Deterministic vs. Stochastic Entry: A Benefit of Running An Auction-Negotiation Hybrid Mechanism

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Abstract

In auctions with costly entry, the entry process matters. This paper compares expected revenues generated by different entry processes. It is shown that an auction with deterministic entry usually generates more expected revenue than that with stochastic entry. Thus influencing the entry process by reducing the randomness of participation is to the seller’s benefit. Based on this insight, we analyze a hybrid mechanism combining both auction and negotiation elements, in which a sole buyer is selected from an auction process, followed by a negotiation stage. In our model, we show that such a hybrid mechanism generates higher expected revenue than the one-stage standard auction, as long as the number of potential buyers is sufficiently large.

1 Introduction

The early literature on optimal auctions and revenue comparison (e.g. Vickrey (1961), Riley and Samuelson (1981), Myerson (1981), Milgrom and Weber (1982)) generally assumes that there is an exogenously specified set of bidders and that these bidders are endowed with information; both endogenous entry and bid preparation costs are ignored. Yet the nature of optimal auctions changes dramatically if this costly entry feature is taken into account.

Two types of entry costs can be identified. The first type relates to information acquisition and processing. Suppose bidders have to spend resources to inspect the object for sale before they

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know their own valuation, then once the valuation is known, the cost of inspection is sunk and
does not affect the subsequent bidding behavior. Another type of cost relates to bid-preparation
and documentation. Engaging into the bidding process itself is costly. By the time a final bid is
submitted, the cost of bid-preparation is sunk, and this cost will not affect the bidding behavior
either. The existing literature mostly focuses on the analysis of the first type of entry costs, with the
exceptions of Samuelson (1985), Landsberger (2007), and Gal et al. (2007), who formulate auctions
mainly as participation games.

Among those models that focus on the analysis of the first type of entry costs, the entry process
is formulated with two different approaches. Let \( N \) be the number of potential bidders. Then the
first approach assumes a deterministic entry process in which exactly \( n \) potential bidders enter the
auction and exactly \( (N - n) \) potential bidders stay out (see, for example, Johnson (1979), McAfee and
McMillan (1987a), and Engelbrecht-Wiggans (1993)). The other approach assumes a stochastic entry
process in which every potential bidder randomizes about entering (see, e.g., Harstad (1990), Levin
and Smith (1994), and Ye (2004)).\(^1\) While deterministic entry implies pure strategies for potential
bidders and leads to asymmetric equilibria, stochastic entry implies mixed strategies and in general
leads to full symmetry; while the deterministic process is immune from coordination problem, the
random process typically incurs the cost of over- or under-participation.

Both approaches treat the entry process as exogenously given: despite the significant difference
in strategies on the potential bidders’ side, the role of the seller here is not addressed. In other words,
no work has yet been done concerning the strategy on the seller’s side in this entry game. Intuitively,
if two processes generate different expected revenues, there is no reason to believe that the seller will
be indifferent between the two processes. If the entry process matters, it would be more reasonable
to assume that the seller would have an incentive to influence the entry process, and if possible, the
seller would even “pre-set” the entry process to serve his own interest.

We compare these two entry processes in the first part of the paper. In a simple model where
bidders can only acquire information about valuations after incurring entry cost, we show that the
optimal auction with deterministic entry can always outperform the optimal auction with stochastic
entry. This result is quite intuitive. Since bidders do not possess private information before entry,
they receive zero expected profit in equilibrium due to endogenous entry. Thus in equilibrium the

\(^1\) Symmetric mixed entry strategy is first introduced by Milgrom (1981) in his two-stage auction example.
expected revenue to the seller is the same as the expected surplus generated from the sale. It can be shown that the expected surplus is concave in the number of entrant bidders. Therefore, any auction with stochastic entry is revenue-dominated by a mean-preserving scheme. This mean-preserving scheme is in turn revenue-dominated by the optimal auction with deterministic entry, which gives rise to our comparison result. This result is compelling, suggesting that reducing the randomness in participation can lead to welfare, and hence revenue improvement. Thus the selection of the entry process can be added to the seller’s long list of policy instruments, following the selection of reserve prices, entry fees, and information for revelation, etc.

Based on the insight obtained from the entry processes comparison, in the second part of the paper we analyze a hybrid mechanism mixing both auction and negotiation elements. Such a hybrid mechanism works as follows. First, an auction is conducted and bids are ranked from the highest to the lowest. The bidder with the highest bid is then selected to advance to the negotiation stage. After some intensive information acquisition process, the seller and the buyer then negotiate over the terms of the sale.

The typical process of buying a house in the United States can be regarded as such a hybrid mechanism. Potential buyers first make offers to indicate the prices they are willing to pay for the house. The seller then accepts an offer (typically the highest offer). After signing a contract with the seller, the potential buyer with the accepted offer then proceeds with having the house inspected and appraised. If both parties do not agree then they will likely make counter-offers, by preparing a new contract with different terms. The process repeats until a contract is signed by both parties, or the deal falls through because the two parties cannot agree. Other than in the housing market, this type of hybrid mechanism is also commonly used in privatization, procurement, takeover, and merger and acquisition contests. For example, when Daewoo Motors, Korea’s second largest conglomerate, ran into financial trouble in July, 1999, creditors stepped in and a hybrid mechanism combining both auction and negotiation formats was adopted by the government to dismantle its asset. In the first round conducted in June 2000, the world’s second largest auto company, Ford Motor, offered $6.9 billion for Daewoo while DaimlerChrysler-Hyundai jointly bid $4.5 billion and GM came in at the lowest price. Consequently, Ford was picked to be the preferred bidder to proceed onto the final round.

Both the seller and the buyers are usually represented by real estate agents.
In this paper, we abstract from the real world complexity of hybrid mechanisms and work instead with a simplified version of the hybrid mechanism, which captures the essence of both the costly information acquisition (costly entry) and negotiation features. We show that the hybrid mechanism can generate higher expected revenue than a pure auction counterpart as long as the number of potential bidders is sufficiently large. This result is intuitive. In a pure auction, bidders enter the auction only if their “types” (the private value estimates) are above some threshold. As a result, the participation is stochastic: when the realized number of bidders is too large, the profit accrued to the bidders is low which would discourage entry in the first place. As a result, the realized number of bidders can be extremely low which would lead to lower expected revenue. In a hybrid mechanism, on the contrary, the entry is essentially a deterministic process, with a fixed number of bidders (1 in our model) being selected for the negotiation process. Given the insight obtained from the entry processes comparison, when the randomness of participation in a pure auction is prominent (i.e., when the number of potential buyers is sufficiently large), we should anticipate the revenue-dominance of hybrid mechanisms over pure auctions.

We thus identify a setting in which a hybrid mechanism mixing auction and negotiation elements can outperform a pure auction. A testable implication is that when costly entry and a large number of potential buyers are both present, a hybrid mechanism is more likely to be employed than a pure auction.

The effect of stochastic participation has also been noted in the existing literature. Samuelson (1985) shows that the expected procurement cost does not necessarily decline with increase in the number of potential bidders. Levin and Smith (1994) discuss the coordination costs associated with thick markets in great detail. All these works suggest that policies to limit the number of potential bidders may be welfare improving. In this paper, we take one step further and show that the expected revenue is maximized when the randomness of participation is completely eliminated, i.e., when the entry process is deterministic.

This paper is organized as follows. In section 2, the comparison of the two entry processes is made when bidders do not possess private information prior to entry. Section 3 considers a modified model in which buyers possess private information before entry, and shows how a well designed hybrid

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mechanism can outperform an optimal pure auction. Section 4 concludes.

2 Stochastic versus deterministic entry

We consider a single item offered to a group of $N$ potential bidders. In this section we assume that potential bidders do not possess private information before entry, and each bidder only learns her private signal ($X_i$) about the value of the item ($V_i$) after incurring an entry cost $c$.

The optimal auction problem is posed as a three stage game. At stage 1, the seller selects an auction rule from the class of auctions, in which the reserve price and entry fee are also specified. At stage 2, each potential bidder decides whether to enter. Entry entails a fixed cost $c$. At stage 3, the number of actual bidders who have entered becomes common knowledge, and they play the usual bidding game.

As mentioned in the introduction, in the entry subgame at stage 2, potential bidders’ different strategies give rise to two different types of entry processes: deterministic and stochastic processes.

The main results from the deterministic entry literature are as follows:

1. Potential bidders enter the auction until the rent goes to zero (or near zero due to the integer problem).

2. In equilibrium, the expected revenue equals (or approximately equals) the expected total surplus from trade. Therefore, the optimal reserve price should be set at the seller’s own valuation of the object.

3. In the private value auction, a zero entry fee induces the optimal set of bidders in maximizing the expected total surplus. Thus to maximize expected revenue, the entry fee should only be used to extract the bidders’ residual rent due to the integer problem, while in the common value auction, a zero entry fee induces excessive entry from the social and private points of view, thus a strictly positive entry fee should be imposed to restrict entry.

We now turn to the stochastic entry process. To characterize this, we follow the analysis in Levin and Smith (1994) closely. In their formulation, each potential bidder is assumed to randomize over entering with the same probability of entry; thus the total number of actual bidders is a binomial random variable. Given our focus on the case with large number of potential bidders, we can
approximate the number of actual bidders by a random variable which is distributed according to a Poisson distribution.\footnote{In a market with many potential bidders (think of online auctions like eBay or Onsale), each potential bidder would actually bid for a specific item with very small probability. In such a case the distribution of the number of actual bidders can well be approximated by a Poisson distribution.}

We maintain the following assumptions throughout this section: First, the seller and all potential bidders are risk-neutral. Second, the seller’s valuation is normalized to be zero. The domain of possible values for the item ($V$) and the domain of estimates ($X$) are compact: $v \in [0, \bar{v}]$ and $x \in [0, \bar{x}]$. Third, information is symmetric; all bidders randomly draw values from the same distribution, i.e., ex ante $V_i$’s and $X_i$’s are i.i.d. Finally, the auction mechanism ($m$) and the number of potential bidders ($N$) are common knowledge, and the number of actual bidders, $n$, is revealed before the final auction in stage 3.\footnote{In the independent private value auctions, if bidders are risk-neutral, our equilibrium analysis and results hold even if bidders do not learn the number of their rivals after entering; if bidders are risk-averse, bidders’ ex ante expected utility (and hence the entry equilibrium) is affected by the assumption that $n$ is revealed. (see McAfee and McMillan (1987b), Matthews (1987), Hastad et al. (1990)).}

We allow the seller’s choice of mechanism to include any rule by which a bidder wins and pays for the item only if her bid is the highest. The mechanism might entail an entry fee paid to the seller, or reservation prices $R = \{R_1, ..., R_N\}$, where $R_n$ represents the common-knowledge reserve price enforced by the seller if $n$ bidders enter. For clarity, we will denote the seller’s mechanism by $m(R, e)$. Let $T$ represent the event that trade occurs and let $T_n(R_n)$ represent the probability of trade given $n$ and the seller’s mechanism.

Using symmetry, a bidder’s ex ante expected profit, conditional on entering an auction where trade occurs as one of $n$ bidders, can be written as $(V_n - W_n)/n - (c + e)$, where $V_n$ is the expected value of the item to the highest bidder and $W_n$ is the expected payment this bidder makes to the seller, both conditional on trade occurring under the given mechanism with $n$ bidders. We use $\Omega$ to denote $\{m(R, e), c, N\}$ and $B_i(\Omega)$ to denote the $i$th bidder’s expected profit from entering when all $N - 1$ rivals are using equilibrium mixed strategies about entering. In equilibrium, let $\gamma_n$ be the probability that $n$ actual bidders enter the auction, and let $p_i^n$ be the updated probability that there are $n$ actual bidders from the perspective of the $i$th potential bidder, conditional on herself being one of the actual bidders. Potential bidders randomize about entering since they earn expected profit of
zero when entering. Therefore, the randomization condition for \( i, i = 1, 2, \ldots, N \) is

\[
B_i(\Omega) = \sum_{n=1}^{N} p_n^iT_n(R_n)(V_n - W_n)/n - (c + e) = 0
\]

Thus the bidders’ rent is driven to zero by the process of endogenous entry. Consequently, in equilibrium the seller’s expected revenue equals the expected total surplus. i.e.,

\[
ER = ES = \sum_{n=1}^{N} \gamma_n T_n(R_n)V_n - \bar{nc}
\]

It is straightforward to see that the optimal reserve price \( (R_n) \) from the perspectives of both the society and the seller equals the seller’s reserve value, for all \( n \). Thus any mechanism that maximizes the seller’s expected revenue also induces socially optimal entry. Such a mechanism should never involve a non-trivial reserve price that is different from the seller’s own reserve value. But regarding entry fees, independent private value (IPV) auctions and common value (CV) auctions have different implications.

**Lemma 1** In an IPV auction, the optimal entry fee is zero; in a CV auction, the optimal entry fee is strictly positive, and without an entry fee, the entry will be socially and privately excessive.

Basically, the above findings concur with the results in the case with deterministic entry. The precise zero entry fee result in IPV auctions is due to the smoothing of the integer problem by randomization.

The optimal auction with stochastic entry is thus characterized by the optimal reserve prices and optimal entry fees derived above. To compare the optimal auction with stochastic entry and the optimal auction with deterministic entry, we start with two lemmas below. Denote \( V_{n:k} \) to be the \( k \)th highest order statistic among \( n \) iid draws of \( V \).

**Lemma 2** In IPV auctions, The expected total surplus, \( EV_{n:1} - nc \), given \( n \), is strictly concave in \( n \), the number of bidders.

Let \( M \) be the expected number of actual bidders showing up under the optimal auction with random entry. We first assume that \( M \) is an integer, and define the auction with mean-preserving entry as the one in which exactly \( M \) bidders participate in the auction.
Lemma 3 The optimal auction with stochastic entry is revenue-dominated by its associated counterpart with mean-preserving entry.

Proof. In IPV auctions, the dominance result is a simple application of Jensen’s inequality. Lemma 2 shows that given any number of actual bidders \( n \), the expected total surplus from trade is strictly concave in \( n \). Thus by Jensen’s inequality, the expected total surplus generated from the optimal auction with random entry is strictly less than the expected total surplus generated by the associated auction with mean-preserving entry defined above (the strict inequality is due to the fact that \( n \) is not degenerated almost surely). Using an optimal entry fee (positive or negative) to keep bidders just willing to participate (making zero expected profit), the expected revenue is the same as the expected surplus. Thus the auction with mean-preserving entry and optimal entry fee out-performs the optimal auction with random entry in IPV auctions.

In CV auctions, \( V_n = EV \) for \( n \geq 1 \). Thus the expected revenue generated from the optimal auction with random entry is \( ER = EV(1−γ_0)−Mc = EV−Mc−γ_0EV \), where \( γ_0 \) is the probability of no entry or no trade. Again, this is dominated by the optimal auction with mean-preserving entry which generates expected revenue \( EV−Mc \).

Now we consider the case when \( M \) is not an integer. In this case, we can modify the definition of the auction with mean-preserving entry as follows: with probability \( θ = [M] + 1 − M \), \( [M] \) bidders are selected, and with probability \( 1 − θ = M − [M] \), \( [M] + 1 \) bidders are selected. It is easily seen that with some minor modification, all the arguments above still go through.

We are now ready to state the main result in this section:

Proposition 1 In IPV auctions, any auction with stochastic entry is revenue-dominated by the optimal auction with deterministic entry; in the CV auctions, the same result holds unless the expected number of bidders in the optimal auction with stochastic entry is less than 2.

The proof is immediate by combining Lemma 3 and the fact that any auction with mean-preserving entry is a member in the family of auctions with deterministic entry, and is hence dominated by the optimal auction with deterministic entry. In IPV auctions, the optimal auction with deterministic entry can be characterized by the triple \( \{R^*, e^*, n^*\} \), where \( R^* = 0 \), \( n^* \) is the optimal number of bidders that maximizes the expected total surplus \( EV_{n;1} − nc \), and \( e^* \) is the optimal entry
fee that extracts the residual rent from the bidders caused by the integer problem (see Engelbrecht-Wiggans (1993)). In CV auctions, the optimal auction with deterministic entry includes exactly two bidders in the auction. Under the regularity condition (such that $c$ is not too high and hence $M > 2$), the optimal auction with mean-preserving entry is revenue-dominated by the optimal auction with deterministic entry.

In practice, the optimal number of bidders may not be easily estimated, which leads to difficulties in applying optimal auctions with deterministic entry. However, the average number of bidders might be easier to obtain, especially in auctions with repeated sales. This could favor the use of optimal auctions with mean-preserving entry.

In the real world, we do observe that sellers use some sort of pre-screening schemes to “control” the entry process. For example, in government-contract bidding, bidders are selected from a list of qualified bidders on a rotating basis (McAfee and McMillan (1987c) [Chap. 7]), which could be perfectly consistent with our result.

A finding worth noting is that, in the optimal auctions with mean-preserving entry, partial subsidy could be involved. This happens when a certain number of bidders are selected, but the selected bidders are not willing to participate. The argument above shows that even in this case, it is to the seller’s benefit to partially subsidize bidders to implement deterministic entry. We provide an example to illustrate this point.

An Example

Suppose potential bidders’ valuations about the object are distributed as Uniform [0,1]. Suppose also that under the optimal auction with random entry, the number of actual bidders is distributed as Poisson ($M$), where $M > 2$. Under the auction with mean-preserving entry, zero reserve price, and zero entry fee, a selected bidder’s expected profit conditional on participation would be

$$E\pi_i = \frac{1}{M} (EV_{M:1} - EV_{M:2}) - c = \frac{1}{M(M + 1)} - c. \quad (2)$$

The randomization condition implies that

$$E \left[ \frac{1}{n + 1} (V_{n+1:1} - \max(V_{n+1:2}, 0)) - c \right] = \frac{1}{M^2} (1 - e^{-M} - Me^{-M}) - c = 0. \quad (3)$$
Substituting the expression for $c$ from equation (3) into equation (2), we have

$$E\pi_i = \frac{1}{M(M+1)} - \frac{1}{M^2}(1 - e^{-M} - Me^{-M})$$

$$= \frac{e^{-M}(M+1)^2 - 1}{M^2(M+1)}$$

$$< 0.$$  

Therefore, potential bidders would not participate when selected.

One way to restore the participation incentive is for the seller to subsidize each participant with a subsidy of an amount $S = c - 1/M(M + 1)$. By doing so, the expected profit of actual bidders is zero and the selected bidders are willing to participate. The seller’s expected revenue under this subsidy scheme is given by

$$ER^S = EV_{M:1} - Mc - MS = 1 - 2Mc.$$  

The expected revenue under random participation is given by

$$ER^R = E(V_{n:1} - nc) = 1 - \frac{1}{M}(1 - e^{-M}) - Mc.$$  

Therefore, $ER^S - ER^R = \frac{1}{M}(1 - e^{-M}) - Mc = e^{-M} > 0$ (by equation (3)).

This example illustrates one role of the subsidy in improving expected revenue, though optimal auctions with deterministic entry never involve a subsidy.

## 3 A benefit of running an auction-negotiation hybrid mechanism

Based on the insight obtained from the previous section, we will show a benefit of running an auction-negotiation hybrid mechanism in this section. For ease of demonstration we consider the following modified model.

There is a single, indivisible asset for sale to $N$ potentially interested buyers (firms). Information is revealed in two stages. In the first stage, each potential buyer is endowed with a private value component $X_i$. The $X_i$’s are independent draws from a distribution with CDF $F(\cdot)$. Bidder $i$ knows its own $X_i$ but not its competitors’ (denoted as $X_{-i}$). In the second stage, by incurring an entry cost $c$, each entrant bidder learns a common value signal $Y$. The total value of the asset to entrant $i$ is given by $X_i + Y.$
In traditional common value models, the common value has some known prior distribution and
bidders’ signals are draws conditional on the underlying common value (see, for example, Rothkopf
(1969), and Wilson (1977)). Here we consider an extreme case in which there is no noise about the
signal: all parties learn the common value precisely. Though somewhat special, this formulation can
still be a good approximation for real world situations when the common value component is highly
standardized. For example, if buyers are to learn general conditions of facilities and equipments,
or the level of outstanding debt, then in the limit of the information acquisition process, we may
assume that the signals are so accurate that everyone who have gone through the same due diligence
process learn the common value component precisely.6

For simplicity we assume that the common value component \( Y \) is known to the seller. Note that
all parties agree on this common value component. We assume that a costless revelation technology
(e.g., a perfectly objective evaluation by a third party) is not available to the seller. This is often
the case because the seller’s announcement of his information may not be credible due to the well
known adverse selection problem, that is, he may always want to claim the highest possible common
value to the buyers. In our case, we assume that the only way the buyers can learn \( Y \) is through
costly information acquisition. That is, each buyer needs to incur a cost \( c \) to learn \( Y \).7

We thus assume that each buyer’s signal consists of two parts, one received before the entry
occurs and the other received after the entry occurs. Moreover, we assume that the two signal
components are additive. This formulation is somewhat special, but it captures one key feature in
sales of highly complicated asset with costly entry. That is, a buyer usually does not know the total
value of the asset in the first stage, and the remaining information about the value can only be
learned after due diligence process is completed. In the above formulations, \( X \) can be interpreted
as the valuation about the asset estimated based on each potential buyer’s private attributes such
as operating experiences, managerial skills, labor relations, market power, and corporate citizenship
etc.8 The second-stage signal \( Y \) may reflect information containing detailed engineering assessment

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6 Note that in our model, \( Y \) can be negative. In fact this is the case for Daewoo’s sale introduced at the beginning
of the paper. The actual outstanding debt was identified to be so high that the final transacted price was much lower
than the bids indicated in the auction stage.

7 An alternative way to credibly reveal the private information is through signaling with reserve prices, which is
analyzed by Jullien and Mariotti (2006) and Cai et al. (2007).

8 We suppress the common value component in the first-stage signal formulation, since the common value component
of the asset, environmental conditions, labor commitments, supply contracts, financial status etc., which may only be learned during the due diligence process.

We start with the benchmark analysis in which a pure auction is conducted. The timeline is similar to what is described in the previous section. That is, in time 1, the auction rule (including entry fee and reserve price, if any) is pre-announced. In time 2, potential bidders simultaneously and independently decide whether to enter the auction or not. Once entering the auction, a bidder pays the entry fee (if any), incurs the information acquisition cost \( c \) and learns \( Y \). Finally, the auction is conducted.

We look for pure symmetric equilibrium in which each potential buyer enters the auction if and only if its private value signal is no less than a cut-off point \( (x^*) \).\(^9\)

The number of entrants is now a random variable. Thus a reserve price may be desirable, especially when there is only one entrant. Given the common value component \( Y \), the seller may announce a relative reserve price, \( r \), with the understanding that the final reserve price will be \( r + Y \), once \( Y \) is revealed or learned. Note that the entry cut-off point \( x^* \) should depend on the reserve price. Anticipating this, the seller will set an optimal reserve price to induce an entry cut-off point which leads to maximal expected revenue.

Using methods familiar from the literature on optimal auctions, one can compute the bidders’ conditional expected profits as follows:\(^10\)

\[
\pi_i(x_i) = \pi_i(x^*) + \int_{x^*}^{x_i} F^{-1}(x) \, dx = \int_{x^*}^{x_i} F^{-1}(x) \, dx
\]

since \( \pi_i(x^*) = 0 \) by the definition of the entry cut-off point. Note that \( \pi_i(x_i) \) is strictly increasing in \( x_i \), so bidders will enter the auction if and only if their types are no less than \( x^* \). This property, together with the definition of \( x^* \), can determine the equilibrium entry condition as follows:\(^11\)

\[
(x^* - r) F^{-1}(x^*) - c = 0
\]

\(^9\)This type of cutoff equilibrium is first analyzed by Samuelson (1985) under a procurement auction setting.

\(^10\)Note that \( Y \) does not contribute to the calculations of bidders’ profit since it is a common value term and is bid away in equilibrium.

\(^11\)When a marginal bidder enters the auction, she wins the object if and only if the private value signals of all the rivals are lower than the entry cut-off point \( x^* \). (Tie occurs with probability zero in equilibrium and hence is ignored here.)
Let \( \tilde{n} \) be the number of entrants associated with cut-off point \( x^* \), then \( \tilde{n} \) is distributed as Binomial \((N, 1 - F(x^*))\), hence \( E\tilde{n} = N(1 - F(x^*)) \).

Conditional on \( Y \), the expected surplus from the sale is

\[
ES = EX_{N:1_{X_{N:1} \geq x^*}} - c \cdot E\tilde{n} + Y = \int_{x^*}^{\bar{x}} xNF^{N-1}(x)f(x) \, dx - Nc(1 - F(x^*)) + Y
\]

The bidders’ expected total profit is

\[
E\Pi = NE\pi(X_i) = N \int_{x^*}^{\bar{x}} F^{N-1}(x)(1 - F(x)) \, dx
\]

Therefore, the expected revenue \((E\Pi^A)\) can be calculated as follows:

\[
E\Pi^A = ES - E\Pi = \int_{x^*}^{\bar{x}} xNF^{N-1}(x)f(x) \, dx - Nc(1 - F(x^*)) - N \int_{x^*}^{\bar{x}} F^{N-1}(x)(1 - F(x)) \, dx + Y
\]

Now maximizing \( E\Pi^A \) over \( r \) subject to entry condition (4), we obtain the relation between the optimal (relative) reserve price and the induced entry cut-off point:\(^{12}\)

\[
r = \frac{1 - F(x^*)}{f(x^*)} \quad (5)
\]

Therefore, the optimal (relative) reserve price \( r \) and the induced entry cut-off point \( x^* \) are jointly determined by equations (4) and (5).

For technical convenience we further assume that the density \( f \) satisfying \( 0 < \underline{f} \leq f(v) \leq \bar{f} < +\infty \) on \([\underline{v}, \bar{v}]\). Moreover, \( f(.) \) is differentiable and \( f'(.) \) is bounded on its support.\(^{13}\) Comparative static analysis reveals the following result:

**Lemma 4** As the number of potential bidders \( N \) is sufficiently large, the entry threshold \( x^* \) approaches \( \bar{x} \); in particular, if the hazard rate function \( H(x) = \frac{f(x)}{1 - F(x)} \) is nondecreasing in \( x \), then \( x^* \) is strictly increasing in \( N \).

The implication is that, as the number of potential bidders gets large, the chance of winning the object is low, and it is more likely for a particular bidder not to enter the auction. In particular,

\[\max_{x^*} E\Pi^A \Rightarrow -NF(x^*)[(x^* - \frac{1 - F(x^*)}{f(x^*)})F^{N-1}(x^*) - c] = 0.\] Comparing this to equation (4), the optimal entry is implemented by setting \( r = \frac{1 - F(x^*)}{f(x^*)} \).

\[^{12}\text{max}_{x^*} E\Pi^A \Rightarrow -NF(x^*)[(x^* - \frac{1 - F(x^*)}{f(x^*)})F^{N-1}(x^*) - c] = 0.\] Comparing this to equation (4), the optimal entry is implemented by setting \( r = \frac{1 - F(x^*)}{f(x^*)} \).

\[^{13}\text{We mean left or right derivatives at its endpoints.}\]
when the hazard rate function is nondecreasing (e.g. uniform distribution), the above proposition demonstrates that successively reducing $N$ decreases the entry threshold, and makes a particular potential bidder more likely to participate. The entry process described here is essentially another type of random processes. In view of the insight obtained from the previous section, reducing the randomness in participation may once again lead to higher expected revenue. We thus turn to the hybrid mechanism, which basically implements a deterministic entry process.

This hybrid mechanism works in two stages. In the first stage an auction is conducted and the bidder with the highest bid ($b_{N:1}$) is selected. In the second stage this selected bidder incurs cost $c$ and learns $Y$. The seller will then offer a price equal to $b_{N:2} + Y$, where $b_{N:2}$ is the second highest (or the highest rejected) bid from the auction process.\(^{14}\)

Given our specific model, it can be easily verified that in equilibrium each bidder bids an amount equal to $X_i - c$ in the auction process, and the highest bidder always accepts the price offered by the seller. Given the seller’s offer in the negotiation stage, it is a weakly dominant strategy for each bidder to bid $X_i - c$ in the auction stage; Given each bidder bids $X_i - c$ in the auction stage, it is always optimal for the final bidder to accept the seller’s offer. As a result, the expected revenue generated by the hybrid mechanism is given by

$$ER^H = EX_{N:2} - c + Y$$

We are now ready to compare this simple hybrid mechanism with the pure auction benchmark described above.

**Proposition 2** If $N$ is sufficiently large, then the hybrid mechanism generates higher expected revenue than the optimal pure auction mechanism.

**Proof:** See Appendix.

The proof is somewhat tedious, but the intuition is clear. The driving force for this limiting dominance result is again the difference in entry processes. The hybrid mechanism implements deterministic entry in the sense that a fixed number of bidders (1 in our model) enter the final sale.

\(^{14}\)Setting alternative offer prices such as $b_{N:1} + Y$ should not alter our analysis in a significant way, though the equilibrium bid function in auction stage will be affected.
However, under a pure auction, the number of entrants is a random variable. This difference in entry processes results in different expected total surplus (including expected total entry cost) in equilibrium, especially when \( N \) is large.

The main idea of the proof goes as follows. First, under the pure auction, entry condition 
\[
(x^* - \frac{1-F(x^*)}{f(x^*)})F^{N-1}(x^*) = c \quad \text{and} \quad 0 < f \leq f(x^*) \text{ imply } F(x^*) = 1 - \delta N/N, \text{ for some } \delta_N, \text{ such that } \\
\delta_N \to \delta \text{ (a constant to be determined) as } N \to \infty.
\]
Using this we can show that when \( N \to \infty \), the no trade (or no entry) probability under the optimal reserve price scheme \( F_N(x^*) \) goes to \( c/\bar{x} \), and the expected number of entrant firms \( E\tilde{n} = N(1 - F(x^*)) = \delta N \to \delta = \log(\bar{x}/c) \).

Therefore under the pure auction mechanism, the expected revenue
\[
ER^A = EX_{N:1} - E\Pi - E\tilde{n} \cdot c \to \bar{x}(1 - \frac{c}{\bar{x}}) - 0 - c \cdot \log(\bar{x}/c)
\]

However, under the hybrid mechanism, trade takes place with probability 1 and exactly one firm incurs entry cost. The expected revenue
\[
ER^H = EX_{N:1} - (EX_{N:1} - EX_{N:2}) - c \to \bar{x} - 0 - c
\]

Therefore, \( ER^H - ER^A \to c \cdot \log(\bar{x}/c) > 0 \), which implies that the hybrid mechanism revenue-dominates the optimal pure auction mechanism as long as \( N \) is sufficiently large.

To get some sense about how large \( N \) needs to be for the dominance result to hold, let’s consider the following example. \( X \) is distributed as Uniform \([0,1]\). Let \( N^* \) be the transition point such that whenever \( N \geq N^* \), the hybrid mechanism revenue-dominates the pure auction mechanism with the optimal reserve price. The following results are obtained from numerical computations:
For \( c/\bar{x} \in [0.03,0.09] \), \( N^* = 3 \); for \( c/\bar{x} \in [0.005,0.01] \), \( N^* = 4 \); for \( c/\bar{x} = 0.003 \), \( N^* = 5 \); for \( c/\bar{x} \in [0.0005,0.001] \), \( N^* = 6 \); for \( c/\bar{x} = 0.0003 \), \( N^* = 7 \); for \( c/\bar{x} = 0.0001 \), \( N^* = 8 \). Presumably the range of the entry cost used in the above computation covers the most common cases in the real sales. Therefore, we have confidence to believe that in the general cases (where \( N \) is almost surely larger than 8), the hybrid mechanism performs better than the optimal pure auction mechanism.

The following table summarizes the results of the numerical computation for the range of entry cost levels \( c/\bar{x} = 0.001,0.003,0.005,0.007,0.009 \), respectively. \( r^*,x^*,N^*,\Delta R^* \) denotes the optimal reserve price, entry cut-off point, transition point, and the limiting difference in expected revenue, respectively. \( (\Delta R^* = \lim_{N \to \infty} ER^H - ER^A.) \)
From the above table, we can see that the dominance result is compelling: as long as $N \geq 6$, the hybrid mechanism generates a higher expected revenue in all the cases considered. The expected gain of using the hybrid mechanism when $N$ is large (e.g., $N = 50$) is also striking. For example, if the maximal possible value of the asset is $1$ billion, and the entry cost is $1$ million (i.e., the entry cost accounts for 0.1% of the maximal value of the asset), the hybrid mechanism could generate $6.9$ million more in revenue on average.

4 Conclusion

Entry costs are prevalent in complex and high-valued asset sales. This paper takes the first step in comparing stochastic and deterministic entry processes when participation is costly. In a simple auction model where bidders do not possess private information before entry, we show that the optimal auction with deterministic entry always outperforms its counterpart with stochastic entry in terms of expected revenue generated. We also demonstrate that an auction with stochastic entry can be improved by using a mean-preserving scheme: Screening to minimize variance in participation, even if it is still random (when $M$ is not an integer), increases expected revenue. So when the seller can influence the entry process, it will in general be in his own interest to reduce the randomness of participation.

This insight is employed to analyze an auction-negotiation hybrid mechanism in the second part of the paper, where bidders possess private information before entry and learn a common value component after entry. We show that, when the number of potential buyers is sufficiently large, or when the randomness in participation in a pure auction is sufficiently prominent, a hybrid mechanism can outperform the optimal pure auction in terms of the expected revenue generated. We thus suggest an explanation for the widespread use of this hybrid mechanism mixing both auction and negotiation.
The main restriction of our current analysis is that our information structure about the common value component is somewhat special. Since all parties (buyers and the seller) observe exactly the same common value signal after entry, the negotiation stage appears overly simplified in the sense that the seller’s offer is always accepted in equilibrium. Since sales always occur with probability one, an important feature of real world negotiations is missing in our model. For more general settings, we should allow for different post-entry valuations or signals between the seller and the buyers, in which case negotiation may break down simply because the seller may offer a price higher than the entrant buyer’s reservation value. Such a generalization is presumably more realistic and should be attempted in future research. Nevertheless we believe that the key insight identified in this current model should carry over to the more general setting as well, and despite the possibility of breakdown in negotiation process, the dominance of the hybrid mechanism over a pure auction should still be anticipated when the randomness of participation presents a major concern.
Appendix

Proof of Lemma 1:

1. The IPV auction. Let the total number of actual bidders be distributed as Poisson ($M$). Let $V_{n;i}$ be the $i$th largest order statistic from $n$ i.i.d. valuation samples, and define $V_{2:1} = V_{1:0} = 0$, then

$$E(V_{n;1}) = E \int vnF^{n-1}(v)f(v) \, dv = M \int ve^{-M(1-F(v))}f(v) \, dv$$

$$E(V_{n+1;1}) = E \int v(n+1)F^n(v)f(v) \, dv = M \int v(MF(v)+1)e^{-M(1-F(v))}f(v) \, dv$$

In the IPV model, it can be shown that the randomization condition (1) is equivalent to $E(V_{n+1:1} - V_{n:1}) = c + \epsilon$, so the seller’s problem is

$$\max_{\epsilon} ER = \int vMe^{-M(1-F(v))}f(v) \, dv - Mc$$

s.t. \( \int v(MF(v)+1)e^{-M(1-F(v))}f(v) \, dv = c + \epsilon \)

Taking the partial derivative of the objective function with respect to $M$, we have

$$\frac{\partial ER}{\partial M} = \int v(MF(v)+1-M)e^{-M(1-F(v))}f(v) \, dv - c = \epsilon$$

$$\frac{\partial ER}{\partial \epsilon} (<) = (>0) as \epsilon(\epsilon) = (>0)$$

Therefore, assuming regularity condition such that $\frac{dM}{\epsilon} < 0$ (i.e. the average number of bidders decreases as the entry fee increases), we have

$$\frac{dER}{\epsilon} = \frac{\partial ER}{\partial M} \frac{dM}{\epsilon} + \frac{\partial ER}{\partial \epsilon} \frac{dM}{\partial \epsilon}$$

\( (>0) as \epsilon(\epsilon) = (>0) \), which implies that $ER$ achieves its maximum at $\epsilon = 0$.

2. The common value auction. In the common value model, the value of the item to the winner, $V$, is independent of the number of bidders. With the optimal reserve price $R^* = 0$, $T_n(R^*) = 1$ for all $n, n \geq 1$. Therefore, $ER = ES = EV(1-\gamma_0) - Mc = EV(1-e^{-M}) - Mc$, and

$$\frac{\partial ER}{\partial \epsilon} = EV \cdot e^{-M} - c.$$
Let $M^*$ be such that $EV \cdot e^{-M} - c = 0$, then similarly to the above argument, as long as $\frac{dR}{de} < 0$, we have $\frac{dER}{de}(>) = (0) as M(<) = (>)^*M^*$, or as $e(>) = (<)^*e$. It remains to determine $e^*$.

The entry condition is given by

$$\sum_{n=1}^{\infty} \gamma_n (EV - W_n) - M(c + e) = 0. \tag{6}$$

Substituting $c = EV \cdot e^{-M^*}$ into equation (6), and noting that $W_1 = 0$, we have

$$M^*e^* = \sum_{n=1}^{\infty} \gamma_n (EV - W_n) - M^*EV \cdot e^{-M^*}$$

$$= \gamma_1 (EV - W_1) + \sum_{n=2}^{\infty} \gamma_n (EV - W_n) - M^*EV \cdot e^{-M^*}$$

$$= \sum_{n=2}^{\infty} \gamma_n (EV - W_n)$$

$$> 0$$

Therefore, the optimal entry fee is strictly positive. Hence,

$$\frac{dER}{de}|_{e=0} = \frac{\partial ER}{\partial M}|_{e=0} > 0$$

which implies that zero entry fee leads to excessive entry.

Proof of Lemma 2: We claim that $EV_{n:1}$ is strictly concave in $n$. Equivalently, we claim that the differences $EV_{n+1:1} - EV_{n:1}$ decline in $n$. To see this, let $X,Y,Z$ be random variables. Then for any realization of the triple $(X,Y,Z)$, it is easily verified that

$$\max(X,Y,Z) - \max(X,Y) \leq \max(X,Z) - X$$

The inequality is strict with positive probability. Therefore, taking expectations on both sides, we have

$$E[\max(X,Y,Z)] - E[\max(X,Y)] < E[\max(X,Z)] - EX \tag{7}$$

Now let $X = V_{n-1:1}$, and let $Y,Z,V_1,V_2,\cdots, V_{n-1}$ be i.i.d. variables. Then $E(\max(X,Y,Z)) = EV_{n+1:1}$, $E(\max(X,Y)) = E(\max(X,Z)) = EV_{n:1}$. Substituting these into equation (7), we have $EV_{n+1:1} - EV_{n:1} < EV_{n:1} - EV_{n-1:1}$. So the claim is correct and $EV_{n:1} - nc$ is also strictly concave
Proof of Lemma 4: The first part is easily seen from the entry condition (4):

\[(x^* - H^{-1}(x^*)F^{N-1}(x^*)) = c\]

By assumption, \(0 < f \leq f(.) \leq \bar{f}\), thus \((x^* - H^{-1}(x^*))\) is bounded. As \(N \to \infty\), if \(x^*\) does not approach \(\bar{x}\), then the entry condition cannot hold. For the second part, taking logarithm transformation and then differentiating with respect to \(N\), we have

\[
\frac{1}{x^* - x^*}(1 - \frac{dH^{-1}(x^*)}{dx^*}) + (N - 1)\frac{f(x^*)}{F(x^*)}\frac{dx^*}{dN} = -\log F(x^*)
\]

Thus if \(H'(x^*) \geq 0\), \(\frac{dH^{-1}(x^*)}{dx^*} \leq 0\), and \(\frac{dx^*}{dN} > 0\).

Proof of Proposition 2: Let \(ER^H\) be the expected revenue generated by the hybrid mechanism, and let \(ER^A\) be the expected revenue generated by the pure auction with the optimal reserve price determined in equations (4) and (5). We will actually show the following:

\[
\lim_{N \to \infty} ER^H - ER^A = c \cdot log\left(\frac{\bar{x}}{c}\right)
\]

Since the limit is a strictly positive term, \(ER^H > ER^A\) when \(N\) is sufficiently large.

To begin, note that the entry condition \((x^* - \frac{1-F(x^*)}{f(x^*)})F^{N-1}(x^*) = c\) and \(0 < f \leq f(x^*)\) imply \(F(x^*) = 1 - \delta_N/N\), for some \(\delta_N\), such that \(\delta_N \to \delta\) as \(N \to \infty\). (\(\delta\) is determined below.)

It is also easily verified that \(x^* \to \bar{x}\) as \(N \to \infty\). Using the Taylor expansion, we have

\[
x^* = F^{-1}(1 - \frac{\delta_N}{N})
\]

\[
= \bar{x} - \frac{\delta_N}{N} \frac{1}{f(\bar{x})} + O(1/N^2)
\]

\[
= \bar{x} - \epsilon_N\frac{\bar{x}}{N}
\]

where \(\epsilon_N \to \epsilon = \frac{\delta}{f(\bar{x})}\). Substituting \(F(x^*) = 1 - \delta_N/N\) into the entry condition, and letting \(N \to \infty\), we have

\[
\delta = \log\left(\frac{\bar{x}}{c}\right)
\]
First, we have as $N \to \infty$, \( ER^H = E(X_{N;2}) - c + Y \to \bar{x} - c + Y \).

To determine the limit of
\[
ER^A = \int_{x^*}^{\bar{x}} xN F^{N-1}(x)f(x) \, dx - Nc (1 - F(x^*)) - \int_{x^*}^{\bar{x}} N F^{N-1}(x)(1 - F(x)) \, dx + Y
\]

We compute the limit of each term as follows:

1) \( Nc (1 - F(x^*)) = Nc \delta_N / N \to c \delta \).

2) Since
\[
0 \leq \int_{x^*}^{\bar{x}} N F^{N-1}(x)(1 - F(x)) \, dx \\
\leq N \int_{x^*}^{\bar{x}} (1 - F(x)) \, dx \\
\leq N \int_{x^*}^{\bar{x}} (1 - F(x^*)) \, dx \\
= \int_{x^*}^{\bar{x}} \delta_N \, dx \\
\to 0,
\]

\( \int_{x^*}^{\bar{x}} N F^{N-1}(x)(1 - F(x)) \, dx \to 0 \). This is intuitive since the bidders’ rents are driven toward zero by increasing competition as $N$ increases.

3) Using the Taylor expansion again,
\[
F(x) = F(x^*) + (x - x^*)f(x^*) + O(x - x^*)^2 \\
= 1 - \frac{\delta_N}{N} + (x - \bar{x} + \frac{\epsilon_N}{N}) f(x^*) + O(\frac{1}{N^2}) \\
= 1 + \frac{(\epsilon_N + N(x - \bar{x})) f(x^*) - \delta_N}{N} + O(\frac{1}{N^2})
\]

\[
F^N(x) = \exp[N \log F(x)] \\
= \exp[N \log(1 + \frac{(\epsilon_N + N(x - \bar{x})) f(x^*) - \delta_N}{N} + O(\frac{1}{N^2}))] \\
= \exp[N(\frac{(\epsilon_N + N(x - \bar{x})) f(x^*) - \delta_N}{N} + O(\frac{1}{N^2}))] \\
= \exp[(\epsilon_N + N(x - \bar{x})) f(x^*) - \delta_N + O(\frac{1}{N})]
\]
Therefore,

\[
\int_{x^*}^\bar{x} xNF^{N-1}(x)f(x)\,dx = e^{\epsilon Nf(x^*) - \delta_N + O(1/N)} \int_{x^*}^{\bar{x}} \frac{x f(x)}{F(x)} Ne^{Nf(x^*)(x-\bar{x})}dx
\]

\[
= e^{\epsilon Nf(x^*) - \delta_N + O(1/N)} \xi_N f(\xi_N) \frac{\xi_N f(\xi_N)}{F(\xi_N)} \int_{x^*}^{\bar{x}} Ne^{Nf(x^*)(x-\bar{x})}dx
\]

\[
= e^{\epsilon Nf(x^*) - \delta_N + O(1/N)} \xi_N f(\xi_N) \frac{1}{F(\xi_N)} \frac{1}{f(x^*)} \int_{x^*}^{\bar{x}} e^{Nf(x^*)(x-\bar{x})}dx
\]

\[
\rightarrow e^{f(x)\epsilon - \delta} \frac{\bar{x} f(\bar{x})}{f(\bar{x})} \frac{1}{1 - e^{-f(x)\epsilon}} \int_{x^*}^{\bar{x}} e^{Nf(x^*)(x-\bar{x})}dx
\]

\[
= \bar{x} e^{f(x)\epsilon - \delta} [1 - e^{-f(x)\epsilon}] \int_{x^*}^{\bar{x}} e^{Nf(x^*)(x-\bar{x})}dx
\]

\[
= \bar{x} (1 - e^{-\delta})
\]

The second equality is due to the Mean Value Theorem. \( \xi_N \in [x^*, \bar{x}] \), thus \( \xi_N \rightarrow \bar{x} \) as \( N \rightarrow \infty \).

Therefore,

\[
ER^A \rightarrow \bar{x} (1 - e^{-\delta}) - c\delta - 0 + Y
\]

\[
= \bar{x} (1 - \frac{c}{\bar{x}}) - c \cdot \log(\frac{\bar{x}}{c}) + Y
\]

\[
= \bar{x} - c - c \cdot \log(\frac{\bar{x}}{c}) + Y
\]

Finally, we have

\[
\Delta R = ER^H - ER^A \rightarrow [\bar{x} - c] - [\bar{x} - c - c \cdot \log(\frac{\bar{x}}{c})]
\]

\[
= c \cdot \log(\frac{\bar{x}}{c})
\]
References


