Efficient and Optimal Mechanisms with Private Information Acquisition Costs*

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Abstract

In auctions with private information acquisition costs, we completely characterize (socially) efficient and (revenue) optimal two-stage mechanisms, with the first stage being an entry right allocation mechanism and the second stage being a traditional private good provision mechanism. Both efficiency and revenue optimality require that the second-stage selling mechanism be *ex post* efficient and the number of entry slots be endogenously determined. We show that both efficient and optimal entry can be truthfully implemented in dominant strategies, and can also be implemented via all-pay, though not uniform-price or discriminatory-price, auctions.

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1 Introduction

The earlier literature on optimal auction design and revenue comparison generally assumes that there is an exogenously specified set of bidders and that these bidders are endowed with information about

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the object's valuations.[1] By taking into account endogenous entry, the nature of optimal auctions changes dramatically.[2] For example, in a standard symmetric independent private value setting with a homogeneous and publicly known information acquisition cost, McAfee and McMillan [11] and Levin and Smith [8] show that the optimal auction is characterized by a standard auction without imposing a distortionary reserve price (which is strictly higher than the seller's own valuation). This is in stark contrast with the optimal auctions characterized by Myerson [17].

Most recently, Lu [10] and Moreno and Wooders [16] extend the analysis of McAfee and McMillan [11] and Levin and Smith [8] to a setting where information acquisition costs are heterogeneous and privately known to the bidders. This setting is not just a theoretical extension, but is also more relevant in many real world settings. For example, in some complex and high-valued asset sales, many aspects of pre-bid information acquisition and analyzation are privately known to bidders (Vallen and Bullinger [22]). More specifically, Lu [10] and Moreno and Wooders [16] consider endogenous entry, in which potential bidders decide whether to enter the auction (and incur information acquisition costs) independently and simultaneously. They characterize threshold entry equilibria in which each bidder enters the auction if and only if her entry cost is lower than some endogenously determined entry threshold.

In this research we take one step further to study the (socially) efficient and (revenue) optimal mechanisms. While Lu [10] and Moreno and Wooders [16] consider endogenous entry where the seller does not exercise entry control, we follow the optimal mechanism design approach to allow for entry screening through an entry right allocation mechanism. In effect, we consider two-stage mechanisms with the first stage being the entry right allocation mechanism and the second stage being the (standard) private good provision mechanism.

In the traditional mechanism design setting where agents are passively endowed with private information about their types, the analysis usually focuses on optimal elicitation of private information. When costly entry is taken into account and the information acquisition costs are privately known, mechanism design has to additionally take into account information elicitation at the information acquisition stage. Mechanism design in such a setting is thus potentially challenging as it has to balance information acquisition and information elicitation, which are interdependent: the second-stage selling mechanism has a direct effect on how effectively an (socially) efficient or (revenue) optimal set of

[1] See, e.g., Vickrey [23], Riley and Samuelson [18], Myerson [17], and Milgrom and Weber [15].

[2] See, for example, McAfee and McMillan [11], Tan [21], Engelbrecht-Wiggans [5], Levin and Smith [8], Stegeman [20], Ye [24, 25]), and Lu [9]. Also see Bergemann and Välimäki [2] for a very insightful survey of this growing literature.

entrants can be induced, and the first-stage entry right allocation mechanism will in turn determine whether the final outcome can be socially efficient or revenue maximizing.

Nevertheless, we are able to completely characterize efficient and optimal two-stage mechanisms in our setting. We demonstrate that, for both efficiency and optimality, the second-stage selling mechanism must be *ex post* efficient (so, in particular, no distortionary reserve price is warranted), and the first-stage entry mechanism should admit bidders in ascending order of their information acquisition costs with the number of entry slots being endogenously determined: while the efficient entry rule maximizes the expected total surplus, the optimal entry rule maximizes the expected "virtual" total surplus, which is the total surplus adjusted for the information rent. It turns out that in our setting, efficient and optimal entry can both be implemented in dominant strategies. In particular, efficient entry can be implemented using the VCG (Vickrey-Clarke-Groves) payment rule. Finally, we show that efficient and optimal entry can both be implemented via all-pay auctions. The basic idea is that, under all-pay auctions (with appropriately set reserve prices), a profile of entry costs can be "recovered" from the equilibrium bids, based on which the efficient or optimal entry allocation rule can be implemented. Somewhat surprisingly, efficient and optimal entry cannot be implemented via either uniform-price or discriminatory-price auctions.

By focusing on entry right allocation mechanisms, our research is also closely related to Fullerton and McAfee [6], who introduce auctions for entry rights to select shortlisted contestants for the final tournament.[3] Our current approach differs from theirs in the way the set of finalists is determined: while in their approach the number of finalists to be selected is fixed and pre-announced,[4] in our entry right allocation mechanism the selection of shortlisted bidders is contingent on the reported bid profile; hence, the number of finalists is endogenously determined. For this reason the entry right allocation mechanism examined in this research is more general. In fact, it resemble multi-unit auctions with endogenously determined supply (see, e.g., McAdams [12]).

The paper is organized as follows. Section 2 lays out the model. Section 3 characterizes the socially efficient mechanisms. Section 4 characterizes the revenue-maximizing (optimal) mechanisms. Section 5 concludes.

[3] Ye [25] extends their approach to study a setting of two-stage auctions.

[4] In Fullerton and McAfee [6], they first show that for a large class of tournaments, the optimal number of contestants is two. They then fix it as the number of entry rights to be auctioned.

2 The Model

There are N potential bidders interested in a single item. Let $\mathbf{N} = \{1, 2, ..., N\}$ denote the set of all the potential bidders and $2^{\mathbf{N}}$ denote the collection of all the subsets (subgroups) in $\mathbf{N}$. The seller's own valuation for the item is v_0. Bidder i has to incur an information acquisition (or, simply put, entry) cost of c_i in order to discover her value v_i for the item. We assume that both c_i and v_i are private information to bidder i. Ex ante, c_i follows distribution $G(\cdot)$ with strictly positive density $g(\cdot)$ on support $[\underline{c}, \overline{c}]$, and v_i follows distribution $F(\cdot)$ with a density $f(\cdot)$ over support $[\underline{v}, \overline{v}]$.

The values of N, v_0, and distributions $F(\cdot)$ and $G(\cdot)$ are all common knowledge. We assume mutual independence across i for both c_i and v_i. The seller and bidders are assumed to be risk neutral.

We consider a general mechanism design framework in which the seller also exercises entry control, so that the mechanism is conducted in two stages. The first stage is the entry right allocation mechanism, and the second stage is the private good provision mechanism. A bidder can only acquire information to learn her value once an entry right is granted.

In the first stage, given a profile of reports on their private entry costs, $\mathbf{c} = (\mathbf{c}) = (c_1, c_2, ..., c_N)$, the mechanism specifies the entry right allocation rule and payment rule: the entry rule specifies probabilities with which any given subset of bidders is admitted, $\mathbf{p} = (p^g(\mathbf{c}))$, $\forall g \in 2^{\mathbf{N}}$, and the payment rule $\mathbf{x} = (x_i(\mathbf{c}))$ specifies the payment $x_i(\mathbf{c})$, $i = 1, ..., N$, that each bidder needs to pay.

We assume that bidders can only learn their values after being admitted, and once admitted, an entrant bidder must incur her entry cost before she can participate in the second-stage selling mechanism.[5]

In the second stage, a private good provision mechanism (Ω) is conducted, which is typically a standard auction in which the object is allocated to the bidder who submits the highest bid (provided that her bid meets the reserve price).[6] Suppose $g \in 2^{\mathbf{N}}$ is the set of entrants who are granted the entry rights with type profile $\mathbf{c}^g$. Since entrants are *ex ante* symmetric in terms of their private values, the expected profit to each entrant is the same under a standard auction. Note that in our setting, conditional on entry, the expected payoffs to the seller and entrant bidders depend on g only through

[5] This is the case when the "due diligence" (information acquisition) process, including the access to "data room" or "information room", is highly controlled or closely monitored by the auctioneer (Ye [25]). As we will demonstrate, the equilibrium expected payoff (net of entry cost) for each admitted bidder is greater than zero. Thus every admitted bidder will indeed find it profitable to incur the cost and participate in the second-stage mechanism in equilibrium.

[6] We establish later that there is no loss of generality to consider standard auctions for efficient and optimal mechanisms.

$n = |g|$, the cardinality of g or the number of bidders in g. For that reason, the expected revenue to the seller and the expected profit to each entrant bidder can be written as $\pi_0^{|g|}$ (or π_0^n) and $\pi^{|g|}$ (or π^n), respectively. Given g, the expected *surplus* generated from the second-stage selling mechanism is $S^{|g|} = \pi_0^{|g|} + |g| \cdot \pi^{|g|}$.

Back to the first stage (with the second stage being replaced by its correlated equilibrium payoffs), suppose the reported cost profile is $\mathbf{c}'$ and $g(\mathbf{c}')$ is the resulting set of entrants selected. Then given the true cost profile $\mathbf{c}$, the payoff for bidder i with cost c_i is given by $\pi^{|g(\mathbf{c}')|} - c_i - x_i(\mathbf{c}')$ if $i \in g(\mathbf{c}')$, and $-\,x_i(\mathbf{c}')$ if $i \notin g(\mathbf{c}')$. The seller's total revenue is given by $\sum_{i=1}^{N} x_i(\mathbf{c}') + \pi_0^{|g(\mathbf{c}')|}$. The *social surplus* generated from the (entire) sale is given by $TS^g(\mathbf{c}^g) = S^{|g|} - \sum_{j \in g} c_j$.

Also bear in mind that $\pi_0^{|g|}$, $\pi^{|g|}$, and hence $S^{|g|}$ and TS^g all depend on the second-stage mechanism, Ω. For brevity of notation, Ω is suppressed as it should be clear in the context.

We first characterize efficient mechanisms that maximize the expected total social surplus of the sale.

3 Socially Efficient Mechanisms

For efficiency, the social planner's objective is to maximize the following expected social surplus subject to the bidders' usual individual rationality (IR) and incentive compatibility (IC) constraints:

$$TS = E_{\mathbf{c}} \sum_{g \in 2^{\mathbf{N}}} p^g(\mathbf{c}) TS^g(\mathbf{c}^g) = E_{\mathbf{c}} \sum_{g \in 2^{\mathbf{N}}} p^g(\mathbf{c}) \left[S^{|g|} - \sum_{j \in g} c_j \right]. \tag{1}$$

First of all, it is apparent that to maximize TS, the second-stage mechanism must be *ex post* efficient (so that $S^{|g|}$ is maximized given any g). An immediate implication is that the reserve price in the second-stage mechanism, r^*, must equal the seller's value, v_0; any other reserve price will lead to an inefficient allocation of the asset with positive probability. Conditional on entry, bidders are symmetric so any standard auction with reserve price equal to v_0 is *ex post* efficient. Thus without loss of generality, we can fix the second-stage mechanism to be a second-price sealed-bid auction (SPA) with reserve v_0 for the rest of the analysis in this section.

We now consider the first-best efficient entry right allocation rule. From (1), it is clear that given n, the number of entrants to be admitted, TS is maximized if the seller admits a group g that consists of n bidders with the lowest n costs from $\mathbf{c}$. It remains to determine $\tilde{n}^*$, the number of entrants in socially efficient entry.

It has been established (e.g., Engelbrech-Wiggans [5]) that under a SPA with $r^* = v_0$, the following identify holds:

$$S^n - S^{n-1} = \pi^n, \tag{2}$$

which says that the marginal contribution to the expected surplus from a new entrant equals the expected (private) gain for that entrant. In light of (2), when $|g| = n$, the total social surplus in (1) can be rewritten as the sum of the marginal contributions to the social surplus from all the entrants:

$$S^{|g|} - \sum_{j \in g} c_j = \sum_{k=1}^{n} \left[\left(S^k - S^{k-1} \right) - c_{(k)} \right] = \sum_{k=1}^{n} \left[\pi^k - c_{(k)} \right], \tag{3}$$

where $c_{(k)}$ denotes the kth lowest cost and $S^0 \equiv 0$.[7] Since S^k is concave in k, $\left(S^k - S^{k-1}\right)$ $(= \pi^k)$ is decreasing in k;[8] on the other hand, $c_{(k)}$ is increasing in k. Thus given any cost profile $\mathbf{c}$, the socially efficient number of entrants, $\tilde{n}^*$, is uniquely determined by the following conditions: $\pi^{\tilde{n}^*} - c_{(\tilde{n}^*)} \geq 0$ and $\pi^{\tilde{n}^*+1} - c_{(\tilde{n}^*+1)} < 0$.

Therefore, the **first-best efficient entry right allocation rule**, denoted by $\tilde{\mathbf{p}}^* = (\tilde{p}^{*g}(\mathbf{c}))$, can be characterized as follows: Bidders are admitted in ascending order of their entry costs (the lowest cost first, the second lowest cost second, etc.) provided that the bidder's contribution to social surplus (TS^g) is non-negative. Let $\tilde{g}^*(\mathbf{c})$ denote the group being admitted according to this rule. Then $i \in \tilde{g}^*(\mathbf{c})$ if $c_i < c_{(\tilde{n}^*)}$ and $i \notin \tilde{g}^*(\mathbf{c})$ if $c_i > c_{(\tilde{n}^*)}$. If there is more than one bidder with costs exactly equal to $c_{(\tilde{n}^*)}$, only one of the bidders who tied should be admitted, and we assume a random tie-breaking rule so that each bidder who tied has equal chance to be admitted.

Three properties of efficient entry follow. First, the efficient set of entrants, $\tilde{g}^*(\mathbf{c})$, is essentially unique, as ties at $c_{(\tilde{n}^*)}$ occur with zero probability; Second, given $\mathbf{c}_{-i}$, if bidder i with c_i is admitted with a positive probability, then she is admitted for sure if her cost is c_i' $(< c_i)$; if bidder i with c_i is excluded with a positive probability, then she is excluded for sure if her cost is c_i' $(> c_i)$;[9] Third, given $\mathbf{c}_{-i}$, whenever bidder i is admitted, the number of the efficient entry slots, $\tilde{n}^*$, as well as the type of the first excluded bidder, $c_{(\tilde{n}^*+1)}$, both remain the same (i.e., independent of c_i).

It turns out that the first-best entry right allocation rule can be truthfully implemented in dominant strategies by a **Vickrey-Clarke-Groves** (VCG) **payment rule** (Vickrey [23]; Clarke [4]; and Groves

[7] The surplus generated from the second-stage mechanism is zero when no one is admitted.

[8] See McAfee and McMillan [11].

[9] More precisely, if we define $\hat{c}_i(\mathbf{c}_{-i}) = \max\{c_{k^*;-i}, \pi^{k^*+1}\}$, where k^* is the socially efficient number of entry slots in the absence of bidder i and $c_{k^*;-i}$ is the k^*th lowest cost among $(\mathbf{c}_{-i})$, then bidder i is admitted (excluded) for sure if $c_i < (>)\ \hat{c}_i(\mathbf{c}_{-i})$.

[7]). To see this, we write the (Bernoulli) utility for bidder i given the entry decision g and cost profile $\mathbf{c}$ as follows:

$$u_i(g, c_i) = \begin{cases} \pi^{|g|} - c_i & \text{if} \quad i \in g, \\ 0 & \text{if} \quad i \notin g. \end{cases}$$

Then the expected social surplus generated can be written as $TS^g(\mathbf{c}) = \pi_0^{|g|} + \sum_{i \in \mathbf{N}} u_i(g, c_i)$.[10] Note that $u_i(g, c_i)$ depends on other agents' types only through the entry allocation, so this fits the standard private value setting where the VCG mechanism applies. By making use of the identity (2), one can verify that the VCG payment rule in our setting is given by:

$$\tilde{x}^*(c_i; \mathbf{c}_{-i}) = \begin{cases} \max\{\pi^{\tilde{n}^*} - c_{(\tilde{n}^*+1)}, 0\} & \text{if} \quad i \in \tilde{g}^*(\mathbf{c}), \\ 0 & \text{if} \quad i \notin \tilde{g}^*(\mathbf{c}). \end{cases} \tag{4}$$

The VCG payment rule (4) has a very intuitive interpretation: a bidder who is admitted pays the contribution to social surplus of the bidder that would have been admitted in her place had she been excluded (zero if no other bidder would have been admitted in her place), and a bidder who is not admitted pays zero.

Next we turn to the auction implementation of the efficient entry right allocation rule. We will examine three conventional auction formats: all-pay, uniform-price, and discriminatory-price auctions.

Without loss of generality, we assume $\underline{c} < \pi^N < \pi^1 < \overline{c}$. So, in particular, the highest cost type that can be possibly admitted in efficient entry is $c_E \equiv \pi^1$. Since bidders with costs strictly above c_E should be excluded for sure, in equilibrium, no bidders with costs strictly above c_E would submit bids. In this sense, the efficient entry rule induces a highest cost type requirement for participation.

Define $\tilde{x}^* : [\underline{c}, \overline{c}] \to R_+$ as the expected payment function under the payment rule (4):

$$\tilde{x}^*(c_i) = E_{\mathbf{c}_{-i}} \tilde{x}^*(c_i, \mathbf{c}_{-i}),\ c_i \in [\underline{c}, \overline{c}]. \tag{5}$$

Clearly, $\tilde{x}^*(c_i) = 0$ for $c_i \geq c_E$, as even when a bidder with cost c_E is admitted, her expected gain from entry (net of her cost) would be zero. When $c_i < c_E$, $x^*(\cdot)$ is strictly decreasing. To see this, consider two cost types c_i' and c_i'', where $c_i' < c_i'' \leq c_E$. Given the reported $\mathbf{c}_{-i}$, whenever bidder i with c_i'' is admitted, she is also admitted with c_i' in a group with the same cardinality and the type of the first excluded bidder remains the same, in which case the payment will also be the same (properties of efficient entry). However, there are events of $\mathbf{c}_{-i}$ (with a strictly positive probability measure) where she is not admitted with c_i'' (and thus pays zero), while she is admitted for sure with c_i' and makes

[10] The seller is treated as one of the agents, whose utility ($\pi_0^{|g|}$) only depends on the entry decision g.

strictly positive payments. Thus, the expected payment $\tilde{x}^*(c_i)$ is strictly higher with a lower entry cost when $c_i \leq c_E$.

We are now ready to describe the all-pay auction: bidders submit bids (if they so choose), and they need to pay what they bid regardless of being admitted or not.[11] Given the bid profile $\mathbf{b} = (b_i)$, a cost profile $\mathbf{c} = (c_i)$ will be recovered according to $c_i = \tilde{x}^{*-1}(b_i)$ ($\tilde{x}^{*-1}(\cdot)$ is well defined since $\tilde{x}^*(\cdot)$ is strictly decreasing).[12] The efficient entry right allocation rule will then be implemented accordingly. However, somewhat surprisingly, the efficient entry rule cannot be implemented via a uniform-price or discriminatory-price auction.

Theorem 1 *Socially efficient entry can be implemented in Bayesian Nash equilibrium via an all-pay auction described in the preceding paragraph, but not via a uniform-price or discriminatory-price auction.*

Proof. See Appendix. ∎

That the all-pay auction works in implementing efficient entry is relatively easy to understand. Basically, under an all-pay auction bidders bid the expected VCG payment, which is strictly decreasing in their costs. Thus, in equilibrium their true cost profile (for those who participate in the auction) can be recovered and efficient entry can be implemented. However, that neither a uniform-price auction nor a discriminatory-price auction works has to do with the endogenously determined nature of the number of entry slots in our setting. Under a uniform-price auction, in equilibrium each bidder is supposed to bid an amount equal to her expected gain from entry conditional on being the marginal entrant. But this implies that whenever $c_i = \pi^k$, $k = 1, 2, ..., N$, bidder i should bid zero, as whenever she is admitted as the marginal entrant, the number of entry slots is exactly k (hence zero gain from entry). So no equilibrium bid function under a uniform-price auction can be strictly monotonic and the implementation fails. Under a discriminatory-price auction, on the other hand, we can show that any equilibrium bid function should exhibit jumps at $c_i = \pi^k$ in a way that the required direction of monotonicity is violated. This is, again, due to the fact that the number of entry slots is endogenously determined: the probability of being admitted for a bidder exhibits jumps at points $c_i = \pi^k$, $k = 1, 2, ..., N$. When the number of entry slots is fixed, however, one can verify that all of the three conventional auctions, including the

[11] Note that for an all-pay auction to implement efficient entry, a trivial reserve price (at zero) is necessary: any other reserve price may not guarantee efficiency in our setting.

[12] If bidder i does not bid, $b_i = \emptyset$. We define $\tilde{x}^{*-1}(\emptyset) = \overline{c}$, and $\tilde{x}^{*-1}(b) = \underline{c}$ for $b > \tilde{x}^*(\underline{c})$. Note that not everyone submits a bid and the true cost profile recovered may be truncated; nevertheless, efficient entry can be implemented in equilibrium.

uniform-price and discriminatory-price auctions, work in implementing the desirable entry.[13]

4 Revenue-Maximizing Mechanisms

Let g_i denote a generic subset in $2^{\mathbf{N}}$ that includes bidder i. Given type c_i and report c_i', bidder i's expected payoff is given by

$$\Pi_i(c_i'; c_i) = E_{\mathbf{c}_{-i}}\left[\sum_{g_i \in 2^{\mathbf{N}}} p^{g_i}(c_i'; \mathbf{c}_{-i})(\pi^{|g_i|} - c_i) - x_i(c_i'; \mathbf{c}_{-i})\right].$$

We consider incentive compatible (IC) direct mechanisms in which bidders report their types truthfully, i.e. $c_i' = c_i$. Thus in equilibrium,

$$\Pi_i(c_i) \equiv \Pi_i(c_i; c_i) = E_{\mathbf{c}_{-i}}\left[\sum_{g_i \in 2^{\mathbf{N}}} p^{g_i}(c_i; \mathbf{c}_{-i})(\pi^{|g_i|} - c_i) - x_i(c_i; \mathbf{c}_{-i})\right]. \tag{6}$$

By the envelope theorem,[14] IC implies

$$\Pi_i(c_i) = \Pi_i(\overline{c}) + \int_{c_i}^{\overline{c}} Q_i(\tilde{c}_i) d\tilde{c}_i, \tag{7}$$

where

$$Q_i(c_i) = E_{\mathbf{c}_{-i}} \sum_{g_i \in 2^{\mathbf{N}}} p^{g_i}(c_i; \mathbf{c}_{-i})$$

is the probability that bidder i is admitted with type c_i.

The following lemma provides a necessary and sufficient condition for the IC mechanism, the proof of which is standard and is hence omitted.

Lemma 1 *A mechanism $(\mathbf{p}, \mathbf{x})$ is incentive compatible if and only if (i) $Q_i(c_i)$ decreases in c_i, and (ii) (7) holds.*

[13] To guarantee a fixed number, say, k, entry slots, we need to modify the rule so that negative bids are allowed (which are meant for entry subsidies). It is easily verified that under a uniform-price auction, each bidder bids $\pi^k - c_i$ in equilibrium, which is strictly decreasing. Invoking the payoff equivalence, one can verify that the equilibrium bid functions under the all-pay and discriminatory-price auctions are also strictly decreasing.

[14] $\Pi_i(c_i'; c_i)$ may not be differentiable in c_i' (e.g., when $x_i(c_i'; \mathbf{c}_{-i})$ exhibits jumps at some c_i'). But $\Pi_i(c_i'; \cdot)$ is linear and hence absolutely continuous on $[\underline{c}, \overline{c}]$ for all $c_i' \in [\underline{c}, \overline{c}]$; besides, $\partial \Pi_i(c_i'; c_i)/\partial c_i$ is bounded. Thus the envelope theorem still applies (Milgrom and Segal [14, Theorem 2]).

The seller's total expected revenue consists of the second-stage (auction) revenue and first-stage entry fee payments, thus is given by

$$ER = E_{\mathbf{c}}\left[\sum_{g \in 2^{\mathbf{N}}} p^g(\mathbf{c})\pi_0^{|g|} + \sum_{i \in \mathbf{N}} x_i(\mathbf{c})\right]. \tag{8}$$

By Lemma 1, we have

$$\begin{aligned} E_{c_i}\Pi_i(c_i) &= \Pi_i(\overline{c}) + \int_{\underline{c}}^{\overline{c}}\left[\int_{c_i}^{\overline{c}} Q_i(\tilde{c}_i)d\tilde{c}_i\right]g(c_i)dc_i = \Pi_i(\overline{c}) + \int_{\underline{c}}^{\overline{c}} \frac{G(c_i)}{g(c_i)}Q_i(c_i)g(c_i)dc_i \\ &= \Pi_i(\overline{c}) + E_{\mathbf{c}}\left[\frac{G(c_i)}{g(c_i)}\sum_{g_i \in 2^{\mathbf{N}}} p^{g_i}(\mathbf{c})\right]. \end{aligned} \tag{9}$$

From (6), we have

$$\sum_i E_{c_i}\Pi_i(c_i) = E_{\mathbf{c}}\left[\sum_{g \in 2^{\mathbf{N}}} p^g(\mathbf{c})\sum_{j \in g}(\pi^{|g|} - c_j) - \sum_i x_i(\mathbf{c})\right]. \tag{10}$$

Using (9) into (10), we have

$$E_{\mathbf{c}}\sum_i x_i(\mathbf{c}) = E_{\mathbf{c}}\sum_{g \in 2^{\mathbf{N}}} p^g(\mathbf{c})\sum_{j \in g}\left[\pi^{|g|} - \frac{G(c_j)}{g(c_j)} - c_j\right] - \sum_i \Pi_i(\overline{c}). \tag{11}$$

Substituting (11) into (8), we have

$$ER = E_{\mathbf{c}}\sum_{g \in 2^{\mathbf{N}}} p^g(\mathbf{c})\left[S^{|g|} - \sum_{j \in g}\left(c_j + \frac{G(c_j)}{g(c_j)}\right)\right] - \sum_i \Pi_i(\overline{c}). \tag{12}$$

To maximize the expected revenue (12) while maintaining the IR condition, we should set $\Pi_i(\overline{c}) = 0$ for all i (by (7)). Thus a bidder with the highest cost type should make zero expected profit.

It is worth noting that $S^{|g|}$ and $\sum_{j \in g}(c_j + G(c_j)/g(c_j))$ are separable in the expression of expected revenue (12). The implication is that regardless of the set of entrants, to maximize expected revenue, the second-stage selling mechanism must be (*ex post*) efficient. As a direct consequence, the reserve price in the second-stage auction, r^*, should be v_0.

Theorem 2 *To maximize expected revenue, the second-stage mechanism must be ex post efficient (e.g., a standard auction with reserve equal to v_0).*

In a general private value setting where agents can acquire information before participating in a mechanism, Bergemann and Välimäki [1] show that the VCG mechanism provides efficient incentives for information acquisition *ex ante* and implements the efficient allocation conditional on the private information *ex post*. Theorem 2 suggests that in our setting, the VCG mechanism (which is *ex post* efficient) can even provide "optimal" incentives for information acquisition to maximize expected revenue *ex-ante*.

Thus, when entry is taken into account, no distortionary reserve price is needed for the sale. This result is quite robust as it is also identified in the settings with publicly known and identical information acquisition costs (e.g., McAfee and McMillan [11], and Levin and Smith [8]). In those settings, bidders will enter until their expected profits (net of entry costs) are driven down to zero, so the expected revenue is the same as the expected total surplus, which invalidates the need to introduce a distortionary reserve price. In our setting with private and heterogeneous information acquisition costs, bidders who enter do make positive expected profits due to their private information about their costs (which is reflected by the total information rent $\sum_{j\in g} G(c_j)/g(c_j)$), so the previous intuition does not apply. However, in our current setting, the total information rent ($\sum_{j\in g} G(c_j)/g(c_j)$) is independent of the second-stage mechanism (as the values are independent of the costs). As such, the seller can choose the second-stage mechanism independently to maximize the total surplus ($TS^g(\mathbf{c}^g) = S^{|g|} - \sum_{j\in g} c_j$) given any set of entrants,[15] leading to the same choice of second-stage mechanism as if the bidders' private information rent were not a concern.[16]

In what follows, we fix the second-stage mechanism as an *ex post* efficient mechanism. By the revenue equivalence, without loss of generality we may simply assume that the second-stage mechanism is a SPA with a reserve equal to v_0. Define the expected virtual surplus in the spirit of Myerson [17]:

$$v^g(\mathbf{c}) = S^{|g|} - \sum_{j\in g}\left(c_j + \frac{G(c_j)}{g(c_j)}\right) = TS^g(\mathbf{c}^g) - \sum_{j\in g}\frac{G(c_j)}{g(c_j)}, \tag{13}$$

which is the expected social surplus generated from the sale less the information rent to the bidders given the set of entrants, g.

To maximize ER, it is optimal to allocate the entry rights to a group $g^*(\mathbf{c})$, such that

$$g^*(\mathbf{c}) \in \arg\max_g v^g(\mathbf{c}),$$

provided $v^{g^*(\mathbf{c})}(\mathbf{c}) \geq 0$.[17]

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[15] Note that the total cost $\sum_{j\in g} c_j$ is subtracted from the revenue expression, thus the seller indirectly bears all the information acquisition cost in equilibrium. This is a common feature in the literature with costly entry (see, for example, Ye [25]).

[16] In settings where the information rent is correlated with the mechanism, it is no longer clear whether *ex post* mechanism remains to be optimal.

[17] If $v^{g(\mathbf{c})}(\mathbf{c}) < 0$ for any $g(\mathbf{c})$, no bidder should be admitted.

We denote this **optimal entry rule** by $\mathbf{p}^* = (p^{*g}(\mathbf{c}))$. To facilitate our characterization of the optimal entry right allocation rule, we maintain the following regularity condition for the rest of the paper:[18]

ASSUMPTION 1 *The virtual cost* $H(c) = c + \frac{G(c)}{g(c)}$ *increases with* c.

With Assumption 1, the optimal entry right allocation rule is characterized below.

THEOREM 3 *To maximize expected revenue, the bidders should be admitted in ascending order of their costs or virtual cost (the lowest cost first, the second lowest cost second, etc.) provided that the bidder's contribution to the expected virtual surplus defined in (13) is positive.*

Proof. Suppose the entry slots are awarded to any group g with n bidders. Then the expected virtual surplus is given by

$$v^g(\mathbf{c}) = S^{|g|} - \sum_{j \in g} H(c_j).$$

By Assumption 1, if any group with size n is admitted to maximize $v^g(\mathbf{c})$, this group must consist of the bidders with the n lowest costs. Thus, in the optimal entry right allocation mechanism, bidders should be admitted in ascending order of their costs provided that the bidder's contribution to the expected virtual surplus is non-negative.[19] ■

Note that for a bidder's contribution to the expected virtual surplus, $\pi^{|g_i|} - H(c_i)$, to be non-negative, the contribution to the total surplus, $\pi^{|g_i|} - c_i$, has to be positive as well. We thus have the following result immediately:

COROLLARY 1 *Optimal entry is lower than efficient entry.*

So the revenue-maximizing entry is distorted from the efficient entry, which is a nice illustration of the classical trade-off between efficiency and information rent extraction.

Similarly to efficient entry, for every $\mathbf{c}$, the optimal number of entrants n^* is uniquely determined by the following two conditions: $\pi^{n^*} - H(c_{(n^*)}) \geq 0$ and $\pi^{n^*+1} - H(c_{(n^*+1)}) < 0$.

[18] When this regularity condition fails, the technique of smooth pasting is required to identify optimal mechanisms (Myerson [17]).

[19] Similarly to (3), we can write:

$$v^g(\mathbf{c}) = \sum_{k=1}^{n} \left[\left(S^k - S^{k-1} \right) - H(c_{(k)}) \right] = \sum_{k=1}^{n} \left[\pi^k - H(c_{(k)}) \right]$$

So the kth entrant's contribution to the virtual surplus is $(\pi^k - H(c_{(k)}))$, $k = 1, 2, ..., n$.

Let $g^*(\mathbf{c})$ denote the optimal set of entrants. The three properties of the efficient entry rule can be easily adapted for the optimal entry rule in a straightforward manner.

Define $\hat{c}(n)$ to be the cutoff (cost) type for the positive marginal contribution to the virtual surplus given the number of entry slots n, that is, $\hat{c}(n)$ solves $\pi^n - H(\hat{c}(n)) = 0$.[20] Note that $\hat{c}(n) < \pi^n$.

We now look for an IC payment scheme that truthfully implements the optimal entry right allocation rule. Given that $\Pi_i(\overline{c}) = 0$, we have from (7),

$$\Pi_i(c_i) = \int_{c_i}^{\overline{c}} Q_i(\tilde{c}_i)d\tilde{c}_i = E_{\mathbf{c}_{-i}} \sum_{g_i \in 2^{\mathbf{N}}} \int_{c_i}^{\overline{c}} p^{g_i}(\tilde{c}_i; \mathbf{c}_{-i})d\tilde{c}_i. \tag{14}$$

Substituting (14) into (6), we identify the following payment rule as a natural candidate:

$$x_i^*(c_i; \mathbf{c}_{-i}) = \sum_{g_i \in 2^{\mathbf{N}}} p^{*g_i}(c_i; \mathbf{c}_{-i})(\pi^{|g_i|} - c_i) - \sum_{g_i \in 2^{\mathbf{N}}} \int_{c_i}^{\overline{c}} p^{*g_i}(\tilde{c}_i; \mathbf{c}_{-i})d\tilde{c}_i, \tag{15}$$

where the value of $p^{*g_i}(\mathbf{c})$ is determined by the optimal entry right allocation rule.

When $i \notin g^*(\mathbf{c})$, clearly $x_i^*(c_i; \mathbf{c}_{-i}) = 0$. When $i \in g^*(\mathbf{c})$ with $|g^*(\mathbf{c})| = n^*$, let $\hat{c} = \min\{c_{(n^*+1)}, \hat{c}(n^*)\}$. Given $\mathbf{c}_{-i}$, in equilibrium bidder i will be admitted for sure in a group with cardinality n^* whenever her cost is lower than $\hat{c}$, and will be excluded whenever her cost is higher than $\hat{c}$. We thus have for $i \in g^*(\mathbf{c})$ with $|g^*(\mathbf{c})| = n^*$,

$$x_i^*(\mathbf{c}) = [\pi^{|g^*(\mathbf{c})|} - c_i] - [\min\{c_{(n^*+1)}, \hat{c}(n^*)\} - c_i] = \max\left\{\pi^{n^*} - c_{(n^*+1)}, \pi^{n^*} - \hat{c}(n^*)\right\}. \tag{16}$$

It is easily seen that the above **optimal payment rule** is nondiscriminatory (it depends on the reported profile of $\mathbf{c}$ only). Thus we can drop the subscript i to write

$$x^*(c_i; \mathbf{c}_{-i}) = \begin{cases} \max\left\{\pi^{n^*} - c_{(n^*+1)}, \pi^{n^*} - \hat{c}(n^*)\right\} & \text{if} \quad i \in g^*(\mathbf{c}) \\ 0 & \text{if} \quad i \notin g^*(\mathbf{c}) \end{cases}. \tag{17}$$

The above payment rule $\mathbf{x}^*$ takes a form similar to the VCG payment rule (4), except that a nontrivial "reserve price" $\pi^{n^*} - \hat{c}(n^*)$ is imposed for each realized n^*; as a result, the rent of the marginal type admitted ($\hat{c}(n^*)$) is fully extracted in optimal entry. As under the VCG payment rule, a bidder's report affects her payment only through the entry right allocation under payment rule (17), so not surprisingly, the payment rule $\mathbf{x}^*$ defined in (17) also implements the optimal entry right allocation rule $\mathbf{p}^*$ in dominant strategies.

Theorem 4 *The optimal entry right allocation rule* $\mathbf{p}^*$ *is truthfully implementable by payment rule* $\mathbf{x}^*$ *in dominant strategies.*

[20] Given Assumption 1 and the assumption that $\underline{c} < \pi^N < \pi^1 < \overline{c}$, the solution of $\hat{c}(n)$ is unique for $n = 1, 2, ..., N$.

Proof. See Appendix. ■

We now turn to the auction implementation of the optimal entry rule. In optimal entry, the highest cost type that can be possibly admitted is $c_R \equiv \hat{c}(1) = H^{-1}(\pi^1)$. Thus c_R can be regarded as the highest cost type that would participate in an (optimal) auction for entry rights. Note that $c_R = H^{-1}(\pi^1) < \pi^1 = c_E$ (Corollary 1).

Define $x^* : [\underline{c}, \overline{c}] \to R_+$ as the expected payment function under the payment rule $\mathbf{x}^*$:

$$x^*(c_i) = E_{\mathbf{c}_{-i}} x^*(c_i, \mathbf{c}_{-i}),\ c_i \in [\underline{c}, \overline{c}]. \tag{18}$$

Clearly, $x^*(c_i) = 0$ for $c_i > c_R$. For type c_R, she can only be admitted when all the other bidders have costs strictly higher than c_R (with probability $[1 - G(c_R)]^{N-1}$), in which case she pays $\pi^1 - c_R = G(c_R)/g(c_R)$. Thus $x^*(c_R) = [1 - G(c_R)]^{N-1} G(c_R)/g(c_R)$. When $c_i < c_R$, by following exactly the same procedures for the case of efficiency, we can show that $x^*(\cdot)$ is strictly decreasing.

Thus, just as efficient entry can be implemented by an all-pay auction, the optimal entry can be implemented by an (optimal) all-pay auction (with reserve price $r_A = [1 - G(c_R)]^{N-1} G(c_R)/g(c_R)$) in which bidders with cost lower than c_R bid according to $x^*(\cdot)$ in equilibrium. The following result is the counterpart of Theorem 1 in the context of revenue maximization.

THEOREM 5 *The revenue-maximizing entry rule can be implemented in Bayesian Nash equilibrium via an all-pay auction with reserve price equal to $r_A = [1 - G(c_R)]^{N-1} G(c_R)/g(c_R)$, but not via any uniform-price or discriminatory-price auction.*

Proof. See Appendix. ■

The intuitions for Theorem 5 parallel those for Theorem 1. Basically, while $x^{*-1}(\cdot)$ can serve as the cost recovery function in an optimal all-pay auction, no cost recovery function exists under a uniform/discriminatory price auctions that implements optimal entry: the required monotonicity fails, this time, at marginal entrant's types $c_i = \hat{c}(k) = H^{-1}(\pi^k)$, $k = 1, 2, ..., N$.

5 Conclusion

This paper studies two-stage mechanisms with an emphasis on entry right allocation mechanisms in a setting where bidders' information acquisition costs are privately known to the bidders. In such a setting we are able to completely characterize both the (socially) efficient and (revenue) optimal mechanisms.

We demonstrate that both efficient and optimal entry can be implemented in dominant strategies, and they can also be implemented via all-pay auctions with appropriately set reserve prices.

Our analysis of two-stage selling mechanisms can shed new light on the practice of two-stage auctions, which are commonly used in high-valued and complex asset sales, procurements, takeovers, and merger and acquisition contests.[21] A feature common to all these two-stage auctions is that there is a pre-qualifying stage to screen the bidders,[22] which corresponds to the entry right allocation mechanism analyzed in our framework. While our current model is somewhat special in the sense that bidders' pre-entry "types" are simply their information acquisition costs, we expect that some insights obtained from our current analysis are robust enough to carry over to more general settings where bidders' pre-entry "types" are correlated to their final valuations for the object for sale. A complete characterization of optimal mechanisms in such a setting is technically challenging, which is left for future research.

Appendix

Proof of Theorem 1: To show that the efficient entry rule can be implemented by the described all-pay auction (with a trivial reserve price at zero), it only remains to show that in the (reduced) all-pay auction game, it is a symmetric strictly monotone Bayesian Nash equilibrium (BNE) for each bidder with a cost lower than c_E to bid according to $\tilde{x}^*(\cdot)$ defined in (5).

Given that everyone else bids according to $\tilde{x}^*(\cdot)$, bidder i maximizes the following objective function by choosing her bid b:

$$\widehat{\Pi}(b, c_i) = E_{\mathbf{c}_{-i}} \left(\pi^{|\tilde{g}^*(\tilde{x}^{*-1}(b), \mathbf{c}_{-i})|} - c_i \right) \cdot \mathbf{1}_{\{i \in \tilde{g}^*(\tilde{x}^{*-1}(b), \mathbf{c}_{-i})\}} - b, \tag{19}$$

where the indicator function $\mathbf{1}_{\{i \in \tilde{g}^*(\tilde{x}^{*-1}(b), \mathbf{c}_{-i})\}} = 1$ if $i \in \tilde{g}^*(\tilde{x}^{*-1}(b), \mathbf{c}_{-i})$, and 0 otherwise.

We next apply the constraint simplification theorem[23] to demonstrate that $\tilde{x}^*(\cdot)$ constitutes a symmetric increasing BNE in this (reduced) entry game. This can be verified in the following steps:

1) $\tilde{x}^*(\cdot)$ is strictly decreasing as shown in the text.

2) Since a lower c_i $(\leq c_E)$ has strictly higher probability of being admitted and that $\tilde{x}^*(\cdot)$ is strictly decreasing, $\partial\widehat{\Pi}/\partial c_i = \Pr\left\{i \in \tilde{g}^*(\tilde{x}^{*-1}(b), \mathbf{c}_{-i})\right\}$ is strictly increasing in b, i.e., $\widehat{\Pi}(b, c_i)$ satisfies the strict and smooth single crossing differences property.

[21] See Ye [25] for industry examples of two-stage auctions.

[22] See Boone and Goeree [3] for an interesting analysis of pre-qualifying auctions.

[23] See, for example, Theorem 4.3 in Milgrom [13], pp. 105.

3) Let $\Pi(c_i)$ be the equilibrium payoff for type $c_i \in [\underline{c}, c_E]$ under the VCG mechanism with payment rule (4). By the envelope theorem, we have

$$\Pi(c_i) = E_{\mathbf{c}_{-i}} \left[\left(\pi^{|\tilde{g}^*(c_i, \mathbf{c}_{-i})|} - c_i \right) \cdot \mathbf{1}_{\{i \in \tilde{g}^*(c_i, \mathbf{c}_{-i})\}} - \tilde{x}^*(c_i, \mathbf{c}_{-i}) \right] = \int_{c_i}^{c_E} \Pr\left\{ i \in \tilde{g}^*(s, \mathbf{c}_{-i}) \right\} ds. \tag{20}$$

Substituting $b = \tilde{x}^*(c_i)$ into (19), we have $\widehat{\Pi}(\tilde{x}^*(c_i), c_i) = \Pi(c_i)$ because, by construction, $\tilde{x}^*(c_i) = E_{c_{-i}} \tilde{x}^*(c_i, \mathbf{c}_{-i})$. In particular, we have $\widehat{\Pi}(\tilde{x}^*(c_E), c_E) = \Pi(c_E) = 0$. By invoking (20), the following envelope formula is verified for $\tilde{x}^*(\cdot)$:

$$\widehat{\Pi}(\tilde{x}^*(c_i), c_i) = -\int_{c_i}^{c_E} \widehat{\Pi}_2(\tilde{x}^*(s), s) ds.$$

4) It is also easily verified that bidding outside the range of $\tilde{x}^*(\cdot)$ cannot lead to higher expected payoff.

Thus all the sufficiency conditions for the constraint simplification theorem are satisfied and $\tilde{x}^*(\cdot)$ indeed constitutes a symmetric monotone BNE.

We next show that efficient entry cannot be implemented by a uniform-price auction (equipped with a reserve price equal to zero). This is equivalent to showing that in the (reduced) uniform-price auction of entry rights, there does not exist an equilibrium in which each bidder (with a cost lower than c_E) bids according to a symmetric strictly monotone bid function, say, $\beta^U(\cdot) : [\underline{c}, c_E] \to R_+$.

Suppose, in negation, such an equilibrium exists. We consider its associated direct game in which bidders are required to report their costs directly. Suppose that everyone else reports their types $\mathbf{c}_{-i}$ truthfully. Let's consider bidder i's report c_i' given her true cost $c_i \in (\pi^{K+1}, \pi^K]$ for some $K \geq 1$. For necessity of equilibrium we only consider local deviations so $c_i' \in (\pi^{K+1}, \pi^K]$. Let k^* be the number of entry slots under the efficient entry rule in bidder i's absence, and let $c_{k;-i}$ denote the kth lowest cost among $(\mathbf{c}_{-i})$. We consider the following cases:

1) $\max\{c_{k^*;-i}, \pi^{k^*+1}\} \leq \pi^{K+1}$, then bidder i cannot be admitted for sure with any $c_i' \in (\pi^{K+1}, \pi^K]$;

2) $\max\{c_{k^*;-i}, \pi^{k^*+1}\} > \pi^K$, then bidder i must be admitted for sure with any $c_i' \in (\pi^{K+1}, \pi^K]$, and the report does not affect the payment, which is the bid from the first excluded bidder (with type $c_{k^*;-i}$ or $c_{k^*+1;-i}$); moreover, i is admitted in a group whose cardinality does not depend on $c_i' \in (\pi^{K+1}, \pi^K]$;

3) $\max\{c_{k^*;-i}, \pi^{k^*+1}\} = \pi^K$, so either $c_{k^*;-i} = \pi^K$ (a zero probability event and hence can be omitted from our consideration) or $k^* = K - 1$ and $c_{k^*;-i} \leq \pi^{k^*+1} < c_{k^*+1;-i}$. In the latter case, bidder i is admitted for sure with any $c_i' \in (\pi^{K+1}, \pi^K]$, in which case $c_{k^*+1;-i}$ is the type of the first excluded bidder;

4) $\max\{c_{k^*;-i}, \pi^{k^*+1}\} \in (\pi^{K+1}, \pi^K)$, i.e., $c_{k^*;-i} \in (\pi^{K+1}, \pi^K)$ and $k^* = K$. In this case, bidder i can

only be admitted with $c_i' \in (\pi^{K+1}, c_{k^*;-i}]$, in which case she crowds out the bidder with $c_{k^*;-i}$ and hence pays $\beta^U(c_{k^*;-i})$.

This shows that bidder i's report $c_i' \in (\pi^{K+1}, \pi^K]$ affects her expected payoff only in case 4) above. Thus the expected payoff for bidder i can be written as follows:

$$\begin{aligned}\Pi(c_i', c_i) &= E_{\mathbf{c}_{-i}}\left[\pi^K - c_i - \beta^U(c_{K;-i})\right] \cdot \mathbf{1}_{\left\{c_i' < c_{K;-i} < \pi^K\right\}} + \text{ (Terms independent of } c_i') \\ &= \int_{c_i'}^{\pi^K} \left[\pi^K - c_i - \beta^U(c)\right] g_{K;N-1}(c)dc + \text{ (Terms independent of } c_i'),\end{aligned}$$

where $g_{K;N-1}(\cdot)$ is the density function of $c_{K;-i}$. $g_{K;N-1}(c) > 0$ over $(\underline{c}, \overline{c})$ as we assume $g(c) > 0$ over $[\underline{c}, \overline{c}]$.

Incentive compatibility thus implies the following first order condition:

$$\frac{\partial \Pi}{\partial c_i'}|_{c_i'=c_i} = -\left[\pi^K - c_i - \beta^U(c_i)\right] g_{K;N-1}(c_i) = 0.$$

So $\beta^U(c_i) = \pi^K - c_i$ for any $c_i \in (\pi^{K+1}, \pi^K]$, $K \geq 1$. But this implies that $\beta^U(\pi^K) = 0$ for all $K = 1, 2, ..., N$. This is a violation of the required strict monotonicity and hence efficient entry cannot be implemented via a uniform-price auction.

Finally, we demonstrate that efficient entry cannot be implemented via a discriminatory-price auction either. We will show this by invoking the payoff equivalence theorem (Myerson [17]), which implies, in our setting, that the expected payment for any given type is the same in any efficient entry mechanism where a bidder with the highest possible cost (c_E) to participate makes zero expected payoff. Suppose $\beta^D : [\underline{c}, c_E] \to R_+$ constitutes a symmetric strictly decreasing bid function in the equilibrium in the (reduced) entry game induced by a discriminatory-price auction (with reserve price equal to zero). Since the reserve price is zero, the expected payoff for type c_E bidder is also zero. Thus as long as β^D constitutes a symmetric monotone BNE, by the payoff equivalence, we must have:

$$\beta^D(c_i) = \frac{\tilde{x}^*(c_i)}{\widetilde{Q}(c_i)},\ c_i \in [\underline{c}, c_E], \tag{21}$$

where $\widetilde{Q}(c_i) = \Pr\{i \in \tilde{g}^*(c_i, \mathbf{c}_{-i})\}$ is the probability of winning an entry slot for a bidder with type c_i in equilibrium, and $\tilde{x}^*(\cdot)$ is given by (5).

Notice that by the efficient entry allocation rule, the probability of winning, $\widetilde{Q}(c_i)$, must exhibit jumps at $c_i = \pi^1, \pi^2, ..., \pi^N$ (these are also the only points where $\widetilde{Q}(\cdot)$ is discontinuous). More specifically, we have $\widetilde{Q}(\pi^{1-}) = \widetilde{Q}(\pi^{1+}) + \Pr\left\{c_{1;-i} > \pi^1\right\}$, $\widetilde{Q}(\pi^{k-}) = \widetilde{Q}(\pi^{k+}) + \Pr\left\{c_{k-1;-i} < \pi^{k-1} \text{ and } \pi^k < c_{k;-i}\right\}$, $k = 2, ..., N-1$, and $\widetilde{Q}(\pi^{N-}) = \widetilde{Q}(\pi^{N+}) + \Pr\left\{c_{N-1;-i} < \pi^{N-1}\right\}$.

However, it can be verified that $\tilde{x}^*(\cdot)$ is continuous over $[\underline{c}, c_E]$.[24] But then by (21), $\beta^D(\cdot)$ has to exhibit jumps at $c_i = \pi^k$, $k = 1, 2, ..., N$, and for those points, we must have $\beta^D(c_i^+) > \beta^D(c_i^-)$, which is a clear violation of the required monotonicity. So discriminatory-price auctions also fail to implement efficient entry. ■

Proof of Theorem 4: First consider the case in which, given $\mathbf{c}_{-i}$, the reported cost profile from all but bidder i, bidder i is admitted to a group with, say, n^*, entrants by reporting c_i truthfully. By misreporting c_i', as long as she is still admitted, she is admitted to the same group with n^* entrants, and the first excluded type also remains the same (properties of optimal entry). Thus her expected payoff remains the same. If, however, by misreporting c_i' she is excluded, then her payoff will be zero. This shows that she has no incentive to misreport in this case.

If by reporting truthfully bidder i is excluded according to $\mathbf{p}^*$, then $c_i \geq c_{(n^*+1)} > \hat{c}(n^*+1)$, where n^* is the number of admitted bidders, and her payoff will be zero. If she misreports a cost such that she is still excluded (and thus pays zero), her payoff remains zero. Suppose she reports a cost c_i' that is low enough for her to be included in $g^*(c_i', \mathbf{c}_{-i})$. We must have $|(g^*(c_i', \mathbf{c}_{-i}))| = n^*$ (the previously admitted bidder with type $c_{(n^*)}$ is crowded out) or $|(g^*(c_i', \mathbf{c}_{-i}))| = n^* + 1$ (no previously admitted bidders are crowded out). Note that these are the only two possible cases, as all the other bidders with costs higher than $c_{(n^*+1)}$ can never be admitted according to $\mathbf{p}^*$. If $|(g^*(c_i', \mathbf{c}_{-i}))| = n^*$, bidder i would pay $\max\{\pi^{n^*} - \hat{c}(n^*),\ \pi^{n^*} - c_{(n^*)}\} \geq \pi^{n^*} - c_{(n^*+1)} \geq \pi^{n^*} - c_i$. This results in a negative expected payoff for her. If $|(g^*(c_i', \mathbf{c}_{-i}))| = n^* + 1$, bidder i would pay $\max\{\pi^{n^*+1} - \hat{c}(n^*+1),\ \pi^{n^*+1} - \tilde{c}\}$, where $\tilde{c} = c_{(n^*+1)}$ if $c_i > c_{(n^*+1)}$, and $\tilde{c} = c_{(n^*+2)}$ if $c_i = c_{(n^*+1)}$. Recall that $c_i \geq c_{(n^*+1)} > \hat{c}(n^*+1)$. Thus if $c_i > c_{(n^*+1)}$, $\min\{\hat{c}(n^*+1), \tilde{c}\} \leq c_{(n^*+1)} < c_i$, which results in a negative expected payoff for bidder i; if $c_i = c_{(n^*+1)}$, $\min\{\hat{c}(n^*+1), \tilde{c}\} \leq c_{(n^*+2)} < c_i$, which also results in a negative expected payoff. ■

Proof of Theorem 5: The proof for the all-pay auction implementation should look almost the same as in the proof of Theorem 1, with c_E being replaced by c_R; in particular, the envelope formula is verified for $x^*(\cdot)$ as it satisfies (6) by construction.

[24] The verification is somewhat tedious, but the intuition is clear: Since $\widetilde{Q}(\cdot)$ only jumps at $c_i = \pi^1, \pi^2, ..., \pi^N$, we only need to check the continuity of $\tilde{x}^*(\cdot)$ at those points. Take a sufficiently small $\epsilon > 0$, and consider a bidder with types $c_i^+ = \pi^k + \epsilon$ and $c_i^- = \pi^k - \epsilon$. Clearly, there is a jump in winning probability when moving from type c_i^+ to type c_i^-. But when she wins only because of the cost reduction, she must have become the last one to join the group with k entrants. Therefore her payoff is $\pi^k - c_i^- = \epsilon$, which goes to zero as $\epsilon \to 0$. As such, it does not take a jump in the expected payment (the bid) in order to guarantee the continuity in expected payoff (an equilibrium requirement).

For uniform-price and discriminatory-price auctions,[25] we can consider $c_i, c_i^{'} \in (\hat{c}(k+1), \hat{c}(k)]$ and demonstrate that the required monotonicity fails at points $c_i = \hat{c}(k) = H^{-1}(\pi^k)$, $k = 1, 2, ..., N$. The rest of the arguments should look exactly the same as in the proof of Theorem 1. ■

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