Optimal Two-stage Auctions with Costly Information Acquisition*

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September 2013

Abstract

We consider a two-stage auction environment with costly information acquisition. Bidders are endowed with original estimates (“types”) about their private values, and can further learn their true values of the object for sale by incurring an information acquisition cost. We show that optimality of the generalized Myerson allocation rule is robust to this sequential screening setting with costly entry, which implies that optimal entry is to admit the set of bidders that maximizes expected virtual surplus adjusted by both the second-stage signal and entry cost. We also show that the optimal mechanism can be implemented via a two-stage auction that is essentially a handicap auction augmented with an entry mechanism.

Keywords: Two-stage auctions, entry, information acquisition, sequential screening, handicap auctions, optimal mechanisms.

JEL Classification: D44, D80, D82.

1 INTRODUCTION

In high-valued asset sales, buyers often need to go through a due diligence processes before developing final bids. Due diligence is usually a process to update or acquire information about their values of the asset for sale, or to prepare for the bidding process. This process is costly and is usually modeled as entry as it is closely monitored by the auctioneer. For a sale of an asset with billion dollars, the entry cost can run from tens of thousands to millions of dollars.1

Given the substantial entry cost, it is unrealistic to assume that whoever is interested would necessarily go through the costly entry process. The success of a sale thus very much relies on whether the

*We thank seminar participants at the Conference in Honor of Paul Milgrom’s 65th Birthday, Asian Meeting of the Econometric Society, and in particularly, Dirk Bergemann, Yeon-Koo Che, Jeff Ely, Li Hao, Preston McAfee, and Ilya Segal for very helpful comments and suggestions. All remaining errors are our own.

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1See Vallen and Bullinger (1999) for a detailed description of the information acquisition process in a typical electric power asset sale.
most qualified bidders would commit to the due diligence process and participate in the final sale. Mainly motivated by the need for entry screening, variants of two-stage selling mechanisms have emerged in the real world. A leading example of the two-stage auction procedure is known as indicative bidding, which is commonly used in sales of complicated business assets with very high values. It works as follows: the auctioneer actively markets the assets to a large group of potentially interested buyers. The potential buyers are then asked to submit non-binding bids, based on which a final set of bidders is shortlisted to advance to the second stage. The auctioneer then communicates only with these final bidders, providing them with extensive access to information about the assets, and finally runs the auction (typically using binding sealed bids). The use of this two-stage auction procedure is quite widespread. For example, in response to the restructuring of the electric power industry in the U.S. – which was designed to separate power generation from transmission and distribution – billions of dollars of electrical generating assets were divested through this two-stage auction procedure over the last two decades. This two-stage auction procedure is also commonly used in privatization, takeover, merger and acquisition contests.

Finally, it is commonly used in the institutional real estate market, which has an annual sales volume in the order of $60 to $100 billion.

Ye (2007) was the first study of indicative bidding using the assumption of costly information acquisition. Ye’s analysis suggests that the current design of indicative bidding cannot reliably select the most qualified bidders for the final sale, as there does not exist a symmetric, strictly increasing equilibrium bid function in the indicative bidding stage. In a more recent paper, by restricting indicative bids to a finite discrete domain, Quint and Hendricks (2013) show that a symmetric equilibrium exists in weakly-monotone strategies. But again, the highest-value bidders are not always selected, as bidder types “pool” over a finite number of bids. Without safely selecting the highest bidders for the final sale, the mechanism is hardly optimal in maximizing expected revenue. What is the optimal mechanism in this two-stage auction environment remains an open question in the literature, and this paper seeks to provide an answer.

We model the two-stage auction environment as follows. Before entry, each potential bidder is endowed with a private signal, $a_i$, which can be regarded as her pre-entry “type”. After entry (by incurring a common entry cost, $c$), each bidder acquires another private signal, $z_i$, which is typically correlated to her pre-entry type. We assume that the (private) value of the asset to bidder $i$, $v_i$, is completely determined given $a_i$ and $z_i$. Given costly entry, it is not feasible for all potential bidders to be included in the final sale. As such, the optimal selling procedure in this setting takes the form of two-stage mechanisms,

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2 A list of industry examples using this two-stage auction design can be found in Ye (2007).
3 Leading examples include the privatization of the Italian Oil and Energy Corporation (ENI), the acquisition of Ireland’s largest cable television provider Cablelink Limited, and the takeover contest for South Korea’s second largest conglomerate Daewoo Motors.
4 See Foley (2003) for a detailed account.
5 Compte and Jehiel (2007) offer a simple two-stage auction model to illustrate a benefit of running multi-stage mechanisms. Boone and Goeree (2009) provide an analysis of pre-qualifying auctions, which are similar to indicative bidding.
with the first stage being the entry right allocation mechanism and the second stage being the traditional private good allocation mechanism.

Despite the potential complication due to both sequential screening and costly information acquisition, we are able to completely characterize the optimal revenue-maximizing mechanisms. Our analysis benefits greatly from recent developments in the literature of sequential screening (e.g., Courty and Li, 2000; Esö and Szentes, 2007; Pavan, Segal, and Toikka, 2012; and Bergemann and Wambach, 2013). In particular, our analysis follows Esö and Szentes closely, and our technical contribution is to extend their analysis to dynamic auctions with costly information acquisition. Using a signal orthogonalization technique pioneered by Esö and Szentes, we show that the optimal allocation rule of the asset in our second stage is the same as that identified by Esö and Szentes, which requires that, among the shortlisted bidders, the asset be allocated to the bidder with the highest virtual value adjusted by the second-stage signal. Our analysis thus suggests that the optimality of the generalized Myerson optimal allocation rule (adjusted by second-round signals) is even robust to the dynamic auction setting with costly entry. The first-stage entry right allocation mechanism is new to the original Esö-Szentes framework, and we show that the optimal entry rule is to admit the set of bidders that gives rise to the maximum expected virtual surplus (adjusted by both the second-stage signal and entry cost). Alternatively, given that buyers are ex ante symmetric in our model, the optimal entry rule is to admit the bidders in descending order of their pre-entry “types”, the highest type first, the second highest type second, etc., provided that their marginal contribution to the expected virtual surplus is positive. Therefore the optimal number of shortlisted bidders typically depends on the reported type profile from the potential bidders, which is endogenously determined.

For an important setting where one’s value is linear in her pre-entry type, Esö and Szentes show that their optimal mechanism can be implemented over two rounds via a so-called handicap auction: in the first round (before observing the second-stage signals), each buyer selects a “price premium” by paying a fee according to a pre-announced schedule. In the second round (after observing the second-stage signals), buyers compete in a second-price or English auction, where the winner obtains the object at a price equal to the second-highest bid plus the price premium selected from the first round. In our setting with entry, the implementation is presumably more complicated, as optimal entry needs to be implemented prior to the final auction. Indeed, now the implementation requires that an (optimal) entry rule be augmented to the handicap auction. So in our case the optimal mechanism is implemented via a two-stage auction, with the first stage being an auction for entry rights (as well as the price premia) and the second stage being a second-price or English auction for the asset.

Other than the connection with sequential screening and dynamic auctions mentioned above, our paper is closely related to the growing literature on auctions with costly entry.6 This literature can be summarized into three branches. In the first branch, bidders are assumed to possess no private infor-

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6See Bergemann and Välimäki (2006) for a thoughtful survey of this literature.
mation before entry and they learn their private values or signals only after entry (see, for example, Milgrom, 1981; McAfee and McMillan, 1987; Engelbrecht-Wiggans, 1993; Tan, 1992; Levin and Smith, 1994; and Ye, 2004). In the second branch, it is assumed that bidders are endowed with private information about their values but have to incur entry costs to participate in an auction (see, for example, Samuelson, 1985; Stegeman, 1996; Campbell, 1998; Menezes and Monteiro, 2000; Tan and Yilankaya, 2006; Cao and Tian, 2009; and Lu, 2009). Finally, in the third branch, bidders are endowed with some private information before entry, and are able to acquire additional private information after entry (Ye, 2007; Quint and Hendricks, 2013). The framework in this current paper nests all the models mentioned above as special cases. Our paper thus characterizes optimal mechanisms for a framework that is most general in the existing literature on auctions with costly entry.

Our research is also related to a small literature on auctions of entry rights. In their seminal work, Fullerton and McAfee (1999) introduce auctions for entry rights to select shortlisted contestants for a final tournament. Ye (2007) extends their approach to the setting of two-stage auctions described above. Our current approach differs from theirs in the way the set of finalists is determined: while in their approach the number of finalists to be selected is fixed and pre-announced, in our entry right allocation mechanism the selection of shortlisted bidders is contingent on the reported bid profile, making the number of finalists endogenously determined. For this reason the entry right allocation mechanism examined in this research is more general.7

In another relevant paper, Lu and Ye (2013) explore optimal two-stage mechanisms in an environment where bidders are characterized by heterogenous and private information acquisition costs before entry. In Lu and Ye (2013), the pre-entry “type” is the entry cost, which is neither correlated to or part of the value of the asset for sale. As such, there is no benefit to make the second-stage mechanism contingent on the reports of the pre-entry types, resulting in a much simpler characterization of optimal mechanisms. The setting in this current paper is different, as the pre-entry “type” is correlated to the value of the asset, hence there are potential gains to make the second-stage mechanism contingent on first-stage reports. Indeed, in our current setting, the optimal allocation and payment rules in the second stage do depend on the first-stage reports. Therefore the characterization of optimal mechanisms is more demanding in this work, and the implementation of the optimal mechanism is also more sophisticated.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 identifies the optimal mechanism and its auction implementation. Section 4 provides concluding remarks.

2 The Model

The information structure in our model is closest to that in Esö and Szentes (2007). The main difference is that in Esö and Szentes, the additional information is controlled by the seller and they focus on the

7In fact, it resemble multi-unit auctions with endogenously determined supply (see, e.g., McAdams, 2007).
seller’s incentive to disclose (without observing) additional signals to the buyers. In our setting, however, it is costly for the bidders to acquire additional information and we focus on the bidders’ incentive for information acquisition (entry). In addition, all buyers are included in the final sale in Esö and Szentes, but due to costly entry in our setting not all the buyers will be willing to participate in the final auction. As such, we will additionally consider entry mechanisms – which is the major difference from the analysis in Esö and Szentes.

Formally, a single indivisible asset is offered for sale to \( N \) potentially interested buyers. The seller and bidders are assumed to be risk neutral. The seller’s own valuation for the asset is normalized to 0. Buyer \( i \)’s true valuation for the asset is \( v_i \). However, initially she only observes a noisy signal of it, \( \alpha_i \), which is her private information and can be interpreted as her original “type.” After incurring a common information acquisition cost (or entry cost) of \( c \), bidder \( i \) acquires an additional private signal, \( z_i \). We assume that \( E[v_i|\alpha_i,z_i] \) is strictly increasing in \( \alpha_i \) and \( z_i \). For example, \( z_i = v_i \).

Ex ante, \( \alpha_i \) follows distribution \( F(\cdot) \) with its associated density \( f(\cdot) \) on support \([\alpha,\overline{\alpha}]\).\(^8\) We assume that \( f \) is positive on the interval \([\alpha,\overline{\alpha}]\) and satisfies the monotone hazard rate condition, that is, \( f/(1-F) \) is weakly increasing.

Buyer \( i \)'s initial signal and post-entry signal, \( \alpha_i \) and \( z_i \), need not be independent or conditionally independent given \( v_i \). The pairs \((\alpha_i, z_i)\) are assumed to be independent across \( i \).\(^9\)

Since buyers are assumed to be risk neutral, we can assume without loss of generality that \( v_i = E[v_i|\alpha_i,z_i] \), that is, after acquiring the signal \( z_i \), the buyer knows her true value of the asset (e.g. \( z_i = v_i \) or \( z_i = v_i - \alpha_i \)). Given \( \alpha_i, v_i \) follows distribution \( H_{\alpha_i} = H(\cdot|\alpha_i) \) with its density \( h_{\alpha_i} = h(\cdot|\alpha_i) \) over support \([\underline{v}(\alpha_i),\overline{v}(\alpha_i)]\).

The values of \( N, c \), and distributions \( F(\cdot) \) and \( H_{\alpha_i} \) are all common knowledge.

Following the signal orthogonalization technique introduced by Esö and Szentes (2007), we define a new random variable

\[
s_i = H(v_i|\alpha_i),
\]

which is the percentile of the value realization to bidder \( i \). It is easily seen that \( s_i \) is uniformly distributed over \([0,1]\), and is hence statistically independent of the initial information \( \alpha_i \). Given type \( \alpha_i \) and signal \( s_i \), the valuation can be computed as

\[
v_i = H^{-1}_{\alpha_i}(s_i) = u(\alpha_i,s_i).
\]

After observing \( s_i \), a type-\( \alpha_i \) bidder knows her true value \( v_i \). As in Esö and Szentes, we will show that the optimal mechanism in our setting will require bidders to report \( s_i \)'s, rather than \( z_i \)'s or \( v_i \)'s.

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\(^8\)Esö and Szentes allow \( \alpha_i \)'s to be drawn from different distributions. This is not essential to our analysis, so we assume that bidders are ex ante symmetric for simplicity.

\(^9\)This assumption rules out the possibility of full rent extraction (Crémer and McLean, 1988).
Following Esö and Szentes, we maintain the following assumptions:

**Assumption 1.** \((\partial H_{\alpha_i}(v_i)/\partial v_i)/h_{\alpha_i}(v_i)\) is increasing in \(v_i\).

**Assumption 2.** \((\partial H_{\alpha_i}(v_i)/\partial v_i)/h_{\alpha_i}(v_i)\) is increasing in \(\alpha_i\).

Esö and Szentes show that these assumptions can be interpreted as a kind of substitutability in buyer \(i\)'s posterior valuation between \(\alpha_i\) and \(s_i\).

**Lemma 1.** (Esö and Szentes, 2007) Assumption 1 is equivalent to \(u_{12} \leq 0\) and Assumption 2 is equivalent to \(u_{11}/u_{1} \leq u_{12}/u_{2}\).

Assumption 1 thus says that the marginal impact of the new information on buyer \(i\)'s value is decreasing in her type \(\alpha_i\). As demonstrated in Esö and Szentes, Assumption 2 implies that an increase in \(\alpha_i\), holding \(u(\alpha_i,s_i)\) constant, weakly decreases the marginal value of \(\alpha_i\).

**Example 1.** (Ye, 2007): Each potential bidder is endowed with a private value component \(\alpha_i\) before entry; after entry, each buyer learns another private value component \(z_i\), where \(z_i\) is independent of \(\alpha_i\). The transformed model is obtained by setting \(s_i = z_i\), and \(u(\alpha_i,s_i) = \alpha_i + s_i\). By the linearity of \(u(\alpha_i,s_i)\), Assumptions 1 and 2 hold.

**Example 2.** (Adapted from Esö and Szentes, 2007): \(v_i\) is drawn from a normal distribution with mean \(\mu\) and precision (inverse variance) \(\tau_0\). The pre-entry type, \(\alpha_i\), is normally distributed with mean \(v_i\) and precision \(\tau_v\). After entry, the buyer can observe her true value, \(z_i = v_i\). It is clear that \(v_i, z_i\), and \(\alpha_i\) are strictly affiliated. The distribution of \(\alpha_i\), which is normal, satisfies the hazard rate condition. The cdf of \(v_i\) conditional on \(\alpha_i\), \(H_{\alpha_i}\), is normal with mean \((\tau_0 \mu + \tau_v \alpha_i)/(\tau_0 + \tau_v)\) and precision \(\tau_0 + \tau_v\). Define \(s_i = H_{\alpha_i}(z_i)\) and let \(u(\alpha_i,s_i) = H^{-1}_{\alpha_i}(s_i) = v_i\). Obviously \(u\) is strictly increasing in \(s_i\). Following the argument in Esö and Szentes, it can be verified that \(u_1(\alpha_i,s_i) = \tau_v/(\tau_0 + \tau_v)\), which is a constant. Therefore, \(u\) is linear and strictly increasing in \(\alpha_i\). Hence Assumptions 1 and 2 hold.

We consider a general mechanism design framework in which the seller also exercises entry control, so that the mechanism is conducted in two stages: the first stage is the entry right allocation mechanism and the second stage is the private good provision mechanism. We should emphasize that in this general mechanism design framework, the second-stage mechanism can be made contingent on the first-stage reports.

In our model, only the bidders who are granted entry rights can go through the costly information acquisition process and participate in the auction, and once admitted, each bidder incurs the same information acquisition cost. This assumption is mainly motivated by the institutional constraint in the real world two-stage auction settings. For a typical sale of complicated and high-valued assets (e.g., electrical power plants), the “due diligence” process, including the access to “data room,” is highly controlled and
closely monitored by the auctioneer. As such, a bidder can only obtain substantial information about the value of the asset from this due diligence process, and the information acquisition cost is more or less the same for each bidder.\textsuperscript{10}

Let $N = \{1, 2, ..., N\}$ denote the set of all the potential buyers and $2^N$ denote the collection of all the subsets (subgroups) in $N$, including the empty set $\phi$. The first-stage mechanism is characterized by the shortlisting rule $A^\beta(\alpha)$ and payment rule $x_i(\alpha), i = 1, 2, ..., N$. Given the reported profile $\alpha$, the shortlisting rule, $A^\beta : [\underline{\alpha}, \overline{\alpha}]^N \rightarrow \{0, 1\}$, assigns a probability to each subgroup $g \in 2^N$, where $\sum_{g \in 2^N} A^\beta(\alpha) = 1$. The payment rule $x_i : [\underline{\alpha}, \overline{\alpha}]^N \rightarrow \mathbb{R}$, specifies bidder $i$'s first-stage payment given the reported profile $\alpha$.

We assume that the first-stage reported profile $\alpha$ is revealed to all the entrant bidders. Given the first-stage report profile $\alpha$, the second-stage mechanism is characterized by $p^s_i(\alpha, s^g)$, the probability that the asset is allocated to buyer $i \in g$ and $t^s_i(\alpha, s^g)$, the payment to the seller made by buyer $i \in g, \forall g \in 2^N$.

3 The Analysis

We start with the second stage. Suppose group $g$ is shortlisted, and the profile $\hat{\alpha}$ reported in the first stage is revealed as public information.

First, suppose $\alpha$ is truthfully reported at the first stage and group $g$ is shortlisted. Assume that they follow the recommendation and incur the information acquisition cost $c$ to discover $s^g$.\textsuperscript{11}

Given the announced $\alpha$ and $s_i$, define the interim winning probability and expected payment rule $P^s_i(\alpha, s_i) = E_{s_i, \hat{\alpha}}[p^s_i(\alpha, s^g) \text{ and } T^s_i(\alpha, s_i) = E_{s_i, \hat{\alpha}}[t^s_i(\alpha, s^g)$, where $s^g_i = s^g \setminus \{s_i\}$, for $\forall i \in g, \forall g \in 2^N$. Then bidder $i$'s second-stage interim expected payoff when she observes $s_i$ but reports $\hat{s}_i$ is as follows:

$$\overline{\pi}^s_i(\alpha; s_i, \hat{s}_i) = E_{\hat{s}_i}[u(\alpha_i, s_i)p^s_i(\alpha, \hat{s}_i, s^g_i) - t^s_i(\alpha, \hat{s}_i, s^g_i)] = u(\alpha, s_i)p^s_i(\alpha, \hat{s}_i) - T^s_i(\alpha, \hat{s}_i).$$

The second-stage incentive compatibility (IC) conditions require that

$$\overline{\pi}^s_i(\alpha; s_i, \hat{s}_i) \leq \overline{\pi}^s_i(\alpha; s_i, s_i), \forall g, \alpha, s_i, \hat{s}_i.$$  \hspace{1cm} (1)

First, the following lemma is standard from the traditional screening literature:

**Lemma 2.** Suppose $\alpha$ is truthfully revealed from the first stage and $P^s_i(\alpha, s_i), \forall i \in g$ is continuous and weakly increasing in $s_i$ where $g$ denotes the group being shortlisted, then the second-stage incentive compatibility condition (1) holds if and only if

$$\overline{\pi}^s_i(\alpha; s_i, s_i) = \overline{\pi}^s_i(\alpha; \hat{s}_i, \hat{s}_i) + \int_{s_i}^{\hat{s}_i} u_2(\alpha_i, \tau)P^s_i(\alpha, \tau)d\tau, \forall s_i > \hat{s}_i, \forall i \in g.$$  \hspace{1cm} (2)

\textsuperscript{10}More details are provided in Section 7 of Ye (2007).
\textsuperscript{11}As will be shown, the equilibrium expected profit from going forward is positive for a buyer upon entry, so on the equilibrium path a buyer does have an incentive to follow the recommendation to acquire (costly) information and participate in the final auction.
Next, we consider the case when $\hat{\alpha}_i$ instead of $\alpha_i$ is reported by bidder $i$ while others report their types truthfully. Paralleling to Esö and Szentes (2007), we can establish Lemmas 3 and 4 below. The proofs are similar to those in Esö and Szentes and are hence omitted. Our analysis differs from Esö and Szentes in that we accommodate entry in the first stage, and the first-stage reports are revealed and made public.

**Lemma 3.** Suppose $\alpha_{-i}$ is truthfully revealed from the first stage and the second-stage mechanism is incentive-compatible given a truthfully revealed $\alpha$. Then buyer $i$ of type $\alpha_i$ who reported $\hat{\alpha}_i$ in the first round will report $\hat{s}_i = \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$ if she observes $s_i$ in the second stage such that

$$u(\alpha_i, s_i) = u(\hat{\alpha}_i, \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)).$$

(3)

Note that $\hat{s}_i$ does not depend on $\alpha_{-i}, g$, or $s^g_i$. Lemma 3 thus says that whenever a shortlisted bidder had misrepresented her type in the first stage, she would “correct” her lie in the second stage. Define

$$\bar{\pi}^g_i(\alpha, \hat{\alpha}_i; s_i, \hat{s}_i) = E_{s^g_i} [u(\alpha, s_i)p^g(\alpha_{-i}, \hat{\alpha}_i, \hat{s}_i, s^g_{-i}) - t^g(\alpha_{-i}, \hat{\alpha}_i, \hat{s}_i, s^g_{-i})]$$

$$= u(\alpha, s_i)p^g(\alpha_{-i}, \hat{\alpha}_i, \hat{s}_i) - T^g(\alpha_{-i}, \hat{\alpha}_i, \hat{s}_i);$$

$$\bar{\pi}^g_i(\alpha, \hat{\alpha}_i; \alpha_{-i}) = E_{s_i} \bar{\pi}^g_i(\alpha, \hat{\alpha}_i; s_i, \hat{s}_i = \sigma_i(\alpha, \hat{\alpha}_i, s_i)).$$

$\bar{\pi}^g_i(\alpha, \hat{\alpha}_i; \alpha_{-i})$ is the expected second-stage payoff for type-$\alpha_i$ bidder if she reported $\hat{\alpha}_i$ in the first stage (and everyone else reported truthfully).

**Lemma 4.** Suppose $\alpha_{-i}$ is truthfully revealed from the first stage and the second-stage mechanism is incentive-compatible given a truthfully revealed $\alpha$. If buyer $i$ of type $\alpha_i$ who reported $\hat{\alpha}_i$ in the first stage is shortlisted in group $g_i$, her expected payoff from the second stage is given by

$$\bar{\pi}^g_i(\alpha_i, \hat{\alpha}_i; \alpha_{-i}) = \bar{\pi}^g_i(\hat{\alpha}_i, \hat{\alpha}_i; \alpha_{-i}) + \int_0^1 \int_{\hat{\alpha}_i}^1 u_i(y, s_i)p^g_i(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i))dydG_i(s_i).$$

(4)

We are now ready to consider the first-stage IC mechanism.

Let $\pi_i(\alpha_i, \hat{\alpha}_i)$ be the expected payoff (net of the entry cost) for a type-$\alpha_i$ bidder who reports $\hat{\alpha}_i$ in the first stage. By Lemma 3, we have

$$\pi_i(\alpha_i, \hat{\alpha}_i) = E_{\alpha_{-i}} \left\{ \sum_{g_i} A^g_i(\hat{\alpha}_i, \alpha_{-i})[\bar{\pi}^g_i(\alpha_i, \hat{\alpha}_i; \alpha_{-i}) - c] - x_i(\hat{\alpha}_i, \alpha_{-i}) \right\}$$

(5)
Pavan, Segal, and Toikka (2012) extend the envelope theorem to a dynamic mechanism design setting.

We are now ready to derive the seller's expected payoff from an IC two-stage mechanism. By Lemma 5,

Lemma 5.

The first-stage IC requires that

\[ \alpha \text{ continuous in } s_i \]

and

\[ \pi_i(\alpha_i, \hat{\alpha}_i) = \pi_i(\alpha_i, \alpha_i) \]

where \( \hat{\alpha}_i = \alpha_i(\alpha_i, \hat{\alpha}_i, s_i) \) and \( \pi_i(\alpha_i) = E_{\alpha \sim \alpha_i} \pi_i(\alpha_i, \alpha_i) \).

The following lemma characterizes the bidder's expected payoff in an IC two-stage mechanism with costly entry.

**Lemma 5.** If the two-stage mechanism is incentive compatible and \( E_{\alpha \sim \alpha_i} A^g(\alpha_i, \alpha_{-i}) P^g_i(\alpha_i, \alpha_{-i}, s_i) \) is continuous in \( \alpha_i \) then buyer \( i \)'s expected payoff (as a function of her pre-entry type) can be expressed as

\[
\pi_i(\alpha_i, \alpha_{-i}) = \pi_i(\alpha, \alpha) + \int_a^{\hat{\alpha}_i} \left\{ \sum_{g} A^g(\alpha, \alpha_{-i}) \left[ \int_0^1 u_1(y, s_i) P^g_i(y, \alpha_{-i}, s_i) dG_i(s_i) \right] \right\} dy \cdot F(\alpha_i)
\]

**Proof.** See Appendix.

(7) is basically an envelope formula adapted to two-stage mechanisms with entry. In a recent paper, Pavan, Segal, and Toikka (2012) extend the envelope theorem to a dynamic mechanism design setting.

### 3.1 The Optimal Two-stage Mechanisms

We are now ready to derive the seller's expected payoff from an IC two-stage mechanism. By Lemma 5, we have

\[
E \pi_i(\alpha_i, \alpha_{-i})
\]

\[
= \pi_i(\alpha, \alpha) + \int_a^{\hat{\alpha}_i} \left\{ \sum_{g} A^g(\alpha, \alpha_{-i}) \left[ \int_0^1 u_1(y, s_i) P^g_i(y, \alpha_{-i}, s_i) dG_i(s_i) \right] \right\} dy \cdot F(\alpha_i)
\]

\[
= \pi_i(\alpha, \alpha) + \int_a^{\hat{\alpha}_i} \left\{ \sum_{g} A^g(\alpha, \alpha_{-i}) \left[ \int_0^1 u_1(y, s_i) P^g_i(y, \alpha_{-i}, s_i) dG_i(s_i) \right] \right\} f(\alpha_i) d\alpha_i
\]

\[
= \pi_i(\alpha, \alpha) + E[ \sum_{g} A^g(\alpha, \alpha_{-i}) \left[ \int_0^1 \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha, \alpha_{-i}) P^g_i(\alpha, \alpha_{-i}, s_i) dG_i(s_i) \right] ]
\]

The second equality above is due to Fubini's Theorem. Thus

\[
\sum_{i=1}^N E \pi_i(\alpha_i, \alpha_{-i}) = \sum_{i=1}^N \pi_i(\alpha, \alpha) + E[ \sum_{g} A^g(\alpha) E_s \left[ \sum_{i \in g} P^g_i(\alpha, s_i) \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i) \right] ]
\]
The total expected surplus from the two-stage mechanism is

$$TS = \sum_g \left\{ A^g(\alpha)E_s \left[ \sum_{i \in g} p_i^g(\alpha, s^g)u(\alpha_i, s_i) - |g|c \right] \right\},$$

The seller’s expected revenue is thus given by

$$ER = TS - \sum_{i=1}^N E\pi_i(\alpha, \alpha_i),$$

where $A^g(\alpha)$ is the shortlisting rule and $p_i^g(\alpha, s^g)$ is the second-stage allocation rule. To maximize $ER$ subject to IC and IR (individual rationality), the seller sets $\pi_i(\alpha, \alpha) = 0$ for all $i = 1, 2, \ldots, N$.

Define the virtual value adjusted by the second-stage signal as follows:

$$w(\alpha_i, s_i) = u(\alpha_i, s_i) - 1 - F(\alpha_i) \frac{\partial u(\alpha_i, s_i)}{\partial \alpha_i}.$$  \hfill (9)

From the expression of the expected revenue, we can derive the optimal allocation rules in both stages as follows. At the second stage, given the revealed $\alpha$ and the shortlisted group $g$, $\forall s^g$,

$$p_i^*(\alpha, s^g) = \begin{cases} 1 & \text{if } i \in \arg\max_{j \in g} \{w(\alpha_j, s_j)\} \text{ and } w(\alpha_i, s_i) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$ \hfill (10)

So as originally identified by Esö and Szentes, the object should be awarded to the bidder with the highest non-negative virtual value adjusted by the second-stage signal, which is a generalization of the optimal allocation rule in Myerson (1981). Our analysis shows that the generalized Myerson allocation rule is robust to settings with costly entry. This should be intuitive given Lemma 5: a buyer’s expected payoff does not depend on the entry cost, which implies that the seller bears all the entry costs (indirectly) in equilibrium. As such, costly entry will affect the final allocation only through its effect on the entry right allocation rule.

Define the expected virtual surplus (the virtual value less the entry cost) as follows:

$$w^*(\alpha) = E_s \left[ \sum_{i \in g} p_i^*(\alpha, s^g)w(\alpha_i, s_i) - |g|c \right].$$
Then at the first stage, contingent on the revealed $\alpha$, the optimal shortlisting rule is as follows:

$$A^{*g}(\alpha) = \begin{cases} 
1 & \text{if } g \in \arg \max_g \{w^{*g}(\alpha)\} \text{ and } w^{*g}(\alpha) \geq 0, \forall g. \\
0 & \text{otherwise.} 
\end{cases} \tag{11}$$

The optimal shortlisting rule admits the set of bidders that gives rise to the maximal expected virtual surplus. Alternatively, the optimal shortlisting rule admits the bidders in descending order of their marginal contribution to the expected virtual surplus, the bidder with the highest contribution first, the bidder with the second-highest contribution second, etc., provided that their marginal contribution is positive.

Similarly to Esö and Szentes, we can also establish the following properties of the optimal second-stage allocation rule: 12

**Corollary 1.** (i) $p_i^{*g}(\alpha,s^g_i)$ increases in both $\alpha_i$ and $s_i, \forall i \in g, \forall g$, $\alpha_{-i}$, and $s^g_{-i}$, which implies that $P_i^{*g}(\alpha_i,\alpha_{-i},s_i)$ increases in both $\alpha_i$ and $s_i, \forall g$, $\alpha_{-i}$.

(ii) If $\alpha_i > \hat{\alpha}_i, s_i < \hat{s}_i$ and $u(\alpha_i,s_i) = u(\hat{\alpha}_i,\hat{s}_i)$, then $p_i^{*g}(\alpha_i,\alpha_{-i},s_i,s^g_{-i}) \geq p_i^{*g}(\hat{\alpha}_i,\alpha_{-i},\hat{s}_i,s^g_{-i})$, which implies $P_i^{*g}(\alpha_i,\alpha_{-i},s_i) \geq P_i^{*g}(\hat{\alpha}_i,\alpha_{-i},\hat{s}_i), \forall g$, $\alpha_{-i}$.

We are now ready to show that the optimal final good allocation and entry right allocation rules (10) and (11) are truthfully implementable.

**Theorem 1.** The incentive compatible optimal two-stage mechanisms are characterized by the final good allocation and entry right allocation rules (10) and (11).

**Proof.** Following the derivations of the optimal final good allocation and entry rules (10) and (11), it remains to show that they are truthfully implementable under Assumptions 1 and 2.

By Assumption 1, $u(\alpha_i,s_i)$ increases with $s_i$ and $u_1(\alpha_i,s_i)$ (weakly) decreases with $s_i$. This implies that $w(\alpha_i,s_i)$ increases with $s_i$. By the final good allocation rule (10), the winning probability $P_i^{*g}(\alpha,s_i)$ is weakly increasing in $s_i$. By Lemma 2, the second-stage mechanism is incentive compatible (given $\alpha$ and $g$). Thus, given the truthfully revealed $\alpha$ and shortlisted group $g$, a second-stage payment rule, say, $t_i^{*g}(\alpha,s^g), \forall i \in g, \forall g$, can be constructed to implement the second-stage allocation rule $p_i^{*g}(\alpha,s^g), \forall i \in g, \forall g$ while maintaining the second-stage IR constraints. This resembles the Myerson (1981) setting with asymmetric bidders. We use $\pi_i^{*g}(\alpha_i,\hat{\alpha}_i;\alpha_{-i})$ to denote the second-stage expected payoff to buyer $i$ of type $\alpha_i$ if she announces $\hat{\alpha}_i$ and is shortlisted in group $g_i$, given that everyone else announces $\alpha_{-i}$ truthfully at the first stage. $\pi_i^{*g}(\alpha_i,\hat{\alpha}_i;\alpha_{-i})$ is well defined given Lemma 4. Therefore, when buyer $i$ of type $\alpha_i$ announces $\hat{\alpha}_i$ while others reveal $\alpha_{-i}$ truthfully, her first-stage expected payoff can be written as follows:

$$\pi_i^{*}(\alpha_i,\hat{\alpha}_i) = E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{*g_i}(\hat{\alpha}_i,\alpha_{-i})[\pi_i^{*g_i}(\alpha_i,\hat{\alpha}_i;\alpha_{-i}) - c] - x_i^{*}(\hat{\alpha}_i,\alpha_{-i}) \right\},$$

---

12As in Esö and Szentes, Assumption 2 is applied for showing part (ii).
where \( x_i^* \) is the first-stage payment rule.

Next, we will show that the first-stage mechanism is truthfully implementable by a properly chosen \( x_i^* \). More specifically, we will show that (11) and (10) are truthfully implementable via properly constructed first-stage payment rules together with the chosen second-stage payment rules \( r_i^*(a, s^g), \forall i \in g, \forall g \).

Note that by (5), we have

\[
\pi_i^*(\alpha_i, \alpha_i) = E_{\alpha_i, \pi_i^*}[\sum_{g_i} A^{g_i}(\alpha_i, \alpha_i)\tilde{\pi}_i^{g_i}(\alpha_i, \alpha_i - \alpha_i) - c - x_i^*(\alpha_i, \alpha_i)]. \tag{12}
\]

Construct the first-stage payment rule as follows:

\[
x_i^*(a) = \sum_{g_i} A^{g_i}(\alpha_i, \alpha_i)\tilde{\pi}_i^{g_i}(\alpha_i, \alpha_i - \alpha_i) - c - \int_{\alpha_i}^{\pi_i^*} A^{g_i}(y, \alpha_i) \left[ \int_0^{1} u_1(y, s_i) P_i^{g_i}(y, \alpha_i - s_i, s_i) dG_i(s_i) \right] dy. \tag{13}
\]

Substituting (13) into (12), we can verify that

\[
\pi_i^*(\alpha_i, \alpha_i) = E_{\alpha_i, \pi_i^*}[\sum_{g_i} A^{g_i}(y, \alpha_i) \left[ \int_0^{1} u_1(y, s_i) P_i^{g_i}(y, \alpha_i - s_i, s_i) dG_i(s_i) \right] dy],
\]

which is precisely equation (7) with \( \pi_i^*(\alpha_i, \alpha_i) = 0 \) (the optimality requirement).

Suppose that all buyers except \( i \) report their types \( \alpha_i \) truthfully. Consider buyer \( i \) with \( \alpha_i \) contemplating to misreport to \( \hat{\alpha}_i < \alpha_i \). The deviation payoff is

\[
\Delta = \pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\alpha_i, \alpha_i) = [\pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\hat{\alpha}_i, \hat{\alpha}_i)] + [\pi_i^*(\hat{\alpha}_i, \hat{\alpha}_i) - \pi_i^*(\alpha_i, \alpha_i)].
\]

Since (7) is satisfied by the construction of \( x_i^*(\alpha) \), we have

\[
\pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\alpha_i, \alpha_i) = -\int_{\hat{\alpha}_i}^{\alpha_i} E_{\alpha_i, \pi_i^*}[\sum_{g_i} A^{g_i}(y, \alpha_i) \int_0^{1} u_1(y, s_i) P_i^{g_i}(y, \alpha_i - s_i, s_i) dG_i(s_i) dy].
\]

Recall the definitions of \( \pi_i^*(\alpha_i, \hat{\alpha}_i) \) above, we have from Lemma 4 that

\[
\pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\hat{\alpha}_i, \hat{\alpha}_i) = \int_{\hat{\alpha}_i}^{\alpha_i} E_{\alpha_i, \pi_i^*}[\sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_i) \int_0^{1} u_1(y, s_i) P_i^{g_i}(\hat{\alpha}_i, \alpha_i - s_i, \sigma_i(y, \hat{\alpha}_i, s_i)) dG_i(s_i) dy].
\]

Therefore, we have

\[
\Delta = \int_{\hat{\alpha}_i}^{\alpha_i} E_{\alpha_i, \pi_i^*}[\sum_{g_i} A^{g_i}(y, \alpha_i) \int_0^{1} u_1(y, s_i) [P_i^{g_i}(\hat{\alpha}_i, \alpha_i - s_i, \sigma_i(y, \hat{\alpha}_i, s_i)) - P_i^{g_i}(\hat{\alpha}_i, \alpha_i - s_i, s_i)] dG_i(s_i) dy
\]

\[
+ \int_{\hat{\alpha}_i}^{\alpha_i} E_{\alpha_i, \pi_i^*}[\sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_i) - A^{g_i}(y, \alpha_i - s_i)] \int_0^{1} u_1(y, s_i) P_i^{g_i}(\hat{\alpha}_i, \alpha_i - s_i, \sigma_i(y, \hat{\alpha}_i, s_i)) dG_i(s_i) dy. \tag{14}
\]
From Corollary 1 (ii), we have $P^*_{\hat{g}_i}(\hat{a}_i, \alpha_{-i}, \sigma_i(y, \hat{a}_i, s)) - P^*_{\hat{g}_i}(y, \alpha_{-i}, s) \leq 0$, which implies that the first term in $\Delta$ is negative.

We now consider the second term in $\Delta$ when $y > \hat{a}_i$. The shortlisting rule implies that given $\alpha_{-i}$, when buyer $i$ is admitted with a higher $a_i$, she must be admitted to a group with a weakly smaller size. If $y$ and $\hat{a}_i$ are admitted in the same group, then $A^*_{\hat{g}_i}(\hat{a}_i, \alpha_{-i}) = A^*_{\hat{g}_i}(y, \alpha_{-i})$ and this term in $\Delta$ is zero. If $g^*(\hat{a}_i, \alpha_{-i}) \supset g^*(y, \alpha_{-i}) \supset [i]$, then

$$
\sum_{g_i} [A^*_{\hat{g}_i}(\hat{a}_i, \alpha_{-i}) - A^*_{\hat{g}_i}(y, \alpha_{-i})]u_1(y, s_i)P^*_{\hat{g}_i}(\hat{a}_i, \alpha_{-i}, \sigma_i(y, \hat{a}_i, s_i))
$$

$$
= u_1(y, s_i)[P^*_{\hat{g}_i}(\hat{a}_i, \alpha_{-i}, \sigma_i(y, \hat{a}_i, s_i)) - P^*_{\hat{g}_i}(y, \alpha_{-i})] \leq 0,
$$

which implies that the second term in $\Delta$ is negative. A similar argument can be used to rule out deviation to $\hat{a}_i > a_i$.

It is worth noting that Assumptions 1 and 2 are sufficient, but not necessary for the optimal entry rule to be truthfully implementable: the necessary and sufficient condition is that $\Delta$ defined in (14) is non-positive.

**Example 3.** Assumptions 1 and 2 are not necessary for Corollary 1 to hold. One can verify that Corollary 1 holds for the case with $H_\alpha(v) = v^\alpha$. In this case, $v = s^{1/\alpha} = u(\alpha, s)$. Thus

$$
w(\alpha, s) = s^{1/\alpha}\left[1 + \frac{1 - F(\alpha)}{f(\alpha)} \frac{1}{\alpha} \log s^{1/\alpha}\right] = u(\alpha, s) \left[1 + \frac{1 - F(\alpha)}{f(\alpha)} \frac{1}{\alpha} \log u(\alpha, s)\right].
$$

When there is only one potential bidder, the second-stage mechanism can be implemented via a take-it-or-leave it offer with a price $P(\alpha) = s(\alpha)^{1/\alpha}$, where $s(\alpha)$ is defined by $w(\alpha, s(\alpha)) = 0$. Note that $s(\alpha) = \exp\left\{-a^2 F(\alpha) \frac{f(\alpha)}{f'\alpha}\right\}$, which decreases with $\alpha$. Thus $P'(\alpha) < 0$. Define $w(\alpha) = E_s \max(0, w(\alpha, s))$. The optimal shortlisting rule is given by $A(\alpha) = 1$ if and only if $\alpha \geq \alpha^*$, where $w(\alpha^*) = 0$. We next derive the first-stage payment rule. Note that

$$
\pi(\alpha, \hat{a}) = A(\hat{a}) \int_0^1 \max(0, s^{1/\alpha} - P(\hat{a}))ds - c = x(\hat{a}), \ \forall \alpha, \hat{a} \geq \alpha^*.
$$

The FOC requires that $\frac{\partial \pi(\alpha, \hat{a})}{\partial \hat{a}} |_{\hat{a} = \alpha} = 0$. Taking $\hat{a} > \alpha \geq \alpha^*$, we have $s(\hat{a}) < s(\alpha) \leq s(\alpha^*)$. Note $A(\hat{a}) = A(\alpha) = 1$. We thus have

$$
\pi(\alpha, \hat{a}) - \pi(\alpha, \alpha) = \int_{s(\hat{a})}^{s(\alpha)} [s^{1/\alpha} - P(\hat{a})]ds + \int_{s(\alpha)}^{1} [P(\alpha) - P(\hat{a})]ds + [x(\alpha) - x(\hat{a})].
$$
Thus according to the following premium schedule: $p\alpha$ according to the optimal entry rule (11), and a “price premium” is determined for each shortlisted buyer since the buyers compete under unequal conditions: a bidder with a smaller premium has an advantage.

Underlying types will be recovered based on a recovery function, $x\alpha$, such that buyer $i$’s perceived type $\alpha\alpha i$ is $x\alpha 1(b_i), i = 1, 2, ..., N$. Given the recovered type profile $(\alpha\alpha i)^N_{i=1}$, the entry rights are implemented according to the optimal entry rule (11), and a “price premium” is determined for each shortlisted buyer according to the following premium schedule: $p(\alpha\alpha i) = u\alpha 1(1 - F(\alpha\alpha i))/f(\alpha\alpha i)$. Both the recovery function $x\alpha 1$ and the premium determination rule $p$ are made public at the outset of the game, which remain common knowledge throughout the auction process. Upon being admitted, each entrant bidder will incur the information acquisition cost and participate in the second-round bidding. The second stage is a traditional second-price or English auction with a zero reserve price, but the winner is required to pay her premium over the price. This mechanism is referred to as the handicap auction in Esö and Szentes, since the buyers compete under unequal conditions: a bidder with a smaller premium has an advantage. In our setting, the handicap auction is modified so that the optimal entry rule is also implemented after the first-round bidding. In Esö and Szentes, buyers pay fees regardless of winning the final good or not; in our setting, buyers pay $b_i$’s regardless of being admitted to the final sale or not. For this reason, the first-stage auction is a variant of all-pay auctions.

In the second-stage auction, it is a (weakly) dominant strategy for entrant bidder $i$ with a price premium $p\alpha i$ (determined from the first stage) to bid $u(a_i, s_i) - p\alpha i$. Assuming that all the entrant buyers follow this weakly dominant strategy in the second stage, the mechanism can be represented by a pair of functions, $p: [a, \overline{a}] \to \mathbb{R}_+$ and $x**: [a, \overline{a}] \to \mathbb{R}$ for $i = 1, 2, ..., N$, where $p(a_i)$ is the price premium for a buyer

\[ \frac{\partial \pi(a, \alpha\alpha i)}{\partial \alpha\alpha i} |_{\alpha\alpha i = a} = -P'(a)(1 - s(a)) - x'(a) = 0. \]

Thus $x'(a) = -P'(a)(1 - s(a))$, together with boundary condition $x(a\alpha) = \int_{0}^{1} \max(0, s^{1/a\alpha} - P(a\alpha)) ds - \alpha$, jointly determines the payment function $x(a)$ for the first stage.

### 3.2 Implementation of Optimal Mechanisms

When $u(a_i, s_i)$ is linear in $a_i$, i.e., when $u(a_i, s_i) = u\alpha 1 a_i + r(s_i)$ for some constant $u\alpha 1$ and function $r$, we will demonstrate that the optimal mechanism can be implemented via a two-stage auction, with the first stage being an auction for both entry rights and price premia and the second stage being a second-price or English auction for the final good. This two-stage auction can be regarded as a handicap auction introduced in Esö and Szentes, augmented by an additional auction at the entry stage.\(^{13}\)

More specifically, our two-stage auction works as follows. The first stage is an all-pay auction, where bidders need to pay what they bid, regardless of being awarded entry rights or not. Suppose buyer $i$, knowing her type $a_i$, bids an amount $b_i$, $i = 1, 2, ..., N$. After all the first-stage bids are collected, underlying types will be recovered based on a recovery function, $x\alpha 1$, such that buyer $i$’s perceived type $\alpha\alpha i$ is $x\alpha 1(b_i), i = 1, 2, ..., N$. Given the recovered type profile $(\alpha\alpha i)^N_{i=1}$, the entry rights are implemented according to the optimal entry rule (11), and a “price premium” is determined for each shortlisted buyer according to the following premium schedule: $p(\alpha\alpha i) = u\alpha 1(1 - F(\alpha\alpha i))/f(\alpha\alpha i)$. Both the recovery function $x\alpha 1$ and the premium determination rule $p$ are made public at the outset of the game, which remain common knowledge throughout the auction process. Upon being admitted, each entrant bidder will incur the information acquisition cost and participate in the second-round bidding. The second stage is a traditional second-price or English auction with a zero reserve price, but the winner is required to pay her premium over the price.\(^{14}\) This mechanism is referred to as the handicap auction in Esö and Szentes, since the buyers compete under unequal conditions: a bidder with a smaller premium has an advantage. In our setting, the handicap auction is modified so that the optimal entry rule is also implemented after the first-round bidding. In Esö and Szentes, buyers pay fees regardless of winning the final good or not; in our setting, buyers pay $b_i$’s regardless of being admitted to the final sale or not. For this reason, the first-stage auction is a variant of all-pay auctions.

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\(^{13}\)Note that the assumption that $u\alpha 1$ is constant is satisfied in both Examples 1 and 2.

\(^{14}\)Should there be only one entrant, the price premium for this sole entrant becomes the effective reserve price.
who bids an amount of \( b_i = x^*(\alpha_i) \).

**Theorem 2.** If \( u_1 \) is constant then the optimal mechanism of Theorem 1 can be implemented via a two-stage auction described above with the recovery function \( x^{-1} \) and price premium function \( p \) defined as follows:

\[
\begin{align*}
  p(\alpha_i) &= \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1, \\
  x^*(\alpha_i) &= E_{\alpha_i} \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) E_{\alpha_{-i}} \left[ \max\{w(\alpha_i, s_i) - w^{g_i}_{-i}(\alpha_{-i}, s_{-i}), 0\} - c \right] \\
    &\quad - E_{\alpha_i} \int_0^{\alpha_i} \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) E_{\alpha_{-i}} \left[ u_1(y, s_i) 1_{\{w(y, s_i) > w^{g_i}_{-i}(\alpha_{-i}, s_{-i})\}} \right] dy,
\end{align*}
\]

where \( w(\alpha_i, s_i) = u(\alpha_i, s_i) - p(\alpha_i) \) and \( w^{g_i}_{-i}(\alpha_{-i}, s_{-i}) = \max_{j \neq i, j \in g_i} \{w(\alpha_j, s_j), 0\} \).

**Proof.** See Appendix. \( \square \)

The implementation is established by showing that \( x^*(\cdot) \) as defined in (16) constitutes a symmetric (strictly) increasing equilibrium bid function in the (reduced) all-pay auction game, with the second stage being replaced by its correlated equilibrium payoffs. The first step in the proof of Theorem 2 establishes that \( x^*(\alpha_i) \) as defined in (16) is strictly increasing for \( \alpha_i \in [a^*, \overline{\alpha}] \), where \( a^* \in \{\alpha, \overline{\alpha}\} \) is the minimum type that could possibly be allocated with an entry right in equilibrium.\(^{15}\) Thus the recovery function \( x^{-1}(\cdot) \) is well defined over \([a^*, \overline{\alpha}]\) and a (truncated) profile of pre-entry types can be recovered from their bids.\(^{16}\)

Optimal entry can then be implemented based on the recovered type profile according to (11). Upon being selected in a group, say, \( g \), everyone will follow the (weakly) dominant strategies in the second round bidding (to bid their value less the price premium), so buyer \( i \in g \) with pre-entry type \( \alpha_i \) will win the asset if and only if

\[
  u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1 \geq \max \left\{ 0, \max_{j \in g, j \neq i} \left[ u(\alpha_j, s_j) - \frac{1-F(\alpha_j)}{f(\alpha_j)} u_1 \right] \right\}.
\]

Hence the optimal allocation rule (10) can indeed be implemented, provided that bidding according to \( x^*(\cdot) \) constitutes a symmetric equilibrium in the (reduced) first-stage auction game, which is established in the second step of the proof.

When \( u_1 \) is not a constant, in particular, if \( u_1 \) is a function of \( s_i \), then one's price premium also depends on her second-stage signal. As such, one's second-stage bid also affects the (total) price to pay should she win the object even under a second-price auction. A direct consequence is that it is no longer

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\(^{15}\) \( a^* \) is defined such that

\[
E_{\alpha_{-i}} \max \left\{ u(\alpha^*, s_i) - \frac{1-F(\alpha^*)}{f(\alpha^*)} u_1, 0 \right\} \geq c.
\]

That is, \( a^* \) is the minimal type that one can possibly be admitted (as a sole entrant). We assume \( a^* \in (\alpha, \overline{\alpha}) \).

\(^{16}\) We assume that buyers with types below \( a^* \) will stay away from bidding. But that should not affect the implementation of optimal entry as those buyers should not be admitted anyway.

---
a (weakly) dominant strategy for bidder $i$ to bid $w(\alpha_i, s_i)$. To avoid such an inconvenience, as in Esö and Szentes, we also focus on the case in which $u_1$ is a constant for auction implementation.

Since $x^*(\cdot)$ is increasing while $p(\cdot)$ is decreasing, the price premium is decreasing in the first-stage bids. Thus a buyer with a higher pre-entry type bids higher in the first round, which results in a higher probability to be admitted and a lower price premium.

The equilibrium (entry) fee $x^*$ has an intuitive interpretation. As can be seen from (16), one’s entry fee equals her expected profit from entry less her informational rent due to her private information about her type $\alpha_i$. So the additional information from the second-stage signals does not contribute to buyers’ rents: the seller seems to appropriate all rents from entry by “charging” each buyer an upfront entry fee equal to her “value” of entry (or equivalently, the value of additional information).\footnote{As pointed out in Esö and Szentes, the “value” of additional information is not well defined, as it depends on the specific rules in the second-round auction.}

Fixing the auction rules in the second round, it is also clear by the envelope theorem that payoff equivalence holds among all the entry mechanisms in which (1) the same entry rule (11) is implemented, and (2) the buyer with type $\alpha^*$ makes zero expected profit. In light of this equivalence result, based on $x^*(\alpha_i)$ we can derive the candidate equilibrium bid function under any alternative entry mechanism. If the candidate equilibrium bid function so derived is strictly increasing, then the inverse of the equilibrium bid function can serve as the recovery function for the implementation of optimal entry. For example, we can consider a discriminatory-price auction, where only bidders who are awarded entry rights need to pay, and they pay what they bid. Using the payoff equivalence, the candidate equilibrium bid functions under the discriminatory-price auction is given by

$$x^*_D(\alpha_i) = x^*(\alpha_i)/\Pr \{ \alpha_{-i} | i \in g^*(\alpha_i, \alpha_{-i}) \},$$

where $g^*(\alpha_i, \alpha_{-i})$ is the set of shortlisted bidders determined by the optimal entry rule (11), given the reported type profile $(\alpha_i, \alpha_{-i})$. A discriminatory-price auction can implement optimal entry if and only if $x^*_D(\alpha_i)$ so derived is strictly increasing.

3.3 Applications

Our optimal mechanism analysis is general enough to encompass many existing models in the literature on auctions with costly entry. Below we demonstrate how we can apply our general optimal mechanism to special models previously studied.

1. Bidders do not have pre-entry types and only learn about their values after entry (e.g., McAfee and McMillan, 1987; Tan, 1992; and Levin and Smith, 1994). In this case, $u(\alpha_i, s_i) = s_i$. Hence $w(\alpha_i, s_i) = s_i$, which implies that the optimal auction is ex post efficient, and the optimal entry is to select a set of bidders that results in the maximal expected social surplus. Since bidders are
identical before entry, optimal entry is entirely characterized by \( n^* \), the optimal number of bidders to be selected. The implementation is somewhat simple: the second round is a standard auction (first-price, second-price, or English auction – no price premium is involved). The first round (entry stage) is to select exactly \( n^* \) bidders, and whomever selected is required to pay an upfront entry fee \( e^* \), which is set so that no rent is left for the entrants \textit{ex ante}.

2. Bidders know their values before entry, and entry is merely a bid preparation process (without value updating) (e.g. Samuelson, 1985; Stegeman, 1996; Campbell, 1998; Menezes and Monteiro, 2000; Tan and Yilankaya, 2006; Cao and Tian, 2009; and Lu, 2009). In this setting, \( u(a_i, s_i) = a_i \), and hence \( w(a_i, s_i) = a_i \cdot (1 - F(a_i)) / f(a_i) \). It is easily verified that according to Theorem 1, the optimal allocation rules can be described as follows: the bidder with the highest “type” \( a_i \) is admitted as the sole entrant to win the item, as long as her contribution to the virtual surplus \( w(a_i, s_i) - c \) is positive. The optimal mechanism can be implemented as follows: each buyer pays what she bids in the first stage (regardless of being admitted or not), and the only entrant wins the item at a price equal to her price premium determined from her first-round bid. For an illustration, below we derive the equilibrium first-stage bid function \( x^* \).

Consider a bidder with type \( a_i > a^* \), where \( a^* \cdot (1 - F(a^*)) / f(a^*) = c \). Suppose, in the (reduced) first-stage direct game, bidder \( i \) reports \( \hat{a}_i < a_i \), a sufficiently small deviation from \( a_i \). Her expected payoff is then given by

\[
\pi_i(a_i, \hat{a}_i) = \left[ a_i - \frac{1 - F(\hat{a}_i)}{f(\hat{a}_i)} - c \right] \Pr(\alpha_i^{-1}(1) < \hat{a}_i) - x^*(\hat{a}_i)
\]

where \( \alpha_i^{-1}(1) \) is the highest type among all the buyers other than \( i \).

Incentive compatibility implies

\[
\frac{d\pi_i(a_i, \alpha_i)}{d\alpha_i} = F_{\alpha_i^{-1}}(a_i).
\]

Thus we have

\[
\pi_i(a_i, \alpha_i) = \left[ a_i - \frac{1 - F(a_i)}{f(a_i)} - c \right] F_{\alpha_i^{-1}}(a_i) - x^*(a_i) = \int_{a_i}^{a^*} F_{\alpha_i^{-1}}(\tau) d\tau.
\]

Substituting \( \hat{a}_i = a_i \) into (17) to obtain the equilibrium expected payoff, and then equating that with (18), gives the (symmetric) equilibrium first-stage bid function

\[
x^*(a_i) = \left[ a_i - \frac{1 - F(a_i)}{f(a_i)} - c \right] F_{\alpha_i^{-1}}(a_i) - \int_{a_i}^{a^*} F_{\alpha_i^{-1}}(\tau) d\tau.
\]
It is easily verified that $x^*(\cdot)$ is strictly increasing so types can be recovered from first-round bids in equilibrium.

If a discriminatory-price auction is conducted instead, by payoff equivalence the candidate equilibrium bid function in the first round is given by

$$x^*_D(\alpha_i) = \frac{x^*(\alpha_i)}{F_{\alpha_i}(\alpha_i)} = \left[ \alpha_i - \frac{1 - F(\alpha_i)}{f(\alpha_i)} - c \right] - \frac{1}{F_{\alpha_i}(\alpha_i)} \int_{\alpha_i}^{\tau} F_{\alpha_i}(\tau) \, d\tau. \quad (20)$$

It is also easily verified that $x^*_D(\cdot)$ is strictly increasing. Thus the discriminatory-price auction works to implement optimal entry in this context as well.

3. Each bidder is endowed with pre-entry type $\alpha_i$, and learns an additional private value component $s_i$ (e.g., Ye, 2007; Quint and Hendricks, 2013). The total value is given by $u(\alpha_i, s_i) = \alpha_i + s_i$. Hence $w(\alpha_i, s_i) = \alpha_i + s_i - (1 - F(\alpha_i)) / f(\alpha_i)$. The optimal second-stage allocation rule thus requires that the asset be allocated to the entrant bidder with the highest virtual value $w(\alpha_i, s_i)$ provided that it is nonnegative. The optimal entry rule requires that bidders be admitted in descending order of their pre-entry types, as long as their contribution to the expected virtual surplus is nonnegative.

To further illustrate the optimal entry rule, we assume that $\alpha_i$ is distributed uniformly over $[0, 1]$ and $s_i$ follows a Bernoulli distribution, taking value 1 (“High”) with probability $q$ and 0 (“Low”) with probability $1 - q$. Then $w(\alpha_i, s_i) = 2\alpha_i + s_i - 1$. If only one buyer (the one with the highest type $\alpha(1)$) is admitted, the expected virtual surplus is given by $w_1 = E(2\alpha(1) + s_1 - 1) - c = 2\alpha(1) + q - 1 - c$. So the optimal number of entrants $n^* \geq 1$ if $2\alpha(1) + q - 1 - c \geq 0$. For ease of computation we assume that $\alpha(1) \geq \alpha(2) \geq .5$ (so that the virtual values from the top two bidders are guaranteed to be nonnegative). If two top buyers are admitted, the expected virtual surplus is given by

$$w_2 = E \left[ \max \{2\alpha(1) + s_1 - 1, 2\alpha(2) + s_2 - 1\} \right] - 2c$$

$$= E \left[ \max \{2\alpha(1) + s_1, 2\alpha(2) + s_2\} \right] - 1 - 2c$$

$$= \Pr(s_1 = 1) \cdot (2\alpha(1) + 1) + \Pr(s_1 = s_2 = 0) \cdot 2\alpha(1) + \Pr(s_1 = 0, s_2 = 1) \cdot (2\alpha(2) + 1) - 1 - 2c$$

$$= q \cdot (2\alpha(1) + 1) + (1 - q)^2 \cdot 2\alpha(1) + (1 - q)q \cdot (2\alpha(2) + 1) - 1 - 2c$$

So the optimal number of entrants $n^* \geq 2$ if the incremental expected virtual surplus $\Delta w = w_2 - w_1 = q(1 - q) \left[ 1 - 2(\alpha(1) - \alpha(2)) \right] - c \geq 0$. Continuing this procedure of calculation,\(^\text{18}\) it can be verified that

\(^{18}\) We continue to consider the case $\alpha(1) > \alpha(2) \geq \ldots \geq \alpha(n) \geq .5$ so that the virtual value from these buyers will be positive.
\[ n^* \geq n \text{ if } q(1-q)^{n-1} \left[ 1 - 2 \left( \alpha(1) - \alpha(n) \right) \right] - c \geq 0, \quad \text{or} \]
\[ \alpha(1) - \alpha(n) \leq \frac{1}{2} \left[ 1 - \frac{c}{q(1-q)^{n-1}} \right]. \quad (21) \]

This condition is intuitive: the admission of the \( n \)-th highest buyer is more likely to be justified if
1. the probability that she will turn out to be the winner in the second round is sufficiently high;
2. the entry cost is sufficiently low; or
3. her type is sufficiently close to the highest type. It is thus clear that the optimal number of entrants, \( n^* \), is determined by the following conditions:
\[ \alpha(1) - \alpha(n^*) \leq \frac{1}{2} \left[ 1 - \frac{c}{q(1-q)^{n^*-1}} \right], \quad \alpha(1) - \alpha(n^*+1) > \frac{1}{2} \left[ 1 - \frac{c}{q(1-q)^{n^*}} \right]. \]

4 Concluding Remarks

Our paper contributes to the literature on two fronts. First, it characterizes optimal mechanisms for an environment of two-stage auctions, which are commonly employed in sales of complicated and high-valued business assets, procurements, takeover, and merger and acquisition contests. Our analysis is general enough to nest many existing studies in the literature of auctions with costly entry. Second, our paper contributes to the literature on sequential screening by introducing costly entry into a dynamic auction framework. Entry provides a natural setting for sequential information acquisition; on the other hand, entry also makes the optimal mechanism design more challenging, as now it has to balance information acquisition at the entry stage and information elicitation in the final good allocation stage, which are interdependent.

Other than entry screening, running an entry right allocation mechanism has two other benefits. First, it implements deterministic entry to improve efficiency and expected revenue. As originally identified by Levin and Smith (1994), when the seller does not exercise entry control, endogenous entry would often lead to coordination failure, as the realized number of entrants is stochastic, resulting in too high or too low entry. While a low entry will reduce competition and pose a direct cost to the seller, the possibility of high entry would reduce the entry incentive \textit{ex ante}. As Levin and Smith demonstrate, the seller can maximize efficiency and hence expected revenue by reducing the number of potential bidders until the randomness in entry is completely eliminated. In fact, Milgrom (2004, pp. 225-227) demonstrates that screening to minimize variance in participation, even if it is still random, increases expected revenue. Therefore, employing a two-stage mechanism to exercise entry control to reduce entry randomness is usually in the best interest of the seller. Second, running an entry right allocation mechanism can uniquely implement optimal entry. As demonstrated in Levin and Smith (1994) and Lu (2010), there are usually multiple equilibria in the entry stage with endogenous entry, so potential bidders need to

\[^{19}\text{The addition of the \( n \)-th highest buyer only contributes to the expected virtual surplus when she turns out to be the only one having a good “shot” in the second stage (i.e., } s_n = 1, \text{ while } s_1 = \ldots = s_{n-1} = 0).\]
coordinate over different entry equilibria. Following the optimal mechanism design approach, however, we show that optimal entry can be uniquely implemented.

Our analysis offers a theoretical benchmark for evaluating various two-stage auctions currently used in the real world. The information structure modeled in this research has recently received attention not only from theorists, but also from econometricians and empiricists. For example, Marmer, Shneyerov, and Xu (2013) and Gentry and Li (2013) have successfully proposed nonparametric specification tests on a so-called affiliated-signal (AS) model with entry, and Roberts and Sweeting (2013) estimate a parametric variant of the AS model using data on California timber auctions. The affiliated-signal models can be regarded as a special case in the framework studied in our paper, and the optimal mechanism characterized in this paper can potentially serve as a calibration benchmark for counter-factual simulations for related empirical works to come.
5 Appendix

Proof of Lemma 5: Let $g_i$ denote any subset that includes $i$. By (5) and Lemma 4, we have

\[
\pi_i(\alpha_i, \hat{\alpha}_i) = \pi_i(\hat{\alpha}_i, \hat{\alpha}_i) + \mathbb{E}_{\alpha_i} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_i|\hat{\alpha}_i, \alpha_i) \left[ \hat{\pi}^{g_i}(\hat{\alpha}_i, \alpha_i|\hat{\alpha}_i, \alpha_i) - \hat{\pi}^{g_i}(\hat{\alpha}_i, \alpha_i|\hat{\alpha}_i, \alpha_i) \right] \right\}
\]

Thus for $\hat{\alpha}_i < \alpha_i$, $\pi_i(\alpha_i, \hat{\alpha}_i) \leq \pi_i(\alpha_i, \alpha_i)$ implies that

\[
\pi_i(\alpha_i, \alpha_i) - \pi_i(\hat{\alpha}_i, \hat{\alpha}_i) \geq \mathbb{E}_{\alpha_i} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_i|\hat{\alpha}_i, \alpha_i) \int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) P^{g_i}_i(\hat{\alpha}_i, \alpha_i|\sigma_i(y, \hat{\alpha}_i, s_i)) dy \right\}
\]

Similarly,

\[
\pi_i(\hat{\alpha}_i, \alpha_i) = \pi_i(\alpha_i, \alpha_i) + \mathbb{E}_{\alpha_i} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_i|\alpha_i, \alpha_i) \left[ \hat{\pi}^{g_i}(\alpha_i, \alpha_i|\alpha_i, \alpha_i) - \hat{\pi}^{g_i}(\alpha_i, \alpha_i|\alpha_i, \alpha_i) \right] \right\}
\]

Thus for $\hat{\alpha}_i < \alpha_i$, $\pi_i(\hat{\alpha}_i, \alpha_i) \leq \pi_i(\hat{\alpha}_i, \alpha_i)$ implies that

\[
\pi_i(\alpha_i, \alpha_i) - \pi_i(\hat{\alpha}_i, \hat{\alpha}_i) \leq \mathbb{E}_{\alpha_i} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_i|\alpha_i, \alpha_i) \int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) P^{g_i}_i(\alpha_i, \alpha_i|\sigma_i(y, \alpha_i, s_i)) dy \right\}
\]

So

\[
\mathbb{E}_{\alpha_i} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_i|\hat{\alpha}_i, \alpha_i) \int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) \frac{P^{g_i}_i(\hat{\alpha}_i, \alpha_i|\sigma_i(y, \hat{\alpha}_i, s_i)) dy}{\alpha_i - \hat{\alpha}_i} dG_i(s_i) \right\} \leq \frac{\pi_i(\alpha_i, \alpha_i) - \pi_i(\hat{\alpha}_i, \hat{\alpha}_i)}{\alpha_i - \hat{\alpha}_i} \leq \mathbb{E}_{\alpha_i} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_i|\alpha_i, \alpha_i) \int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) \frac{P^{g_i}_i(\alpha_i, \alpha_i|\sigma_i(y, \alpha_i, s_i)) dy}{\alpha_i - \hat{\alpha}_i} dG_i(s_i) \right\}
\]

By Fubini's Theorem, we have

\[
\mathbb{E}_{\alpha_i} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_i|\hat{\alpha}_i, \alpha_i) \int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) \frac{P^{g_i}_i(\hat{\alpha}_i, \alpha_i|\sigma_i(y, \hat{\alpha}_i, s_i)) dy}{\alpha_i - \hat{\alpha}_i} dG_i(s_i) \right\} = \sum_{g_i} \int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) E_{\alpha_i} \left[ A^{g_i}(\hat{\alpha}_i, \alpha_i|\hat{\alpha}_i, \alpha_i) P^{g_i}_i(\hat{\alpha}_i, \alpha_i|\alpha_i, \alpha_i) \sigma_i(y, \alpha_i, s_i) \right] dy dG_i(s_i).
\]
Since $A^{g_i}, P^{g_i}_i \leq 1$, and $u$ is concave in $\alpha_i$, we have
\[
\frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s)E_{a_{-i}} \left[ A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) P^{g_i}_i(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s)) \right] \, dy}{\alpha_i - \hat{\alpha}_i} \leq u_1(\hat{\alpha}_i, s_i).
\]
By assumption $u_1(\hat{\alpha}_i, s_i)$ has a finite expectation with respect to $s_i$. Hence, by the Lebesgue convergence theorem,
\[
\lim_{\hat{\alpha}_i \to \alpha_i} E_{a_{-i}} \left\{ \sum_{\hat{g}_i} A^{\hat{g}_i}(\hat{\alpha}_i, \alpha_{-i}) \int_0^{\alpha_i} \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s)E_{a_{-i}} \left[ A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) P^{g_i}_i(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s)) \right] \, dy}{\alpha_i - \hat{\alpha}_i} \, dG_i(s_i) \right\}
= \sum_{\hat{g}_i} \int_0^{\alpha_i} \lim_{\hat{\alpha}_i \to \alpha_i} \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s)E_{a_{-i}} \left[ A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) P^{g_i}_i(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s)) \right] \, dy}{\alpha_i - \hat{\alpha}_i} \, dG_i(s_i)
= \sum_{\hat{g}_i} \int_0^{\alpha_i} \lim_{\hat{\alpha}_i \to \alpha_i} \left( u_1(\hat{\alpha}_i, s_i)E_{a_{-i}} \left[ A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) P^{g_i}_i(\hat{\alpha}_i, \alpha_{-i}, s_i) \right] \right) \, dG_i(s_i)
= \sum_{\hat{g}_i} \lim_{\hat{\alpha}_i \to \alpha_i} \left( u_1(\hat{\alpha}_i, s_i)E_{a_{-i}} \left[ A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) \int_0^{\alpha_i} u_1(\alpha_i, s_i)P^{g_i}_i(\alpha_i, \alpha_{-i}, s_i) \, dG_i(s_i) \right] \right) \, dG_i(s_i)
= E_{a_{-i}} \left\{ \sum_{\hat{g}_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int_0^{\alpha_i} u_1(\alpha_i, s_i)P^{g_i}_i(\alpha_i, \alpha_{-i}, s_i) \, dG_i(s_i) \right\}.
\]
The third equality above is due to the assumption that $E_{a_{-i}} \left[ A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) P^{g_i}_i(\hat{\alpha}_i, \alpha_{-i}, s_i) \right]$ is continuous in $\hat{\alpha}_i$ (which is guaranteed as long as both $A^{g_i}$ and $P^{g_i}_i$ are continuous a.e. in $[\underline{a}, \overline{a}]^{l_i}$).

Analogously, we can show that
\[
\lim_{\hat{\alpha}_i \to \alpha_i} E_{a_{-i}} \left\{ \sum_{\hat{g}_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int_0^{\alpha_i} \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s)E_{a_{-i}} \left[ A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) P^{g_i}_i(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \alpha_i, s)) \right] \, dy}{\alpha_i - \hat{\alpha}_i} \, dG_i(s_i) \right\}
= E_{a_{-i}} \left\{ \sum_{\hat{g}_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int_0^{\alpha_i} u_1(\alpha_i, s_i)P^{g_i}_i(\alpha_i, \alpha_{-i}, s_i) \, dG_i(s_i) \right\}.
\]
Thus the left derivative of $\pi_i(\alpha_i, \alpha_i)$ is given by
\[
\frac{d\pi_i^-}{d\alpha_i} = E_{a_{-i}} \left\{ \sum_{\hat{g}_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int_0^{\alpha_i} u_1(\alpha_i, s_i)P^{g_i}_i(\alpha_i, \alpha_{-i}, s_i) \, dG_i(s_i) \right\}.
\]
Working with the case $\hat{\alpha}_i > \alpha_i$, we can obtain the right derivative of $\pi_i(\alpha_i, \alpha_i)$, which is given by
\[
\frac{d\pi_i^+}{d\alpha_i} = E_{a_{-i}} \left\{ \sum_{\hat{g}_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int_0^{\alpha_i} u_1(\alpha_i, s_i)P^{g_i}_i(\alpha_i, \alpha_{-i}, s_i) \, dG_i(s_i) \right\}.
\]
Therefore, we conclude that $\pi_i(\alpha_i) = \pi_j(\alpha_i, \alpha_i)$ is differentiable everywhere, and
\[
\pi_i'(\alpha_i) = E_{a_{-i}} \left\{ \sum_{\hat{g}_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int_0^{\alpha_i} u_1(\alpha_i, s_i)P^{g_i}_i(\alpha_i, \alpha_{-i}, s_i) \, dG_i(s_i) \right\}.
\]
Since $\pi_i'(a_i)$ is bounded over $[a, \overline{a}]$, $\pi_i$ satisfies a Lipschitz condition and hence it can be recovered from its derivative, which gives rise to (7).

**Proof of Theorem 2:** We start by providing some preliminary results that will be used in the proof. We first pin down the contingent expected payoff $\tilde{\pi}_i^g(a_i, a_i; a_{-i})$ for bidder $i$ in the shortlisted group $g$.

Define a truncated random variable as follows:

$$w_i^+(a_i, s_i) = \begin{cases} w(a_i, s_i) & \text{if } w(a_i, s_i) \geq 0 \text{ or equivalently } s_i \geq s(a_i), \forall i. \\ 0 & \text{otherwise.} \end{cases}$$

Note that conditional on $\alpha$, $w_i^+$'s are independent across $i \in g$.

Since the second stage is a second-price auction, the payoff $\tilde{\pi}_i^g(a_i, a_i; a_{-i})$ of $i \in g$ can be calculated as

$$\tilde{\pi}_i^g(a_i, a_i; a_{-i}) = \tilde{S}(a_i^g) - \tilde{S}(a_{-i}^g), i \in g, \forall \alpha^g,$$

where $a_{-i}^g = a^g \setminus \{a_i\}$ and

$$\tilde{S}(a^g) = \mathbb{E}_{a^g} \max_{i \in g} \{w_i^+(a_i, s_i)\}, \forall g, \forall \alpha^g.$$

From the above expression of $\tilde{\pi}_i^g(a_i, a_i; a_{-i})$, the following two properties are obvious:

(P1) $\tilde{\pi}_i^g(a_i, a_i; a_{-i})$ increases with $a_i$, and decreases with $a_j, \forall j \neq i, \forall i \in g, \forall g$.

(P2) $\tilde{\pi}_i^g(a_i, a_i; a_{-i}) \geq \tilde{\pi}_i^g(a_i, a_i; a_{-i}), \forall \alpha_{-i}, \forall i \in g, \forall g \subseteq g'$. The revenue-optimal shortlisting rule can be alternatively described as follows. For given $\alpha$, we can rank all $a_i$ from the highest to the lowest. The seller admits bidders one by one in descending order of $a_i$'s as long as the bidder's marginal contribution to the expected virtual surplus is nonnegative, i.e.

$$\tilde{\pi}_i^g(a_i, a_i; a_{-i}) - c = \tilde{S}(a_i^g) - \tilde{S}(a_{-i}^g) - c \geq 0,$$

where $g$ denotes the group of bidders with the highest $|g|$ types before entry.

Two properties follow immediately from this shortlisting rule:

(P1') Given $a_{-i}$, if bidder $j$ with $a_j$ is admitted, then she would also be admitted with a higher type $\tilde{a}_j(> a_j)$.

(P2') Suppose $i$ is admitted with $a_{-i}$, then bidder $i$ would remain being admitted as long as $a_i$ is higher than a threshold $\tilde{a}_i(a_{-i})$. As $a_i$ increases, the shortlisted group weakly shrinks. As $a_i$ increases from $\tilde{a}_i(a_{-i})$, the bidders in $g^*(a) \setminus \{i\}$ would be excluded one by one (with the lowest type originally admitted being excluded first).

We are now ready to proceed with the proof.

**Step 1:** We will show that $x^*$ as defined in (16) is strictly increasing over $[a^*, \overline{a}]$, where $a^*$ is the minimum type that one can possibly be admitted in optimal entry.

Given $a_i$, let $s(a_i)$ be defined such that $w(a_i, s(a_i)) = 0$. Define $t = \max_{j \neq i, j \in g} w_j^+(a_j, s_j)$ and let $\Psi(t)$
denote the cdf of \( t \). We have

\[
\hat{\pi}^*_i(y, y; \alpha_{-i}) = \int_{s(y)}^1 \int_0^{w(y, s_i)} [w(y, s_i) - t]d\Psi(t)dG_i(s_i) + \int_{s(y)}^1 w(y, s_i)\Psi(0)dG_i(s_i)
\]

\[
= \int_{s(y)}^1 w(y, s_i)\int_0^{w(y, s_i)} d\Psi(t)dG_i(s_i) - \int_{s(y)}^1 \int_0^{w(y, s_i)} td\Psi(t)dG_i(s_i) + \int_{s(y)}^1 w(y, s_i)\Psi(0)dG_i(s_i)
\]

\[
= \int_{s(y)}^1 w(y, s_i)\Psi(w(y, s_i))dG_i(s_i) - \int_{s(y)}^1 \int_0^{w(y, s_i)} td\Psi(t)dG_i(s_i)
\]

\[
= \int_{s(y)}^1 \int_0^{w(y, s_i)} \Psi(t)dtdG_i(s_i).
\]

As \( w(y, s(y)) = 0 \), we have

\[
\frac{d\hat{\pi}^*_i(y, y; \alpha_{-i})}{dy} = \int_{s(y)}^1 \frac{\partial w(y, s_i)}{dy}\Psi(w(y, s_i))dG_i(s_i)
\]

\[
= \int_{s(y)}^1 \left[ u_1(y, s_i) - u_1(y, s_i)\left(\frac{1-F(y)}{f(y)}\right)\right]\Psi(w(y, s_i))dG_i(s_i)
\]

\[
\geq \int_{s(y)}^1 u_1(y, s_i)\Psi(w(y, s_i))dG_i(s_i).
\]

Since

\[
\int_{s(y)}^1 u_1(y, s_i)\hat{P}_i^*(y, \alpha_{-i}, s_i)dG_i(s_i) = \int_{s(y)}^1 u_1(y, s_i)\Psi(w(y, s_i))dG_i(s_i),
\]

we have

\[
\frac{d\hat{\pi}^*_i(y, y; \alpha_{-i})}{dy} \geq \int_{s(y)}^1 u_1(y, s_i)\hat{P}_i^*(y, \alpha_{-i}, s_i)dG_i(s_i) \geq 0. \tag{22}
\]

As in the proof of Theorem 1, we construct a first-stage payment function as given by (13):

\[
x_i^*(\alpha) = \sum_{g_i} A_i^*(\alpha_i, \alpha_{-i})[\hat{\pi}^*_i(y, \alpha_i; \alpha_{-i}) - c]
\]

\[
- \int_{\alpha_i}^{\alpha^*} \sum_{g_i} A_i^*(y, \alpha_{-i})\left[ \int_{0}^{u_1(y, s_i)} u_1(y, s_i)\hat{P}_i^*(y, \alpha_{-i}, s_i)dG_i(s_i) \right] dy. \tag{23}
\]

By construction, it satisfies the envelope formula (7).

Below we will apply the inequality (22) to show that \( x_i^*(\alpha) \geq 0 \) and \( x^*(\alpha_i) = E_{\alpha_{-i}}x_i^*(\alpha) \) is strictly increasing in \( \alpha_i \geq \alpha^* \).

Let \( \alpha_i^*(g_i; \alpha_{-i}) \) be the lower bound for \( i \) to be included in \( g_i \) and \( \alpha_i^h(g_i; \alpha_{-i}) \) be the upper bound for \( i \) to be included in \( g_i \). Recall that \( \tilde{\alpha}_i(\alpha_{-i}) \) denotes the minimum \( \alpha_i \) that can be included given \( \alpha_{-i} \).
Then (23) can be rewritten as

\[ x_i^*(\alpha_i; \alpha_{-i}) = \sum_{g_i} A_{g_i}^s g_i (\alpha_i, \alpha_{-i}) [\tilde{\pi}^s g_i (\alpha_i, \alpha_{-i}) - c] - \sum_{g^*(\alpha_{-i}) \not\subset g_i \subset \tilde{g}^*(\tilde{\alpha}_{-i}, \alpha_{-i}) \forall g \in g_i, \forall \alpha \in \alpha_{-i}} \int_{\min(a_i, a^h(g_i; \alpha_{-i}))}^{\alpha_i} |\tilde{g}_i| \int_{\alpha_i}^{\alpha_i} \frac{d\tilde{\pi}_i^s g_i (y, \alpha_{-i})}{dy} dy \text{ (by (22))} \]

\[ \geq \sum_{g_i} A_{g_i}^s g_i (\alpha_i, \alpha_{-i}) [\tilde{\pi}^s g_i (\alpha_i, \alpha_{-i}) - c] - \sum_{g^*(\alpha_{-i}) \not\subset g_i \subset \tilde{g}^*(\tilde{\alpha}_{-i}, \alpha_{-i}) \forall g \in g_i, \forall \alpha \in \alpha_{-i}} \tilde{\pi}_i^s g_i (\tilde{\alpha}_i, \alpha_{-i}) \min(a_i, a^h(g_i; \alpha_{-i})) \bigg|_{\alpha_i}^{\alpha_i} \]

To simplify notation, we use \( \tilde{\pi}_i^s g_i (\alpha_i, \alpha_{-i}) \) to denote \( \tilde{\pi}_i^s g_i (\alpha_i, \alpha_{-i}) \).

Recall the following two facts: (i) \( \tilde{\pi}_i^s g_i (\alpha) \geq \tilde{\pi}_i^s g_i (\alpha), \forall i \in g \subset \tilde{g} \), and (ii) by the definition of \( \tilde{\alpha}_i(\alpha_{-i}) \),

\[ \tilde{\pi}_i^s g^*(\tilde{\alpha}_i, \alpha_{-i}) (a_i^s (g^*(\tilde{\alpha}_i, \alpha_{-i}); \alpha_{-i}); \alpha_{-i}) \geq c. \]

Thus when \( i \in g^*(\alpha) \),

\[ x_i^*(\alpha_i; \alpha_{-i}) \geq [\tilde{\pi}_i^s g^*(\alpha_i, \alpha_{-i}) (a_i^s (g^*(\alpha_i, \alpha_{-i}); \alpha_{-i}); \alpha_{-i}) - c] - \sum_{g^*(\alpha_{-i}) \not\subset g_i \subset \tilde{g}^*(\tilde{\alpha}_{-i}, \alpha_{-i}) \forall g \in g_i, \forall \alpha \in \alpha_{-i}} \tilde{\pi}_i^s g_i (\tilde{\alpha}_i, \alpha_{-i}) \min(a_i, a^h(g_i; \alpha_{-i})) \bigg|_{\alpha_i}^{\alpha_i} \]

Note that for two neighboring sets \( g_i \) and \( g'_i \) such that \( g_i \not\subset g'_i \) and only the lowest \( \alpha_j \in g'_i \) is excluded from \( g_i \), we have \( a_i^s (g_i) = a^h(g'_i) \). Thus \( x_i^*(\alpha_i; \alpha_{-i}) \) can be written alternatively as follows:

\[ x_i^*(\alpha_i; \alpha_{-i}) \geq [\tilde{\pi}_i^s g^*(\tilde{\alpha}_i, \alpha_{-i}) (a_i^s (g^*(\tilde{\alpha}_i, \alpha_{-i}); \alpha_{-i}); \alpha_{-i}) - c] + \sum_{\forall g_i, g'_i, |g_i| = |g'_i| - 1 \forall g \in g_i, \forall \alpha \in \alpha_{-i}, \forall g \in g'_i, \forall \alpha \in \alpha_{-i}, \forall g \in g_i, \forall \alpha \in \alpha_{-i}} \tilde{\pi}_i^s g_i (a_i^s (g_i; \alpha_{-i}); \alpha_{-i}) - \tilde{\pi}_i^s g_i (a_i^s (g'_i; \alpha_{-i}); \alpha_{-i}) ] \tag{24} \]

\[ ^{20} \text{Note } a_i^s (g^*(\tilde{\alpha}_i, \alpha_{-i}), \alpha_{-i}) = \tilde{\alpha}_i(\alpha_{-i}). \]
Therefore \( x_i^\ast(\alpha_i; \alpha_{-i}) \geq 0 \) because

\[
\tilde{\pi}_i^{g_i}(a_i^1(g_i; \alpha_{-i}); \alpha_{-i}) - \tilde{\pi}_i^{g_i}(a_i^1(g_i; \alpha_{-i}); \alpha_{-i}) > 0,
\]

and

\[
\tilde{\pi}_i^{g_i}(\hat{\alpha}(\alpha_{-i}); \alpha_{-i})(a_i^1(g^\ast(\hat{\alpha}(\alpha_{-i}), \alpha_{-i}); \alpha_{-i}) - c \geq 0.
\]

Recall that \( \alpha^\ast \) is the minimum type that can be possibly admitted, i.e.,

\[
\alpha^\ast = \min_{\forall \alpha_{-i}, s.t. |g^\ast(\hat{\alpha}(\alpha_{-i}), \alpha_{-i})| = 1} \hat{\alpha}(\alpha_{-i}),
\]

we thus have \( \tilde{\pi}_i^{g_i}(\hat{\alpha}(\alpha_{-i}), \alpha_{-i})(\alpha^\ast; \alpha_{-i}) = c \) when \( |g^\ast(\hat{\alpha}(\alpha_{-i}), \alpha_{-i})| = 1 \) or \( g^\ast(\hat{\alpha}(\alpha_{-i}), \alpha_{-i}) = \{i\} \).

Take any two representative bidders, say, bidders 1 and 2, and define

\[
\tilde{a}_2(\alpha_1) = \begin{cases} 
\arg\min_{\alpha_2} [\tilde{\pi}_2^{g_i}(\alpha_1, \alpha_2) = c] & \text{if } \tilde{\pi}_2^{g_i}(\alpha_1, \alpha_2) \geq c,
+\infty & \text{if } \tilde{\pi}_2^{g_i}(\alpha_1, \alpha_2) < c.
\end{cases}
\]

Clearly, \( \tilde{a}_2(\alpha_1) \) strictly increases with \( \alpha_1 \) before it reaches the upper bound \( \bar{\alpha} \). Define \( a_2^\ast(\alpha_1) = \min[\tilde{a}_2(\alpha_1), \alpha_1] \leq \alpha_1 \). Clearly, \( a_2^\ast(\alpha_1) \) strictly increases with \( \alpha_1 \). Thus, only bidder 1 is admitted given \( \alpha_{-1} \) if \( \alpha_1 \geq \alpha^\ast \) and \( \max_{j \neq i}(a_j) \leq a_2^\ast(\alpha_1) \).

We next show that \( \forall \alpha_{-i}, \) we have \( x_i^\ast(\alpha_i; \alpha_{-i}) \geq x_i^\ast(\alpha_i; \alpha_{-i}) \) for \( \alpha_i > \alpha_i \). This is clear if \( i \notin g^\ast(\alpha_i; \alpha_{-i}) \). If \( i \in g^\ast(\alpha_i; \alpha_{-i}) \), we must have \( g^\ast(\alpha_i; \alpha_{-i}) \leq g^\ast(\alpha_i, \alpha_{-i}) \). Consider the difference of the two payments:

\[
\begin{align*}
&x_i^\ast(\alpha_i; \alpha_{-i}) - x_i^\ast(\alpha_i; \alpha_{-i}) \\
&= \sum_{g_i} A^{g_i}(\alpha_i; \alpha_{-i})[\tilde{\pi}_i^{g_i}(\alpha_i^1; \alpha_{-i}) - c] \\
&\quad - \sum_{g_i} \int_{g_i(\alpha_i; \alpha_{-i})} \min(\alpha_i^1, \alpha_i^1) \int_0^1 u_1(y, s_i) P_i^{g_i}(y, \alpha_{-i}, s_i) dG_i(s_i) dy \\
&\quad + \sum_{g_i} \int_{g_i(\alpha_i; \alpha_{-i})} \min(\alpha_i^1, \alpha_i^1) \int_0^1 u_1(y, s_i) P_i^{g_i}(y, \alpha_{-i}, s_i) dG_i(s_i) dy \\
&\quad - \sum_{g_i} A^{g_i}(\alpha_i; \alpha_{-i})[\tilde{\pi}_i^{g_i}(\alpha_i^1; \alpha_{-i}) - c].
\end{align*}
\]
\[-\sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i})[\tilde{\pi}^{*g_i}(\alpha_i; \alpha_{-i}) - c] \geq \sum_{g_i} A^{*g_i}(\alpha_i', \alpha_{-i})[\tilde{\pi}^{*g_i}(\alpha_i', \alpha_{-i}) - c] - \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i})[\tilde{\pi}^{*g_i}(\alpha_i, \alpha_{-i}) - c] \geq \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i})[\tilde{\pi}^{*g_i}(\alpha_i, \alpha_{-i}) - c] \]

\[-\sum_{g_i} \sum_{g_i' \subseteq g_i \cap g_i'} \int_{\alpha'_{(g_i; \alpha_{-i})}}^{\min[\alpha', \alpha^0_i(g_i; \alpha_{-i})]} \frac{d\tilde{\pi}^{*g_i}(y, y; \alpha_{-i})}{dy} dy \leq \sum_{g_i} A^{*g_i}(\alpha_i', \alpha_{-i})[\tilde{\pi}^{*g_i}(\alpha_i, \alpha_{-i}) - c] \geq \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i})[\tilde{\pi}^{*g_i}(\alpha_i, \alpha_{-i}) - c] \geq 0.\]

Thus for \(\alpha_{-i}\) such that both \(\alpha_i'\) and \(\alpha_i\) are admitted, we must have \(x_i^*(\alpha_i'; \alpha_{-i}) \geq x_i^*(\alpha_i; \alpha_{-i})\). In addition, there is a positive measure of \(\alpha_{-i}\) such that \(\alpha_i\) cannot be admitted as a sole entrant, but \(\alpha_i'\) can be admitted at least as a sole entrant. These \(\alpha_{-i}\) types cover the set \(\Omega(\alpha_i') = \{\alpha_{-i}| \alpha_i' \geq \alpha_i > \alpha_i' > \alpha_i > \alpha_i \}\). Note that \(\alpha_i' \geq \alpha_i'\). For any \(\alpha_{-i}\) in \(\Omega(\alpha_i'; \alpha_{-i})\), we have \(x_i^*(\alpha_i'; \alpha_{-i}) > x_i^*(\alpha_i; \alpha_{-i})\) as there are two possibilities spelled out below.

First, consider \(\alpha_{-i}\) in \(\Omega(\alpha_i'; \alpha_{-i})\) such that type \(\alpha_i\) is not admitted. In this case, \(x_i^*(\alpha_i; \alpha_{-i}) = 0\), but \(x_i^*(\alpha_i'; \alpha_{-i}) > 0\) must hold for the following reasons. There are two subcases. Case I: If for such an \(\alpha_{-i}\) in \(\Omega(\alpha_i'; \alpha_{-i})\), only bidder \(i\) is admitted when her type changes from \(\hat{\alpha}_i(\alpha_{-i})\) to \(\alpha_i'\), then \(|g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i})| = 1\) and \(\hat{\alpha}_i(\alpha_{-i}) > \alpha_i\). We have \(\hat{\alpha}_i(\alpha_{-i}) > \alpha_i\), because \(\hat{\alpha}_i(\alpha_{-i}) \geq \max_{j \neq i}(\alpha_j)\). \(\hat{\alpha}_i(\alpha_{-i})\) is admitted as a sole entrant and \(\max_{j \neq i}(\alpha_j) \geq \alpha_i > \alpha_i\). (\(\alpha_i\) is not even admitted as a sole entrant). Since \(\hat{\alpha}_i(\alpha_{-i}) > \alpha_i\), we have \(\tilde{\pi}_i^{*g^*(\hat{\alpha}_i(\alpha_{-i}) \alpha_{-i})}(\hat{\alpha}_i(\alpha_{-i}); \alpha_{-i}) > \tilde{\pi}_i^{*g^*(\alpha_i')}(\alpha_i'; \alpha_{-i}) - \tilde{\pi}_i^{*g^*(\alpha_i)}(\alpha_i; \alpha_{-i})\). Case II: If for such an \(\alpha_{-i}\) in \(\Omega(\alpha_i'; \alpha_{-i})\), there can be more than two bidders to be admitted when her type changes from \(\hat{\alpha}_i(\alpha_{-i})\) to \(\alpha_i'\), then by (25) \(x_i^*(\alpha_i'; \alpha_{-i})\) is larger than strictly positive term \(\tilde{\pi}_i^{*g^*(\alpha_i')}(\alpha_i'; \alpha_{-i}) - \tilde{\pi}_i^{*g^*(\alpha_i)}(\alpha_i; \alpha_{-i})\) where \(g_i = g^*(\alpha_i', \alpha_{-i})\). This means \(x_i^*(\alpha_i'; \alpha_{-i}) > 0\).

Second, consider \(\alpha_{-i}\) in \(\Omega(\alpha_i'; \alpha_{-i})\) such that type \(\alpha_i\) is admitted with at least two bidders. Note that \(\alpha_i\) cannot be admitted as a sole entrant by construction of \(\Omega(\alpha_i'; \alpha_{-i})\). In this case, we have \(x_i^*(\alpha_i; \alpha_{-i}) \geq 0\), but \(x_i^*(\alpha_i'; \alpha_{-i}) > x_i^*(\alpha_i; \alpha_{-i})\) for the following reason. For such an \(\alpha_{-i}\) in \(\Omega(\alpha_i'; \alpha_{-i})\), there can be more than two
bidders admitted when bidder $i$’s type changes from $\hat{\alpha}_i(a_{-i})$ to $\alpha_i$ and further to $\alpha_i'$, but eventually bidder $i$ is only admitted as the sole entrant when her type reaches $\alpha_i'$. Thus by (25) $x_i^*(\alpha_i'; \alpha_{-i})$ is larger than $x_i^*(\alpha_i; \alpha_{-i})$ by at least one strictly positive term $\bar{\pi}_i(g_i^*(\alpha_i; \alpha_{-i}); \alpha_{-i}) - \bar{\pi}_i(g_i^*(\alpha_i; \alpha_{-i}); \alpha_{-i})$ where $g_i = \{i\}$.

This means $x_i^*(\alpha_i'; \alpha_{-i}) > x_i^*(\alpha_i; \alpha_{-i})$.

We thus have proven that given $\alpha_{-i}$, $x_i^*(\alpha_i, \alpha_{-i})$ must be strictly increasing in $\alpha_i$ for $\alpha_i \geq \alpha^*$. Since $A^*(\alpha_i, \alpha_{-i}), \bar{\pi}_i^*(\alpha_i, \alpha_{-i})$, and $P_i^*(y, \alpha_{-i}, s_i)$ are all symmetric across bidders, $x_i^*(\alpha_i)$ is also a symmetric function by inspecting (23). Thus $x^*(\alpha_i) \equiv E_{\alpha_{-i}}x_i^*(\alpha_i)$ is strictly increasing in $\alpha_i$ for $\alpha_i \geq \alpha^*$. Under the handicap auction rules, $x^*(\alpha_i)$ can be rewritten as (16).

**Step 2:** We will show that, given (15) and (16), $x^*$ constitutes a symmetric (strictly) increasing equilibrium in the (reduced) first-stage all-pay auction game, with the second stage being replaced by its correlated equilibrium payoffs.

To that end, we consider the associated direct mechanism described by the entry allocation rule (11) and payment rules (15) and (16). By construction of (16), the envelope formula (7) is satisfied (with $\pi_i(\alpha_i, \alpha_{-i}) = 0$). The second-round allocation rule is dominant strategy implementable, so (4) is also satisfied. Thus we can follow exactly the same procedure as in the proof for Theorem 1 to demonstrate that

\[ \Delta = \pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\alpha_i, \alpha_i) \leq 0, \text{ for any } \alpha_i \text{ and } \hat{\alpha}_i \in [\underline{\alpha}, \bar{\alpha}] \]

That is, the associated direct mechanism is incentive compatible, which in turn implies that in the original (reduced) first-stage all-pay auction game, $x^*$ constitutes a symmetric (strictly) increasing equilibrium.

Hence the two-stage auction can implement the optimal mechanism described in Theorem 1.
REFERENCES


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