new approach generally yields equilibria that differ from those under the traditional approach. In order to explore the exact differences between two approaches, in what follows we will assume that the primitives of the model, i.e., $F(\theta)$, $G(\theta)$, and $\varepsilon(\theta)$, are given such that the following property holds:

$$0 < \int_0^\infty [T_i(t)-A_i]\varepsilon(t)f(t)dt < \frac{\int_\theta^\infty [T_i(t)-A_i]\varepsilon(t)f(t)dt}{1-F(\theta)}, \text{ for } \forall\theta>0. \tag{22}$$

Property (22) should be regarded as a regular property in equilibrium. To see this, note that in close economy with unified taxation, by (6) we have $T_f'(\theta)\ge 0$ with equality only at $\theta=0$. This implies the following property:

$$0 < R = \int_0^\infty \left[T_f(t)-A_f\right]f(t)dt < \frac{\int_\theta^\infty \left[T_f(t)-A_f\right]f(t)dt}{1-F(\theta)}, \text{ for } \forall\theta>0, \tag{23}$$

i.e., the average net tax payment over types $[\theta,+\infty)$ increases in $\theta$. Property (22) should be regarded as a counterpart of Property (23) in the open economy taking into account competition and migration. Note that when $\varepsilon(\theta)$ is a constant, Property (22) is exactly Property (23).

We will now identify a set of conditions under which Property (22) holds. Given $F(\theta)$ and $G(\theta)$, we introduce the following bounding functions:

$$M_i(\theta) = \max\left\{\frac{d\log h(\theta)}{d\theta} | d\log\left[\frac{1-p_i\frac{g(\theta)}{f(\theta)}}{h(\theta)}\right]/d\theta \geq 0\right\}, \tag{24}$$

$$m_i(\theta) = \min\left\{\frac{d\log h(\theta)}{d\theta} | d\log[(T_i(\theta)-A_i)h(\theta)]/d\theta > 0 \text{ and } \int_0^\infty (T_i(\theta)-A_i)h(\theta)dF(\theta) > 0\right\}, \tag{25}$$

where $p_i$, $T_i(\theta)$, and $A_i$ are defined in (20) with $\varepsilon(\theta)$ being replaced by a generic (positive) function $h(\theta)$ here. Note that $M_i$ and $m_i$ are functionals of $F$ and $G$.

**Lemma 2.** Property (22) holds if $g(\theta)/f(\theta)$ is nonincreasing and $m_i(\theta) \leq \frac{d\log\varepsilon(\theta)}{d\theta} \leq M_i(\theta)$ for $\theta \in (0,+\infty)$, where $m_i \leq 0 \leq M_i$, $i \in \{o,n\}$.

*Proof.* See Appendix. □

That $g(\theta)/f(\theta)$ is nonincreasing (likelihood ratio dominance) is a sufficient condition for $G(\theta) \geq F(\theta)$ (first-order stochastic dominance), which can be regarded as a stronger redistributive objective for the government. In the close economy (where $\varepsilon(t) = 0$), $G(\theta) \geq F(\theta)$ is sufficient to induce $T'(\theta) \geq 0$. In the open economy with tax competition, a stronger government redistributive objective is required to insure that $T'(\theta) \geq 0$. In addition, some restriction on the shape of the semi-elasticity of migration function $\varepsilon(\theta)$ is also needed. The above proposition suggests that as long as $\varepsilon(\theta)$ does not vary too much with $\theta$ (neither increasing nor decreasing too sharply), our Property (22) should hold. Since $m_i \leq 0 \leq M_i$, in particular, Property (22) holds if $g(\theta)/f(\theta)$ is nonincreasing and $\varepsilon(\theta) = h$, a constant.

Next we examine taxes on high incomes. In particular, we look at the optimal asymptotic marginal tax rate, which is the limit of marginal tax rates as $\theta$ goes to $\infty$. In the case of unified taxation, recent research has shown that at high incomes, marginal tax rates increase with income, converging to a high level that Diamond and Saez (2012) calculate to be $T'_f(\infty) = .73$. An important assumption behind this calculation is that the social marginal utility of income goes to zero as income goes to infinity. Similarly, we assume that $\lim_{\theta\to\infty} g(\theta)/f(\theta) = 0$. In addition, we assume that $\varepsilon(\theta)$ converges to some constant $h$. With these assumptions in addition to Property (22), we have:

**Lemma 3.** Suppose $F(\theta)$, $G(\theta)$, and $\varepsilon(\theta)$ are given such that Property (22) holds. In addition, we assume that $\lim_{\theta\to\infty} g(\theta)/f(\theta) = 0$ and both $\lim\limits_{\theta\to\infty} \frac{1-F(\theta)}{\theta f(\theta)}$ and $\lim\limits_{\theta\to\infty} \varepsilon(\theta)$ are finite and positive. Then the asymptotic marginal tax rate in the independent regime equals zero, using either the traditional or new approaches.

*Proof.* See Appendix. □

Our finding that competition for residents reduces the asymptotic tax rate to zero may be simply explained. If the rate stayed bounded above some positive number, then tax payments would grow with income without bound. Unless $1/\varepsilon(\theta)$ similarly grew, which is ruled out by our assumptions, the tax payment that a state could obtain by lowering marginal taxes at high incomes would also grow without bound. Thus, states compete the asymptotic tax rate down to zero. This result shows that the effects of competition on marginal tax rates can be especially strong at high incomes.

Consider now the taxes paid by individuals with abilities above some level $\theta^*$. The average taxes paid by these individuals are

$$AT_i\left(\theta^*\right) = \int_{\theta^*}^{\infty} \left(T_i(t) - A_i\right) \frac{f(t)}{1 - F(\theta^*)} dt.$$

We can let $\theta^*$ go to infinity and compute the asymptotic average tax payment. Although the asymptotic marginal tax rate is zero under both approaches, the asymptotic average tax payments differ:

**Lemma 4.** Under the assumptions listed in Lemma 3, the asymptotic average tax payment is lower under the new approach than under the traditional approach.

*Proof.* See Appendix. □

This lemma tells us that for an ability $\theta^*$ sufficiently high, average tax payments for individuals with abilities above $\theta^*$ will be lower under new approach than under the traditional approach. For budget balance, the reverse will be true for individuals with abilities below $\theta^*$.

We are now ready to state our central comparison result.

**Proposition 5.** Under the assumptions listed for Lemma 3, there exists $\widehat{\theta} \in (0, +\infty)$ such that $T_n(\theta) - A_n > T_o(\theta) - A_o$ when $\theta \in [0, \widehat{\theta})$; $T_n(\theta) - A_n = T_o(\theta) - A_o$ when $\theta = \widehat{\theta}$; and $T_n(\theta) - A_n < T_o(\theta) - A_o$ when $\theta \in (\widehat{\theta}, +\infty)$. So compared to the traditional approach, under the new approach the rich pay less taxes and the poor receive less subsidies; in other words, the extent of income redistribution is lower under our new approach.

*Proof.* See Appendix. □

The key intuition of our comparison result can be understood based on the effect of the other state's poll subsidy response to a change in a state's marginal tax rate. As argued before, when raising the marginal tax rate by $\delta T$ over the interval $\theta$ to $\theta + \delta\theta$, this tax change is equivalent to a lump sum tax of $\delta T \delta\theta$ on the $1 - F(\theta)$ individuals whose skill is larger than $\theta$. This leads to a reduction in the rent provision $u(t)$ for all types above $\theta$. In our new approach, the other state will respond by changing its poll subsidy $\overline{A}$. This effect is captured via the Fréchet derivative, which equals

$$\frac{\int_{\theta}^{\infty} [T_n(t) - A_n] \varepsilon(t) f(t) dt}{\int_0^{\infty} [T_n(t) - A_n] \varepsilon(t) f(t) dt - 1}.$$

Since $\int_0^{\infty} [T_n(t) - A_n] \varepsilon(t) f(t) dt < 1/2$, this effect is negative, which means that $\overline{A}$ will increase. As $\overline{A}$ increases, more citizens would emigrate from the state under consideration (compared to the situation under the traditional approach without poll subsidy adjustment). This suggests that the ex ante incentive to raise the marginal tax rate will be reduced, leading to less redistribution under the new approach in equilibrium.

That the income redistribution is reduced under our new approach can also be understood intuitively: moving from traditional approach to new approach is equivalent to a change from $\varepsilon(\theta)$ to

$$\varepsilon(\theta)/\left(1-\int_0^{\infty}[T_n(\theta)-A_n]\varepsilon(\theta)dF(\theta)\right).$$

Since $\varepsilon(\theta)$ captures the degree of competition between two tax authorities, our new approach implies more intense competition and hence lower level of income redistribution under the new approach. Since the unified tax regime can be regarded as an extreme case with $\varepsilon(\theta)=0$, a direct corollary is that the income redistribution under unified regime will be higher than the independent tax regime (under either traditional or new approaches).

To gain some feel about how the comparison of tax schedules looks like, we provide an illustration based on numerical computations. We consider two symmetric states, which design the tax system to maximize the welfare of the worst-off individuals (Rawlsian, or $G(\theta)=1$). Adopting similar numerical environments as used in Lehmann et al., we assume that the income distribution is a Pareto distribution with $\alpha=2.293$, so that the top 1% of the population gets 18% of total income (to proxy the US economy). The disutility of effort is given by $v(l)=l^{1+1/e}$. This specification implies a constant elasticity of gross earnings with respect to the retention rate $e$, as in Diamond (1998) and Saez (2001). We choose $e=0.25$, which is a reasonable value based on the survey by Saez et al. (2012). The semi-elasticity of migration $\varepsilon(\theta)$ is constant throughout the whole skill distribution and is $10^{-6}$; The required per-capital government net revenue $R$ is $0.265\times10^6$ dollars. The comparison of the tax schedules $T_n(\theta)-A_n$ and $T_o(\theta)-A_o$ is illustrated by the following figure:

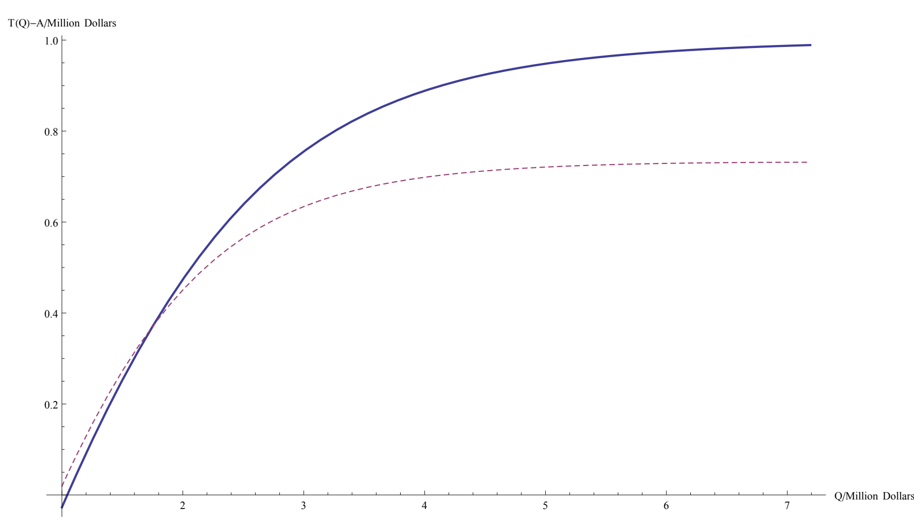


Figure 1: An Illustration of The Comparison of $T_n(Q)-A_n$ (dashed) and $T_o(Q)-A_o$ (solid)

As is clear, the poll subsidy is positive under old approach but turns negative under the new approach, implying that the change in approaches switches the direction of redistribution. From the figure, we can tell that the difference between two tax schedules is quite significant.

More generally, the difference between two tax schedules $T_n(\theta)-A_n$ and $T_o(\theta)-A_o$ depends on the value of $\int_0^\infty [T(\theta)-A]\varepsilon(\theta)dF(\theta)$. Given that $\int_0^\infty [T_n(\theta)-A_n]\varepsilon(\theta)dF(\theta) < \frac{1}{2}$, the bigger is $\int_0^\infty [T_n(\theta)-A_n]\varepsilon(\theta)dF(\theta)$, the larger is the difference between $T_n(\theta)-A_n$ and $T_o(\theta)-A_o$. We can thus evaluate the comparison under three scenarios by varying the semi-elasticity function $\varepsilon(\theta)$ (but maintaining $E\varepsilon(\theta) = h$). First we have

$$\begin{aligned}\int_0^\infty [T_n(\theta)-A_n]\varepsilon(\theta)dF(\theta) &= R\cdot E\varepsilon(\theta) + cov(T_n(\theta)-A_n, \varepsilon(\theta)) \\ &= R\cdot h + cov(T_n(\theta)-A_n, \varepsilon(\theta))\end{aligned}$$

In the first scenario, we assume that $\varepsilon(\theta)$ is decreasing (e.g., it is zero up to the top centile and then decreasing) so that $cov(T_n(\theta)-A_n, \varepsilon(\theta)) < 0$; in the second scenario, we assume that $\varepsilon(\theta)$ is a constant; in the third scenario, we assume that $\varepsilon(\theta)$ is increasing so that $cov(T_n(\theta)-A_n, \varepsilon(\theta)) > 0$. We thus have

$$\begin{aligned}\int_0^\infty \left[T_n^1(\theta)-A_n^1\right]\varepsilon^1(\theta)dF(\theta) &= R\cdot h + cov\left(T_n^1(\theta)-A_n^1, \varepsilon^1(\theta)\right) < R\cdot h \\ \int_0^\infty \left[T_n^2(\theta)-A_n^2\right]\varepsilon^2(\theta)dF(\theta) &= R\cdot h + cov\left(T_n^2(\theta)-A_n^2, \varepsilon^2(\theta)\right) = R\cdot h \\ \int_0^\infty \left[T_n^3(\theta)-A_n^3\right]\varepsilon^3(\theta)dF(\theta) &= R\cdot h + cov\left(T_n^3(\theta)-A_n^3, \varepsilon^3(\theta)\right) > R\cdot h\end{aligned}$$

The difference between the two tax schedules is smaller in the first scenario than in the second scenario. The difference between the two tax schedules is bigger in the third scenario than in the second scenario.

Figure 1 is an illustration of the second scenario. By Lemma 2, $\varepsilon(\theta)$ cannot decrease or increase too much. So $cov(T_n(\theta)-A_n, \varepsilon(\theta))$ must be small. As such the two tax schedules in the first and third scenarios will be very similar to the two schedules in the second scenario.

## 7 Conclusion

In this paper, we have developed a model of competition between states in marginal income tax schedules, where poll subsidies adjust to achieve budget balance, taking into account interstate migration responses to tax changes. In contrast, the traditional literature typically assumes that strategies are the entire tax policies, including poll subsidies. In this case, when one state deviates from its equilibrium tax policy, the competing states' government budgets are typically left unbalanced. Under certain regular conditions, we demonstrate that requiring budget balance (both on and off equilibrium paths) is similar to an increase in migration propensities, causing competition to be more intense, in which case there is less redistribution of income to low-income individuals.

Our model uses a utility function under which there are no income effects in the supply of labor. Although the assumption of no income effects has become common in the optimal income tax literature, it is particularly important in the current model. With income effects, changes in the poll subsidy will affect labor supplies, given the chosen marginal tax schedules. These labor supply effects will change tax revenue, complicating the determination of the budget-balancing poll subsidies. For example, suppose that country 1 redistributes more income to high-income taxpayers. Some of them move to country 2, generating more tax revenue. Country 2's budget-balancing poll subsidy rises, but that causes labor supplies to decline, assuming leisure is a normal good. The resulting revenue loss reduces the budget-balancing rise in 2's poll subsidy. These complications increase the complexity of our new approach, but they still do not justify ignoring budget balance for out-of-equilibrium moves, except maybe as a pragmatic shortcut. But the assumption of no income has been similarly justified.

An alternative method of budget-balance would be to use a public good, $G$, that is separable between private consumption and labor in the utility function (i.e., $U(c,l,G) = u(c,l) + h(G)$ ). We can then model competition between states as a Nash game in income tax policies (including the poll subsidy), with the public good used to balance the government budgets. Adjustments in the public good will induce the same pattern of migration as a change in the poll subsidy, under the assumption of identical preferences for the public good. The separability assumption implies that the public good does not affect labor supplies, regardless of whether there are income effects in the supply of labor. Thus, in the case of income effects, this approach will be simpler to analyze than when the poll subsidy is used for budget balance. Allowing public good preferences to vary with worker ability would alter the equilibrium, since this variation would now affect the distribution of movers resulting from a given change in one state's public good supply. But we already know from the literature on capital tax competition that the choice of strategy variables affects the equilibrium tax policy (see Wildasin (1988)).

Building on our new approach, we can now move on to the debate of which components of a state's tax and expenditure policies should serve as strategies in the Nash game between states, and which should be used to achieve budget balance. An important consideration will be how best to model important institutional features of the process of tax policy design.

## 8 Appendix

**Proof of Proposition 2**: We work with the relaxed program by ignoring the monotonicity constraint. Denoting $p$ the Lagrangian parameter and $\lambda(\theta)$ the co-state variable, the Hamiltonian is:

$$H(u(\theta), l(\theta), \lambda(\theta), \theta, A, p) = (u(\theta) + A) g(\theta) + p[\theta l(\theta) - u(\theta) - v(l(\theta)) - A]\eta_1(\theta, \Delta) + \lambda(\theta) v'(l(\theta)) \frac{l(\theta)}{\theta}$$

The maximization conditions are:

$$p\left[\theta - v'(l(\theta))\right]\eta_1(\theta,\Delta) = -\lambda(\theta)\left(v'(l(\theta))\frac{1}{\theta} + v''(l(\theta))\frac{l(\theta)}{\theta}\right)$$

$$-\lambda'(\theta) = g(\theta) - p\eta_1(\theta,\Delta) + p\left[\theta l(\theta) - u(\theta) - v(l(\theta)) - A\right]\frac{\partial\eta_1(\theta,\Delta)}{\partial\Delta}$$

The trasversality conditions are given by

$$\lambda(0) = 0,\ \lim_{\theta\to\infty}\lambda(\theta) = 0$$

By symmetry of the equilibrium, $u_o(\theta) = \overline{u}(\theta)$ and $A_o = \overline{A}$ (and hence $\Delta = 0$), we have

$$p_o\left[\theta - v'(l_o(\theta))\right]\eta_1(\theta,0) = -\lambda_o(\theta)\left(v'(l_o(\theta))\frac{1}{\theta} + v''(l_o(\theta))\frac{l_o(\theta)}{\theta}\right)$$

$$-\lambda_o'(\theta) = g(\theta) - p_o\eta_1(\theta,0) + p_o\left[\theta l_o(\theta) - u_o(\theta) - v(l_o(\theta)) - A_o\right]\frac{\partial\eta_1(\theta,0)}{\partial\Delta}$$

Or

$$p_o\left[\theta - v'(l_o(\theta))\right]f(\theta) = -\lambda_o(\theta)\left(v'(l_o(\theta))\frac{1}{\theta} + v''(l_o(\theta))\frac{l_o(\theta)}{\theta}\right) \tag{26}$$

$$-\lambda_o'(\theta) = g(\theta) - p_o f(\theta) + p_o\left[\theta l_o(\theta) - u_o(\theta) - v(l_o(\theta)) - A_o\right]\varepsilon(\theta)f(\theta) \tag{27}$$

Integrating (27) from 0 to $\infty$ and using the transversality conditions, we obtain:

$$0 = \int_0^\infty \{g(\theta) - p_o f(\theta) + p_o\left[\theta l_o(\theta) - u_o(\theta) - v(l_o(\theta)) - A_o\right]\varepsilon(\theta)f(\theta)\}\,d\theta$$

Or

$$p_o = \frac{1}{1 - \int_0^\infty [T_o(\theta) - A_o]\varepsilon(\theta)f(\theta)d\theta}$$

Integrating (27) from $\theta$ to $\infty$ and using the transversality conditions, we obtain:

$$\lambda_o(\theta) = \int_\theta^\infty \{g(t) - p_o f(t) + p_o[T_o(t) - A_o]\varepsilon(t)f(t)\}\,dt \tag{28}$$

Plugging (28) into (26), we have

$$p_o\left[\theta - v'(l_o(\theta))\right]f(\theta) = -\left(v'(l_o(\theta))\frac{1}{\theta} + v''(l_o(\theta))\frac{l_o(\theta)}{\theta}\right)\int_\theta^\infty \{g(t) - p_o f(t) + p_o[T_o(t) - A_o]\varepsilon(t)f(t)\}\,dt$$

Using $(1 - T_o'(\theta l_o(\theta)))\cdot l_o(\theta) = v'(l_o(\theta))\frac{l_o(\theta)}{\theta}$ and $e_o(\theta) = \frac{v'(l_o(\theta))}{v''(l_o(\theta))l_o(\theta)}$, the above equation can be rewritten as

$$p_o\left[1-\frac{v'(l_o(\theta)}{\theta})\right]\theta f(\theta)=-\frac{v'(l_o(\theta)}{\theta}\left(1+v''(l_o(\theta))\frac{l_o(\theta)}{v'(l_o(\theta)}\right)\int_\theta^\infty\{g(t)-p_of(t)+p_o[T_o(t)-A_o]\varepsilon(t)f(t)\}dt$$

Or

$$\frac{\left[1-\frac{v'(l_o(\theta)}{\theta})\right]}{\frac{v'(l_o(\theta)}{\theta}}\theta f(\theta)=\left(1+v''(l_o(\theta))\frac{l_o(\theta)}{v'(l_o(\theta)}\right)\int_\theta^\infty\left\{-\frac{1}{p_o}g(t)+f(t)-[T_o(t)-A_o]\varepsilon(t)f(t)\right\}dt$$

Or

$$\frac{T_o'(\theta)}{1-T_o'(\theta)}=\left(1+\frac{1}{e_o(\theta)}\right)\frac{1-F(\theta)}{\theta f(\theta)}\left(\frac{(1-F(\theta))-\frac{1}{p_o}(1-G(\theta))}{1-F(\theta)}-\frac{\int_\theta^\infty(T_o(t)-A_o)\varepsilon(t)f(t)dt}{(1-F(\theta))}\right)$$

**Proof of Proposition 3**: Suppose that the optimal solution to the maximization problem above exists and is given by $u_n(\theta)$ and $A_n$. Let $y(\theta)$ and $A$ denote some other admissible functions. Define $h(\theta)=u(\theta)-u_n(\theta)$ and $\delta A=A-A_n$. Then $y(\theta)=u_n(\theta)+ah(\theta)$, for some constant $a$, and $B=A_n+a\delta A$ are also admissible.

Next, define the objective function evaluated at $(y(\theta),B)$ as follows:

$$\begin{aligned}
k(a) &= \int_0^\infty\left\{(y(\theta)+B)g(\theta)+p\left[M\left(\theta,y(\theta),y'(\theta)\right)-B\right]\eta_1\left(\theta,y(\theta)-\overline{u}(\theta)+B-\overline{A}(y,B)\right)\right\}d\theta-pR \\
&= \int_0^\infty\left\{\begin{array}{c}(u_n(\theta)+ah(\theta)+A_n+a\delta A)g(\theta)\\ +p\left[M\left(\theta,u_n(\theta)+ah(\theta),u_n'(\theta)+ah'(\theta)\right)-(A_n+a\delta A)\right]\\ \cdot\eta_1\left(\theta,u_n(\theta)+ah(\theta)-\overline{u}(\theta)+(A_n+a\delta A)-\overline{A}(u_n+ah,A_n+a\delta A)\right)\end{array}\right\}d\theta-pR.
\end{aligned}$$

By the optimality of $(u_n(\theta),A_n)$, $k(a)$ achieves its maximum at $a=0$. This implies $k'(0)=0$. Differentiating $k(a)$ with respect to $a$, we have:

$$
k'(a)=\int_0^\infty\left\{\begin{array}{c}
(h(\theta)+\delta A)\,g(\theta)+p\left[\begin{array}{c}M_u\left(\theta,u_n(\theta)+ah(\theta),u_n'(\theta)+ah'(\theta)\right)h(\theta)\\ +M_{u'}\left(\theta,u_n(\theta)+ah(\theta),u_n'(\theta)+ah'(\theta)\right)h'(\theta)-\delta A\end{array}\right]\\
\cdot\eta_1\left(\theta,u_n(\theta)+ah(\theta)-\overline{u}(\theta)+(A_n+a\delta A)-\overline{A}(u_n+ah,A_n+a\delta A)\right)\\
+p\left[M\left(\theta,u_n(\theta)+ah(\theta),u_n'(\theta)+ah'(\theta)\right)-(A_n+a\delta A)\right]\\
\cdot\frac{\partial\eta_1\left(\theta,u_n(\theta)+ah(\theta)-\overline{u}(\theta)+(A_n+a\delta A)-\overline{A}(u_n+ah,A_n+a\delta A)\right)}{\partial\Delta}(h(\theta)+\delta A)\\
-\dfrac{\left[\begin{array}{c}p\int_0^\infty\left[\begin{array}{c}\left(M\left(\theta,u_n(\theta)+ah(\theta),u_n'(\theta)+ah'(\theta)\right)-(A_n+a\delta A)\right)\\ \cdot\frac{\partial\eta_1\left(\theta,u_n(\theta)+ah(\theta)-\overline{u}(\theta)+(A_n+a\delta A)-\overline{A}(u_n+ah,A_n+a\delta A)\right)}{\partial\Delta}\end{array}\right]d\theta\\ \cdot\left[M\left(\theta,\overline{u}(\theta),\overline{u}'(\theta)\right)-\overline{A}(u_n+ah,A_n+a\delta A)\right]\\ \cdot\frac{\partial\eta_2\left(\theta,u_n(\theta)+ah(\theta)-\overline{u}(\theta)+(A_n+a\delta A)-\overline{A}(u_n+ah,A_n+a\delta A)\right)}{\partial\Delta}\end{array}\right]}{\begin{array}{c}\int_0^\infty\eta_2\left(\theta,u_n(\theta)+ah(\theta)-\overline{u}(\theta)+(A_n+a\delta A)-\overline{A}(u_n+ah,A_n+a\delta A)\right)d\theta\\ +\int_0^\infty\left[\begin{array}{c}\left(M\left(\theta,\overline{u}(\theta),\overline{u}'(\theta)\right)-\overline{A}(u_n+ah,A_n+a\delta A)\right)\\ \cdot\frac{\partial\eta_2\left(\theta,u_n(\theta)+ah(\theta)-\overline{u}(\theta)+(A_n+a\delta A)-\overline{A}(u_n+ah,A_n+a\delta A)\right)}{\partial\Delta}\end{array}\right]d\theta\end{array}}h(\theta)\\
-\dfrac{\left[\begin{array}{c}p\left[M\left(\theta,u_n(\theta)+ah(\theta),u_n'(\theta)+ah'(\theta)\right)-(A_n+a\delta A)\right]\\ \cdot\frac{\partial\eta_1\left(\theta,u_n(\theta)+ah(\theta)-\overline{u}(\theta)+(A_n+a\delta A)-\overline{A}(u_n+ah,A_n+a\delta A)\right)}{\partial\Delta}\\ \cdot\int_0^\infty\left[\begin{array}{c}\left(M\left(\theta,\overline{u}(\theta),\overline{u}'(\theta)\right)-\overline{A}(u_n+ah,A_n+a\delta A)\right)\\ \cdot\frac{\partial\eta_2\left(\theta,u_n(\theta)+ah(\theta)-\overline{u}(\theta)+(A_n+a\delta A)-\overline{A}(u_n+ah,A_n+a\delta A)\right)}{\partial\Delta}\end{array}\right]d\theta\end{array}\right]}{\begin{array}{c}\int_0^\infty\eta_2\left(\theta,u_n(\theta)+ah(\theta)-\overline{u}(\theta)+(A_n+a\delta A)-\overline{A}(u_n+ah,A_n+a\delta A)\right)d\theta\\ +\int_0^\infty\left[\begin{array}{c}\left(M\left(\theta,\overline{u}(\theta),\overline{u}'(\theta)\right)-\overline{A}(u_n+ah,A_n+a\delta A)\right)\\ \cdot\frac{\partial\eta_2\left(\theta,u_n(\theta)+ah(\theta)-\overline{u}(\theta)+(A_n+a\delta A)-\overline{A}(u_n+ah,A_n+a\delta A)\right)}{\partial\Delta}\end{array}\right]d\theta\end{array}}\delta A,
\end{array}\right\}d\theta.
$$

Evaluating at $a=0$, we have

$$
\begin{aligned}
k'(0) &= \int_0^\infty \left\{ \begin{array}{c}
(h(\theta)+\delta A)\, g(\theta) + p \begin{bmatrix} M_u\left(\theta, u_n(\theta), u_n'(\theta)\right) h(\theta) \\ +M_{u'}\left(\theta, u_n(\theta), u_n'(\theta)\right) h'(\theta) - \delta A \end{bmatrix} \\
\cdot \eta_1\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(u_n, A_n)\right) \\
+p\left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \cdot \frac{\partial \eta_1\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(u_n, A_n)\right)}{\partial \Delta} (h(\theta)+\delta A) \\
-\dfrac{\begin{bmatrix} \int_0^\infty p\left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \cdot \frac{\partial \eta_1\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(u_n, A_n)\right)}{\partial \Delta} d\theta \\ \cdot \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}(u_n, A_n)\right] \frac{\partial \eta_2\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(u_n, A_n)\right)}{\partial \Delta} \end{bmatrix}}{\int_0^\infty \eta_2\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(u_n, A_n)\right) d\theta + \int_0^\infty \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}(u_n, A_n)\right] \frac{\partial \eta_2\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(u_n, A_n)\right)}{\partial \Delta} d\theta} h(\theta) \\
-\dfrac{\begin{bmatrix} p\left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \cdot \frac{\partial \eta_1\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(u_n, A_n)\right)}{\partial \Delta} \\ \cdot \int_0^\infty \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}(u_n, A_n)\right] \frac{\partial \eta_2\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(u_n, A_n)\right)}{\partial \Delta} d\theta \end{bmatrix}}{\int_0^\infty \eta_2\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(u_n, A_n)\right) d\theta + \int_0^\infty \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}(u_n, A_n)\right] \frac{\partial \eta_2\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(u_n, A_n)\right)}{\partial \Delta} d\theta} \delta A,
\end{array} \right\} d\theta, \\
&= \int_0^\infty \left\{ \begin{array}{c}
g(\theta) + p\eta_1\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(\cdot)\right) \cdot M_u\left(\theta, u_n(\theta), u_n'(\theta)\right) \\
+p\left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \cdot \frac{\partial \eta_1\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(\cdot)\right)}{\partial \Delta} \\
-\dfrac{\begin{bmatrix} \int_0^\infty p\left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \cdot \frac{\partial \eta_1\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(\cdot)\right)}{\partial \Delta} d\theta \\ \cdot \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}(\cdot)\right] \frac{\partial \eta_2\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(\cdot)\right)}{\partial \Delta} \end{bmatrix}}{\int_0^\infty \eta_2\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(\cdot)\right) d\theta + \int_0^\infty \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}(\cdot)\right] \frac{\partial \eta_2\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(\cdot)\right)}{\partial \Delta} d\theta}
\end{array} \right\} h(\theta) d\theta \\
&\quad + \int_0^\infty p\eta_1\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(\cdot)\right) \cdot M_{u'}\left(\theta, u_n(\theta), u_n'(\theta)\right) h'(\theta) d\theta \\
&\quad + \left\{ \begin{array}{c}
\int_0^\infty \left[g(\theta) - p\eta_1\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(\cdot)\right)\right] d\theta \\
+p \int_0^\infty \left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \cdot \frac{\partial \eta_1\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(\cdot)\right)}{\partial \Delta} d\theta \\
-\dfrac{\begin{bmatrix} \int_0^\infty p\left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \cdot \frac{\partial \eta_1\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(\cdot)\right)}{\partial \Delta} d\theta \\ \cdot \int_0^\infty \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}(\cdot)\right] \frac{\partial \eta_2\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(\cdot)\right)}{\partial \Delta} d\theta \end{bmatrix}}{\int_0^\infty \eta_2\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(\cdot)\right) d\theta + \int_0^\infty \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}(\cdot)\right] \frac{\partial \eta_2\left(\theta, u_n(\theta) - \overline{u}(\theta) + A_n - \overline{A}(\cdot)\right)}{\partial \Delta} d\theta}
\end{array} \right\} \delta A
\end{aligned}
$$

where $\overline{A}(\cdot) = \overline{A}(u_n(\theta), A_n)$

In the symmetric equilibrium, $\overline{u}(\theta) = u_n(\theta)$. Thus we have

$$
\begin{aligned}
k'(0) &= \int_0^\infty \left\{ \begin{array}{c} g(\theta) + p\eta_1\left(\theta, A_n - \overline{A}(\cdot)\right) \cdot M_u\left(\theta, u_n(\theta), u_n'(\theta)\right) \\ +p\left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \cdot \frac{\partial\eta_1\left(\theta, A_n - \overline{A}(\cdot)\right)}{\partial\Delta} \\ -\frac{\left[\begin{array}{c} \int_0^\infty p\left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \cdot \frac{\partial\eta_1\left(\theta, A_n - \overline{A}(\cdot)\right)}{\partial\Delta} d\theta \\ \cdot\left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - \overline{A}(\cdot)\right] \frac{\partial\eta_2\left(\theta, A_n - \overline{A}(\cdot)\right)}{\partial\Delta} \end{array}\right]}{\int_0^\infty \eta_2(\theta, 0) d\theta + \int_0^\infty \left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - \overline{A}(\cdot)\right] \frac{\partial\eta_2\left(\theta, A_n - \overline{A}(\cdot)\right)}{\partial\Delta} d\theta} \end{array} \right\} h(\theta) d\theta \\
&+ \int_0^\infty p\eta_1\left(\theta, A_n - \overline{A}(\cdot)\right) \cdot M_{u'}\left(\theta, u_n(\theta), u_n'(\theta)\right) h'(\theta) d\theta \\
&+ \left\{ \begin{array}{c} \int_0^\infty \left[g(\theta) - p\eta_1\left(\theta, A_n - \overline{A}(\cdot)\right)\right] d\theta \\ +p\int_0^\infty \left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \cdot \frac{\partial\eta_1\left(\theta, A_n - \overline{A}(\cdot)\right)}{\partial\Delta} d\theta \\ -\frac{\left[\begin{array}{c} \int_0^\infty p\left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \cdot \frac{\partial\eta_1\left(\theta, A_n - \overline{A}(\cdot)\right)}{\partial\Delta} d\theta \\ \cdot\int_0^\infty \left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - \overline{A}(\cdot)\right] \frac{\partial\eta_2\left(\theta, A_n - \overline{A}(\cdot)\right)}{\partial\Delta} d\theta \end{array}\right]}{\int_0^\infty \eta_2\left(\theta, A_n - \overline{A}(\cdot)\right) d\theta + \int_0^\infty \left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - \overline{A}(\cdot)\right] \frac{\partial\eta_2\left(\theta, A_n - \overline{A}(\cdot)\right)}{\partial\Delta} d\theta} \end{array} \right\} \delta A
\end{aligned}
$$

From the two resource constraints, we have

$$
\begin{aligned}
\int_0^\infty \left[M\left(\theta, u(\theta), u'(\theta)\right) - A\right] \eta_1(\theta, 0) d\theta &= R; \\
\int_0^\infty \left[M\left(\theta, \overline{u}(\theta), \overline{u}'(\theta)\right) - \overline{A}\right] \eta_2(\theta, 0) d\theta &= R.
\end{aligned}
$$

Solving $\left(A_n, \overline{A}(u_n(\theta), A_n)\right)$ from the above two equations, we have the solutions:

$$
\overline{A}(u_n(\theta), A_n) = A_n = \int_0^\infty M\left(\theta, u_n(\theta), u_n'(\theta)\right) dF(\theta) - R \tag{29}
$$

Substituting (29) into the expression of $k'(0)$, we have

$$
\begin{aligned}
k'(0) &= \int_0^\infty \left\{ \begin{array}{c} g(\theta) + p\eta_1(\theta, 0) \cdot M_u\left(\theta, u_n(\theta), u_n'(\theta)\right) \\ +p\left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \cdot \frac{\partial\eta_1(\theta, 0)}{\partial\Delta} \\ -\frac{\left[\begin{array}{c} \int_0^\infty p\left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \cdot \frac{\partial\eta_1(\theta, 0)}{\partial\Delta} d\theta \\ \cdot\left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \frac{\partial\eta_2(\theta, 0)}{\partial\Delta} \end{array}\right]}{\int_0^\infty \eta_2(\theta, 0) d\theta + \int_0^\infty \left[M\left(\theta, u_n(\theta), u_n'(\theta)\right) - A_n\right] \frac{\partial\eta_2(\theta, 0)}{\partial\Delta} d\theta} \end{array} \right\} h(\theta) d\theta \\
&+ \int_0^\infty p\eta_1(\theta, 0) \cdot M_{u'}\left(\theta, u_n(\theta), u_n'(\theta)\right) h'(\theta) d\theta
\end{aligned}
$$

$$+\left\{\begin{array}{c}\int_0^\infty\left[g(\theta)-p\eta_1(\theta,0)\right]d\theta\\+p\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\cdot\frac{\partial\eta_1(\theta,0)}{\partial\Delta}d\theta\\-\frac{\left[\begin{array}{c}\int_0^\infty p\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\cdot\frac{\partial\eta_1(\theta,0)}{\partial\Delta}d\theta\\\cdot\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\frac{\partial\eta_2(\theta,0)}{\partial\Delta}d\theta\end{array}\right]}{\int_0^\infty\eta_2(\theta,0)d\theta+\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\frac{\partial\eta_2(\theta,0)}{\partial\Delta}d\theta}\end{array}\right\}\delta A$$

Or

$$\begin{aligned}k'(0) &= p\lim_{\overline{\theta}\to\infty}\left[M_{u'}\left(\theta,u_n(\theta),u_n'(\theta)\right)f(\theta)h(\theta)\right]_0^{\overline{\theta}}\\&+\int_0^\infty\left\{\begin{array}{c}g(\theta)+p\cdot\left[M_u\left(\theta,u_n(\theta),u_n'(\theta)\right)-\frac{d\{M_{u'}(\theta,u_n(\theta),u_n'(\theta))f(\theta)\}}{f(\theta)d\theta}\right]f(\theta)\\+p\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\cdot\varepsilon(\theta)f(\theta)\\+\frac{\left[\begin{array}{c}p\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta\\\cdot\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)\end{array}\right]}{1-\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta}\end{array}\right\}h(\theta)d\theta\\&+\left\{\begin{array}{c}\int_0^\infty[g(\theta)-pf(\theta)]d\theta\\+p\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta\\+\frac{\left[\begin{array}{c}p\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta\\\cdot\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta\end{array}\right]}{1-\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta}\end{array}\right\}\delta A.\end{aligned}$$

Since $\delta A$ is arbitrary, we have

$$\left\{\begin{array}{c}\int_0^\infty[g(\theta)-pf(\theta)]d\theta\\+p\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta\\+\frac{\left[\begin{array}{c}p\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta\\\cdot\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta\end{array}\right]}{1-\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta}\end{array}\right\}=0,$$

which implies

$$p=\frac{1}{1-\frac{\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta}{1-\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta}}.\tag{30}$$

The function $h(\cdot)$ is an arbitrary admissible deviation. This first implies the following transversality conditions:

$$\lim_{\overline{\theta}\to\infty}\left[M_{u'}\left(\overline{\theta},u_n(\overline{\theta}),u_n'(\overline{\theta})\right)f\left(\overline{\theta}\right)\right] = 0;\tag{31}$$

$$M_{u'}\left(0,u_n(0),u_n'(0)\right)f(0) = 0.\tag{32}$$

That $h(\cdot)$ is an arbitrary admissible deviation implies the following:

$$\begin{aligned}
&\frac{1}{p}g(\theta)+\left[M_u\left(\theta,u_n(\theta),u_n^{'}(\theta)\right)-\frac{d\{M_{u'}\left(\theta,u_n(\theta),u_n^{'}(\theta)\right)f(\theta)\}}{f(\theta)d\theta}\right]f(\theta)\\
=&\quad\frac{\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)}{\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta-1},
\end{aligned}$$

which further implies

$$\begin{aligned}
\frac{d\{M_{u'}\left(\theta,u_n(\theta),u_n^{'}(\theta)\right)f(\theta)\}}{d\theta} &= \frac{1}{p}g(\theta)+M_u\left(\theta,u_n(\theta),u_n^{'}(\theta)\right)f(\theta)\\
&\quad-\frac{\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)}{\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta-1};\\
-M_{u'}\left(\theta,u_n(\theta),u_n^{'}(\theta)\right)f(\theta) &= \int_\theta^\infty\{\frac{1}{p}g(t)+M_u\left(t,u_n(t),u_n'(t)\right)f(t)\}dt\\
&\quad-\int_\theta^\infty\frac{\left[M\left(t,u_n(t),u_n'(t)\right)-A_n\right]\varepsilon(t)f(t)}{\int_0^\infty\left[M\left(\theta,u_n(\theta),u_n'(\theta)\right)-A_n\right]\varepsilon(\theta)f(\theta)d\theta-1}dt;\\
M_{u'}\left(\theta,u_n(\theta),u_n^{'}(\theta)\right) &= -\frac{\frac{1}{p}(1-G(\theta))-(1-F(\theta))-\frac{\int_\theta^\infty[M(t,u_n(t),u_n'(t))-A_n]\varepsilon(t)f(t)dt}{\int_0^\infty[M(\theta,u_n(\theta),u_n'(\theta))-A_n]\varepsilon(\theta)f(\theta)d\theta-1}}{f(\theta)}\\
&= \frac{(1-F(\theta))-\frac{1}{p}(1-G(\theta))+\frac{\int_\theta^\infty[M(t,u_n(t),u_n'(t))-A_n]\varepsilon(t)f(t)dt}{\int_0^\infty[M(\theta,u_n(\theta),u_n'(\theta))-A_n]\varepsilon(\theta)f(\theta)d\theta-1}}{f(\theta)}. \qquad (33)
\end{aligned}$$

Since $\{\theta\xi\left(u_n'(\theta),\theta\right)-u_n(\theta)-v(\xi\left(u_n'(\theta),\theta\right))\}=M\left(\theta,u_n(\theta),u_n^{'}(\theta)\right)$, we have $M_u\left(\theta,u_n(\theta),u_n^{'}(\theta)\right)=-1$, and

$$\begin{aligned}
M_{u'}\left(\theta,u_n(\theta),u_n^{'}(\theta)\right) &= \{\theta-v'(\xi[u_n'(\theta),\theta])\}\xi_{u_n'(\theta)}[u_n'(\theta),\theta]\\
&= \{\theta-v'(\xi[u_n'(\theta),\theta])\}\frac{\theta}{v'\{\xi[u_n'(\theta),\theta]\}+v''\{\xi[u_n'(\theta),\theta]\}\xi[u_n'(\theta),\theta]}\\
&= \{\frac{\theta}{v'\{\xi[u_n'(\theta),\theta]\}}-1\}\frac{\theta}{1+\frac{v''\{\xi[u_n'(\theta),\theta]\}\xi[u_n'(\theta),\theta]}{v'\{\xi[u_n'(\theta),\theta]\}}}.
\end{aligned}$$

Define

$$T_n(\theta)=M\left(\theta,u_n(\theta),u_n^{'}(\theta)\right).$$

With this notation, we have $l_n(\theta)=\xi[u_n'(\theta)]$ and $v'\left(l_n(\theta)\right)\frac{l_n(\theta)}{\theta}=(1-T'\left(\theta l_n(\theta)\right))\cdot l_n(\theta)$. Let $e_n\left(\theta\right)=\frac{v'[l_n(\theta)]}{v''[l_n(\theta)]l_n(\theta)}$. Then

$$M_{x'}\left(\theta,x_n(\theta),x_n'(\theta)\right)=\frac{T'\left(\theta l_n(\theta)\right)}{1-T'\left(\theta l_n(\theta)\right)}\frac{\theta}{1+e_n^{-1}(\theta)}.$$

Equation (33) implies:

$$\frac{T'(\theta l_n(\theta))}{1-T'(\theta l_n(\theta))}\frac{\theta}{1+e_n^{-1}(\theta)} = \frac{(1-F(\theta))-\frac{1}{p_n}(1-G(\theta))+\frac{\int_\theta^\infty [T_n(t)-A_n]\varepsilon(t)f(t)dt}{\int_0^\infty [T_n(t)-A_n]\varepsilon(t)f(t)dt-1}}{f(\theta)}.$$

So

$$\begin{aligned}\frac{T_n'(\theta)}{1-T_n'(\theta)} &= \left(1+\frac{1}{e_n(\theta)}\right)\frac{1-F(\theta)}{\theta f(\theta)}\\ &\quad \cdot\left(\frac{(1-F(\theta))-\frac{1}{p_n}(1-G(\theta))}{1-F(\theta)} - \frac{\int_\theta^\infty [T_n(t)-A_n]\varepsilon(t)f(t)dt}{\left(1-\int_0^\infty [T_n(t)-A_n]\varepsilon(t)f(t)dt\right)(1-F(\theta))}\right),\end{aligned}$$

where

$$\begin{aligned} T'(\theta l_n(\theta)) &= T_n'(\theta), e_n(\theta)=\frac{v'[l_n(\theta)]}{v''[l_n(\theta)]l_n(\theta)};\\ p_n &= \frac{1}{1-\frac{\int_0^\infty [T_n(t)-A_n]\varepsilon(t)f(t)dt}{1-\int_0^\infty [T_n(t)-A_n]\varepsilon(t)f(t)dt}};\\ A_n &= \int_0^\infty T_n(t)f(t)dt-R.\end{aligned}$$

**Proof of Lemma 2:** Define

$$\Omega_i(\theta)=\alpha_i\frac{1-\frac{1}{p_i}\frac{g(\theta)}{f(\theta)}}{\varepsilon(\theta)},$$

where $\alpha_i$ and $p_i$ are given previously. We can rewrite (21) into

$$\frac{T_i'(\theta)}{1-T_i'(\theta)}=\left(1+\frac{1}{e_i(\theta)}\right)\frac{1}{\alpha_i\theta f(\theta)}\int_\theta^\infty [\Omega_i(t)-(T_i(t)-A_i)]\varepsilon(t)f(t)dt.$$

By adapting the proof of Proposition 3 in Lehmann et al.,[7] we can show that if $d\Omega_i(\theta)/d\theta \geq 0$ then $T_i'(\theta)>0$.

Since $d\left(\frac{g(\theta)}{f(\theta)}\right)/d\theta \leq 0$ and $p_i>0$, we have $d\log\left[1-\frac{1}{p_i}\frac{g(\theta)}{f(\theta)}\right]/d\theta \geq 0$. Since $d\log\varepsilon(\theta)/d\theta \leq M_i(\theta)$, by the definition of $M_i(\theta)$ we have $M_i(\theta)\geq 0$ and $d\log\left[\frac{1-\frac{1}{p_i}\frac{g(\theta)}{f(\theta)}}{\varepsilon(\theta)}\right]/d\theta \geq 0$. Hence $d\Omega_i(\theta)/d\theta \geq 0$ (as $\alpha_i>0$). Using the result mentioned above, we have $T_i'(\theta)>0$, and hence $d(T_i(\theta)-A_i)/d\theta>0$. Since $d\log\varepsilon(\theta)/d\theta \geq m_i(\theta)$, by the definition of $m_i(\theta)$ we have $m_i(\theta)\leq 0$, $d\log[(T_i(\theta)-A_i)\varepsilon(\theta)]/d\theta>0$, and $\int_0^\infty (T_i(\theta)-A_i)\varepsilon(\theta)dF(\theta)>0$. Given that $[(T_i(\theta)-A_i)\varepsilon(\theta)]$ is strictly increasing, it is easily verified that Property (22) holds.

[7]The proof is tedious, which is available upon request.

**Proof of Lemma 3:** Let $\beta_i = \lim_{\theta\to\infty}\left(1+\frac{1}{e_i(\theta)}\right)\frac{1-F(\theta)}{\theta f(\theta)}$. Then

$$\begin{aligned}
\lim_{\theta\to\infty}\frac{T_i'(\theta)}{1-T_i'(\theta)} &= \lim_{\theta\to\infty}\left(1+\frac{1}{e_i(\theta)}\right)\frac{1-F(\theta)}{\theta f(\theta)} \\
&\quad\cdot\left(\frac{G(\theta)-F(\theta)}{1-F(\theta)}+\left(1-\frac{1}{p_i}\right)\left[\frac{1-G(\theta)}{1-F(\theta)}-\frac{\int_\theta^\infty [T_i(t)-A_i]\varepsilon(t)\frac{f(t)}{1-F(\theta)}dt}{\int_0^\infty [T_i(t)-A_i]\varepsilon(t)f(t)\,dt}\right]\right) \\
&= \beta_i\left(1+\left(1-\frac{1}{p_i}\right)\lim_{\theta\to\infty}\left[\frac{1-G(\theta)}{1-F(\theta)}-\frac{\int_\theta^\infty [T_i(t)-A_i]\varepsilon(t)\frac{f(t)}{1-F(\theta)}dt}{\int_0^\infty [T_i(t)-A_i]\varepsilon(t)f(t)\,dt}\right]\right) \\
&= \beta_i\left(1-\left(1-\frac{1}{p_i}\right)\lim_{\theta\to\infty}\frac{\int_\theta^\infty [T_i(t)-A_i]\varepsilon(t)\frac{f(t)}{1-F(\theta)}dt}{\int_0^\infty [T_i(t)-A_i]\varepsilon(t)f(t)\,dt}\right) \\
&= \beta_i\left[1-\frac{1-\frac{1}{p_i}}{\int_0^\infty [T_i(t)-A_i]\varepsilon(t)f(t)\,dt}\lim_{\theta\to\infty}\int_\theta^\infty [T_i(t)-A_i]\varepsilon(t)\frac{f(t)}{1-F(\theta)}dt\right] \\
&= \beta_i\left[1-\frac{1-\frac{1}{p_i}}{\int_0^\infty [T_i(t)-A_i]\varepsilon(t)f(t)\,dt}\lim_{\theta\to\infty}(T_i(\theta)-A_i)\varepsilon(\theta)\right] \qquad (34)
\end{aligned}$$

Given the expressions of $p_o$ and $p_n$, and the Properties (17) and (22), it is easily verified that

$$\frac{1-\frac{1}{p_i}}{\int_0^\infty [T_i(t)-A_i]\varepsilon(t)f(t)\,dt} > 0$$

If $T_i'(\theta)$ stays bounded above some positive number as income goes to infinity with $\theta$, then tax payments for individuals with ability $\theta$, $T_i(\theta)$, will go to infinity. But then the RHS of (34) must be negative, a contradiction. This shows that $\lim_{\theta\to\infty} T_i'(\theta) \leq 0$. Symmetric reasoning rules out the case $\lim_{\theta\to\infty} T_i'(\theta) < 0$. We thus have $\lim_{\theta\to\infty} T_i'(\theta) = 0$.

**Proof of Lemma 4:** Using the finding that the asymptotic marginal tax rate is zero, (34) and the formulas for $p_o$ and $p_n$ give

$$\lim_{\theta\to\infty}[(T_o(\theta)-A_o)\varepsilon(\theta)] = 1$$

and

$$\lim_{\theta\to\infty}[T_n(\theta)-A_n]\varepsilon(\theta) = 1-\int_0^\infty [T_n(t)-A_n]\varepsilon(t)f(t)\,dt$$

Thus,

$$\lim_{\theta\to\infty}[(T_n(\theta)-A_n)\varepsilon(\theta)] < \lim_{\theta\to\infty}[(T_o(\theta)-A_o)\varepsilon(\theta)].$$

Under the assumption that $\varepsilon(\theta)$ converges to a positive number as $\theta$ goes to infinity, this inequality gives

$$\lim_{\theta\to\infty}[T_n(\theta)-A_n] < \lim_{\theta\to\infty}[T_o(\theta)-A_o], \qquad (35)$$

which can be used to conclude that $AT_n(\theta^*) < AT_o(\theta^*)$ for sufficiently high $\theta^*$.

**Proof of Proposition 5:** First, by Lemma 4, there exists $\theta^*$ sufficiently large so that $AT_n(\theta^*) < AT_o(\theta^*)$. We proceed in the following steps:

**Step 1:** By (35), there exists $\theta' > \theta^*$ s.t. $T_n(\theta) - A_n < T_o(\theta) - A_o$ for all $\theta \geq \theta'$.

**Step 2:** $A_n < A_o$, i.e., the equilibrium poll subsidy is lower under the new approach.

From the tax rule (19), we have

$$\frac{T_i'(\theta)}{1-T_i'(\theta)} = \left(1+\frac{1}{e_i(\theta)}\right)\left(\frac{G(\theta)-F(\theta)}{\theta f(\theta)} + \frac{1}{\alpha_i}\frac{(1-G(\theta))\int_0^\infty [T_i(t)-A_i]\varepsilon(t)f(t)dt - \int_\theta^\infty [T_i(t)-A_i]\varepsilon(t)f(t)dt}{\theta f(\theta)}\right),$$

where $\alpha_o = 1$ and $\alpha_n = 1 - \int_0^\infty [T_n(t)-A_n]\varepsilon(t)f(t)dt < 1$.

Define function $\Phi(z) \equiv \frac{z}{1-z}$ over $(0,1)$, and $\Psi_i$:

$$\Psi_i(\theta, T(\theta)) \equiv \left(1+\frac{1}{e(\theta)}\right)\left(\frac{G(\theta)-F(\theta)}{\theta f(\theta)} + \frac{1}{\alpha_i}\frac{(1-G(\theta))\int_0^\infty [T(t)-A]\varepsilon(t)f(t)dt - \int_\theta^\infty [T(t)-A]\varepsilon(t)f(t)dt}{\theta f(\theta)}\right)$$

over $[0,\theta^*] \times \left[0,\overline{T}\right]$, where $\overline{T}$ is an upper bound of $T_o(\theta^*)$ and $T_n(\theta^*)$, $A = \int_0^\infty T(t)f(t)dt$, $e(\theta) = \frac{v'(l(\theta))}{v''(l(\theta))l(\theta)}$, and $i = o,n$.

Since $\Phi$ is monotone increasing, $\Phi^{-1}$ over $(0,1/2)$ is also monotone increasing.

Since $G(\theta) \geqslant F(\theta)$, Property (22) implies

$$(1-G(\theta))\int_0^\infty [T_i(t)-A_i]\varepsilon(t)f(t)dt - \int_\theta^\infty [T_i(t)-A_i]\varepsilon(t)f(t)dt < 0 \text{ for } \theta > 0.$$

We thus have $\Psi_n(\theta, T(\theta)) < \Psi_o(\theta, T(\theta))$ for $\theta \in (0,\theta^*]$ (since $\alpha_n < \alpha_o$).

Define $\Gamma_i(\theta, T(\theta)) \equiv \Phi^{-1}(\Psi_i(\theta, T(\theta)))$ over $[0,\theta^*] \times \left[0,\overline{T}\right]$. Then $T_n'(\theta) = \Gamma_n(\theta, T_n(\theta))$, $T_o'(\theta) = \Gamma_o(\theta, T_o(\theta))$.

Since $\Phi^{-1}$ is monotone increasing, we have

$$\Gamma_n(\theta, T(\theta)) < \Gamma_o(\theta, T(\theta)) \text{ for } \theta \in \left(0,\theta^*\right].$$

It can be verified that $\Gamma_i$ is continuously differentiable in the bounded closed convex domain $D = [0,\theta^*] \times \left[0,\overline{T}\right]$. By Lemma 1 (Birkhoff and Rota,1989, pp. 26), $\Gamma_i$ satisfies the Lipschitz condition (with $L = \sup_D |\partial\Gamma_i/\partial T(\theta)|$).

We also have $T_n(0) = T_o(0) = 0$. We can thus invoke a variant of Theorem 8 in Birkhoff and Rota (page 30, the comparison theorem) to conclude that $T_n(\theta) < T_o(\theta)$ for $\theta \in (0,\theta^*]$, which also implies

$$\int_0^{\theta^*} T_n(\theta) dF(\theta) < \int_0^{\theta^*} T_o(\theta) dF(\theta). \tag{36}$$

Given $AT_n(\theta^*) < AT_o(\theta^*)$, i.e.,

$$\int_{\theta^*}^{\infty} [T_n(t) - A_n] \frac{f(t)}{1-F(\theta^*)} dt < \int_{\theta^*}^{\infty} [T_o(t) - A_o] \frac{f(t)}{1-F(\theta^*)} dt,$$

which implies

$$F(\theta^*) \int_{\theta^*}^{\infty} T_n(\theta) dF(\theta) - \left(1 - F(\theta^*)\right) \int_0^{\theta^*} T_n(\theta) dF(\theta) < F(\theta^*) \int_{\theta^*}^{\infty} T_o(\theta) dF(\theta) - \left(1 - F(\theta^*)\right) \int_0^{\theta^*} T_o(\theta) dF(\theta) \tag{37}$$

(36) and (37) imply

$$\int_0^{\infty} T_n(\theta) dF(\theta) < \int_0^{\infty} T_o(\theta) dF(\theta),$$

which in turn implies $A_n < A_0$.

**Step 3:** $\exists \widehat{\theta} \in \left(0,\theta'\right)$ s.t. $T_n(\widehat{\theta}) - A_n = T_o(\widehat{\theta}) - A_o$.

Since $T_n(0) = T_o(0) = 0$, we also have $T_n(0) - A_n > T_o(0) - A_o$ by Step 2. Combining this with Step 1, there exists $\widehat{\theta} \in \left(0,\theta'\right)$ s.t. $T_n(\widehat{\theta}) - A_n = T_o(\widehat{\theta}) - A_o$ (by the continuity of function $T_i(\theta) - A_i$).

**Step 4:** $\forall \theta'' > \theta'$, $T_n(\theta) - A_n > T_o(\theta) - A_o$ if $\theta \in [0,\widehat{\theta})$ and $T_n(\theta) - A_n < T_o(\theta) - A_o$ if $\theta \in (\widehat{\theta},\theta'']$.

Let $w_i(\theta) \equiv T_i(\theta) - A_i$, i.e., $w_i(\theta)$ is the net tax from type-$\theta$ agent, $i = o,n$.

The tax rule (19) can be rewritten as

$$\frac{w_i'(\theta)}{1 - w_i'(\theta)} = \left(1 + \frac{1}{e_i(\theta)}\right)\left(\frac{G(\theta) - F(\theta)}{\theta f(\theta)} + \frac{1}{\alpha_i} \frac{(1 - G(\theta)) \int_0^{\infty} w_i(t)\varepsilon(t) f(t) dt - \int_{\theta}^{\infty} w_i(t)\varepsilon(t) f(t) dt}{\theta f(\theta)}\right),$$

where $\alpha_o = 1$, $\alpha_n = 1 - \int_0^{\infty} w_n(t)\varepsilon(t) f(t) dt < 1$.

Define $\Upsilon_i$:

$$\Upsilon_i(\theta, w(\theta)) \equiv \left(1 + \frac{1}{e(\theta)}\right)\left(\frac{G(\theta) - F(\theta)}{\theta f(\theta)} + \frac{1}{\alpha_i} \frac{(1 - G(\theta)) \int_0^{\infty} w(t)\varepsilon(t) f(t) dt - \int_{\theta}^{\infty} w(t)\varepsilon(t) f(t) dt}{\theta f(\theta)}\right),$$

over $\left[0,\theta''\right] \times \left[\underline{w},\overline{w}\right]$ where $\underline{w}$ is a lower bound of $w_o(0)$ and $w_n(0)$ and $\overline{w}$ is an upper bound of $w_o(\theta'')$ and $w_n(\theta'')$, and $e(\theta) = \frac{v'(l(\theta))}{v''(l(\theta))l(\theta)}$.

Also define $\Lambda_i(\theta, w(\theta)) \equiv \Phi^{-1}(\Upsilon_i(\theta, w(\theta)))$ over $\left[0,\theta''\right] \times \left[\underline{w},\overline{w}\right]$. Then $w_i'(\theta) = \Lambda_i(\theta, w_i(\theta))$, $i = o,n$.

Following similar arguments in Step 2, we have $\Lambda_n(\theta, T(\theta)) < \Lambda_o(\theta, T(\theta))$ for $\theta \in (0,\theta'']$. It can be verified that $\Lambda_i$ is continuously differentiable in the bounded closed convex domain $D' = \left[0,\theta''\right] \times \left[\underline{w},\overline{w}\right]$.

Invoking Lemma 1 (Birkhoff and Rota,1989, pp. 26) again, $\Lambda_i$ satisfies the Lipschitz condition (with $L = \sup_{D'} |\partial\Lambda_i/\partial w(\theta)|$). From Step 3, we have $w_n(\widehat{\theta}) = w_o(\widehat{\theta}) = 0$. By a variant of Theorem 8 in Birkhoff and Rota (page 30) again, we can conclude that $w_n(\theta) > w_o(\theta)$ when $\theta \in [0, \widehat{\theta})$ and $w_n(\theta) < w_o(\theta)$ when $\theta \in (\widehat{\theta}, \theta'']$; Since $\theta''$ is arbitrarily given and $\theta'' > \theta'$, we have $T_n(\theta) - A_n > T_o(\theta) - A_o$ when $\theta \in [0, \widehat{\theta})$ and $T_n(\theta) - A_n < T_o(\theta) - A_o$ when $\theta \in (\widehat{\theta}, +\infty)$.

## References


[1] Bassetto, M., “A Game-Theoretical View of The Fiscal Theory of The Price Level,” Econometrica 70 (6), 2002: 2167–2195.

[2] Bassetto, M., “Equilibrium and government commitment,” J. Econ. Theory 124 (1), 2005: 79–105.

[3] Bierbrauer, Felix, Craig Brett, and John A. Weymark, “Strategic Nonlinear Income Tax Competition with Perfect Labor Mobility,” Games and Economic Behavior 82, 2013: 292-311.

[4] Birkhoff, Garrett and Gian-Carlo Rota, Ordinary Differential Equations, New York: John Wiley & Sons, 1989.

[5] Buiter, W., “The fiscal theory of the price level: A critique,” Econ. J. 112 (481), 2002: 459–480.

[6] Cochrane, J., “A frictionless view of U. S. inflation,” In: NBER Macroeconomics Annual 1998.

[7] Gordon, Roger, and Julie Cullen, “Income Distribution in a Federal System of Governments,” Journal of Public Economics 96, 2012: 1100-1109.

[8] Gruber, Jonathan and Emmanuel Saez, “The Elasticity of Taxable Income: Evidence and Implications,” Journal of Public Economics 84, 2002: 1-32.

[9] Leeper, E., “Equilibria under active and passive monetary policies,” J. Monet. Econ. 27, 1991: 129–147.

[10] Lehmann, Etienne, Laurent Simula, and Alain Trannoy, “Tax Me If You Can! Optimal Nonlinear Income Tax Between Competing Governments,” forthcoming, Quarterly Journal of Economics.

[11] Lipatov, Vilen, and Alfons Weichenrieder, “Optimal Income Taxation with Tax Competition,” Working Paper No. 3108, CESifo, 2012.

[12] Luenberger, David G. Optimization by Vector Space Methods, New York: John Wiley & Sons, 1969.

[13] Mirrlees, James, “Migration and Optimal Income Taxes,” Journal of Public Economics 18, 1982: 319-341.

[14] Morelli, Massimo, Huanxing Yang, and Lixin Ye, “Competitive Nonlinear Taxation and Constitutional Choice,” American Economic Journal: Microeconomics 4(1), 2012: 142-175.

[15] Niepelt, D., “The fiscal myth of the price level,” Q. J. Econ. 119 (1), 2004: 277–300.

[16] Piaser, Gwenael, “Labor Mobility and Income Tax Competition,” In: Gregoriou, G., Read, C. (Eds.), International Taxation Handbook: Policy, Practice, Standards and Regulation. CIMA Publishing, Oxford, 2007: 73–94.

[17] Salanie, Bernard, “The Economics of Contracts,” 2nd edition, 2005, The MIT Press.

[18] Sims, C., “A simple model for study of the price level and the interaction of monetary and fiscal policy,” J. Econ. Theory 4, 1994: 381–399.

[19] Wilson, John D., “Income Taxation and Skilled Migration: The Analytical Issues,” In: Bhagwati, J., Hanson, G. (Eds.), Skilled Immigration Today: Prospects, Problems, and Policies. Oxford University Press, Oxford, 2009: 285-314.

[20] Wilson, John D., “Tax Competition in a Federation,” In: Brosio, G., Ahmad, E. (Eds.), Handbook on Fiscal Federalism II. Edward Elgar, forthcoming.

[21] Wildasin, David E., “Nash Equilibria in Models of Fiscal Competition,” Journal of Public Economics 35, 1988: 393-421.

[22] Woodford, M., “Monetary policy and price level determinacy in a cash-in-advance economy,” J. Econ. Theory 4, 1994: 345–380.

[23] Woodford, M., “Price level determinacy without control of a monetary aggregate,” Carnegie Rochester Conference Series on Public Policy 43, 1995: 1–46.