Competitive Nonlinear Pricing and Contract Variety*

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Abstract

We analyze how entry, or increased competition, affects the product line or contract variety in markets with both horizontally and vertically differentiated products. In the base model with two consumer types, we identify sufficient conditions under which entry prompts an incumbent to expand or contract the low end of its product line. We also extend our analysis to three types of consumers and show that entry may lead the incumbent to expand or contract the middle range of its product line (middle contracts). Our paper is the first to explain how entry or increased competition affects the product line or contract variety based on interactions between horizontal differentiation (competition) and screening of consumers along vertical (type) dimension.

Keywords: Nonlinear pricing, contract variety, product line, fighting brands, product pruning.

JEL Classification: D42, D43, D82, L15

1 INTRODUCTION

As more Japanese car makers enter the US market, will GM or Ford offer more models targeting different types of consumers? As more competitors enter the cellular phone market, will Verizon or Sprint offer more calling plans? A number of empirical studies and anecdotal evidence suggest that, when competition increased, firms responded by adjusting their product line or contract variety offered. For example, in the face of increased competition, American Express introduced 12 to 15 new credit cards per year, targeted at different customer segments. A similar pattern

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Forbes, July 1, 1996, “The Battle of Credit Cards”.
is observed in the airline industry as well. Borenstein and Rose (1994) find that on routes with more competition, each airline offers a larger variety of air tickets. On the other hand, increased competition may also lead to fewer contracts to be offered. For example, in response to private label brands entering in the early 1990’s, Procter & Gamble removed some weak products from its product line. More recently, Honda decided to eliminate its Element SUV after 2011 due to a mix of internal and external competition.

In this paper, we offer a framework to analyze how entry, or increased competition, affects the product line or contract variety offered. Specifically, in our model consumers are both vertically and horizontally differentiated: in the vertical dimension, they have different marginal utilities of quality and, in the horizontal dimension, they have different tastes over firms’ (horizontally differentiated) products. Thus each consumer is characterized by a horizontal type and a vertical type. Firms’ products are horizontally differentiated. Furthermore, each firm offers a range of products with different qualities in the vertical dimension. In the case with multiple firms, firms compete by offering a menu of nonlinear pricing contracts (or, equivalently, price-quality schedules).

In the base model, we focus on the case where consumers only have two vertical types, high \( h \) and low \( l \). We first show that, with a minimum quality requirement – which is reasonable in real world settings and is assumed throughout our analysis – pooling never arises in the optimal (monopolistic) or equilibrium (duopolistic) nonlinear pricing contracts. Thus each firm either offers a single contract targeting high-type consumers (the high or \( h \) contract hereafter), or two separating contracts targeting both types of consumers: one for the high type and one for the low type (the low or \( l \) contract hereafter). In other words, entry may only affect the lower end of the product line (leading to the removal or introduction of the low contract targeting type-\( l \) consumers). We compare the optimal menu of nonlinear pricing contracts under monopoly to the equilibrium menu of nonlinear pricing contracts in the symmetric equilibrium under duopoly. Our main result is that when the degree of horizontal differentiation (measured by the per unit transportation cost \( k \)) is low, entry will never lead to the removal of the low contract, but it may lead to the introduction of the low contract; on the other hand, when the degree of horizontal differentiation is high, entry will never lead to the introduction of the low contract, but it may lead to the removal of the low contract.

The intuition of the above results is as follows. First consider the case when the degree of horizontal differentiation \( (k) \) is low. Under monopoly, in order to reduce the informational rent enjoyed by type \( h \), the low type can be excluded, in which case only an \( h \) contract is offered.

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2 “Procter & Gamble to cut some items”, The Sun, August 3, 1992.
Under duopoly, competition for type $h$ leads to higher rent for type $h$. This relaxes the incentive compatibility (IC) constraint (the screening condition) in the sense that it is now less likely for type $h$ to mimic type $l$ by choosing the $l$ contract. Hence the informational rent consideration becomes less important as type $h$ will secure higher rent regardless. This implies that offering a contract to low-type consumers can be profitable, leading to the introduction of the low contract. The reason that entry may lead to the removal of the $l$ contract, however, is more subtle and involves the effect of entry on market shares as well. When $k$ is relatively large, the incentive for a firm to “steal” the other firm’s market share for type $h$ is low, thus entry leads to a lower rent provision to type $h$. This makes the screening condition more binding. On the other hand, entry reduces the incumbent firm’s market share for type $h$, which makes firms more willing to cover low-type consumers. When $k$ is sufficiently large, the first effect dominates, leading the incumbent to remove its low-quality product. It is thus clear that although $k$ does not enter the IC constraint directly, it affects the IC condition through the rent provision for type-$h$ consumers. In this sense, our main results are driven by the interaction between horizontal differentiation and vertical screening.

Similar results hold when further entry occurs, as our analysis of the two-type case can be easily translated to the case with $n \geq 3$ firms, which is standard for a Salop circular city model. Our analysis of the $n$-firm case suggests that if the initial competition is strong, further entry can only lead to the introduction of the low contract; if the initial competition is weak, however, further entry can lead to the removal or the introduction of the low contract. Intuitively, as the initial degree of competition changes, the informational rent required by type $h$ changes, which tightens or relaxes the incentive compatibility constraint, leading to the introduction or removal of the low contract.

In our two-type model, the expansion or reduction of contract variety only occurs at the lower end of the product line. To examine the robustness of our findings, we extend our analysis to the case with three vertical types: $h$, $m$, and $l$. Unlike the two-type model, (partial) pooling may now occur wherein the middle and the low types may choose the same contract. When the degree of horizontal differentiation is low, we identify conditions under which a monopolist offers two contracts, one targeting the high type and the other targeting both the low and the middle types (pooling), while in a duopoly both firms offer three contracts, targeting each type separately (fully separating). In other words, entry leads to the addition of the middle contract (the middle-quality product). When the degree of horizontal differentiation is high, on the other hand, we identify conditions under which the opposite occurs: a monopoly firm offers fully separating contracts,  

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4The degree of competition is measured by the horizontal differentiation parameter $k$ as demonstrated in Yang and Ye (2008).
but in a duopoly, each firm offers only two contracts with the middle and low types pooled at the low contract. In this case, entry leads to the removal of a middle contract or a middle-quality product. The driving force of these results, once again, is the interaction between horizontal differentiation (competition) and vertical screening (on vertical types).

Our paper is first related to the literature on nonlinear pricing, and, particularly, the literature on competitive nonlinear pricing. Since the seminal work by Mussa and Rosen (1978) and Maskin and Riley (1984) on monopolistic nonlinear pricing, there is a growing literature on nonlinear pricing in competitive settings, see, for example, Spulber (1989), Champsaur and Rochet (1989), Wilson (1993), Gilbert and Matutes (1993), Stole (1995), Verboven (1999), Villas-Boas and Schmidt-Mohr (1999), Armstrong and Vickers (2001, 2006), Rochet and Stole (1997, 2002), Ellison (2005). However, all these papers assume that all the (vertical) types of consumers are served in the market. This full market coverage assumption greatly simplifies the analysis, but precludes the effect of competition on the number of contracts offered on the vertical dimension, which is the main focus of our analysis.

An exception is provided by Yang and Ye (2008), who allow for partial market coverage on the vertical dimension. By focusing on the case where the lowest type of consumer being served is endogenously determined, they are able to study the effect of varying the horizontal differentiation (competition) on market coverage. However, Yang and Ye (2008) assume a continuous type space along the vertical dimension. This prevents an analysis of how increased competition can affect the number of contracts offered, a main task left for current research with discrete (vertical) types of consumers.

In our base model with two-type consumers, we show that the change in the contract variety only occurs at the lower end of the product line. This relates our work to Johnson and Myatt (2003) on fighting brands (expansion of product line at the lower end) and product pruning (contraction of product line at the lower end). In Johnson and Myatt (2003), a single firm enters a market originally dominated by a monopolist. The duopolists then compete in quantities, each potentially offering a range of quality-differentiated products. They show that whether the incumbent will choose to extend or contract its product line depends on the shape of the marginal revenue curves in the market. When marginal revenue is decreasing, the incumbent responds to entry by pruning low-quality products. However, when marginal revenue is increasing in

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5In Yang and Ye (2008), although the coverage of the vertical types changes moving from monopoly to duopoly, the quality range offered is always the same.

6Another exception allowing for partial coverage is Monteiro and Page (2008), who study a “catalog” game in which firms compete for a buyer of unknown type by offering the buyer a catalog of products and prices. Their analysis focuses on the existence of a Nash equilibrium in mixed-strategy catalog games, which is quite different from our focus in this paper.

7Abundant examples of fighting brands and product pruning can be found in Johnson and Myatt (2003).
some regions, upon entry an incumbent may find it optimal to introduce a lower-quality product (brand fighting). The base model in our paper offers an alternative explanation for fighting brands and product line pruning: in Johnson and Myatt, whether fighting brand or product pruning will occur depends on the shape of the marginal revenue curve, which in turn depends on the distribution of consumer types; while in our model, it is the degree of horizontal differentiation (intensity of competition) that determines whether fighting brands or product pruning will occur.

Our paper is also related to the common-agency literature in which the contract is non-exclusive. The most relevant setting is the delegated common-agency game. Specifically, in such a game consumers have multi-unit demand and each consumer can either buy from one store or split her purchase from competing sellers offering nonlinear pricing schedules. Bond and Gresik (1997) studies a duopoly game in which one of the firms is perfectly informed. When neither firm observes consumers’ types, Martimort and Stole (2009) show that in the case of substitutes, the participation rate is higher in the duopoly market than in the monopoly market, and the quality distortion is lower in the duopoly market. However, they do not consider horizontal differentiation, which is a key element in our modeling.

The rest of the paper is organized as follows. The next section lays down the base model with two consumer types. Section 3 extends the analysis to the case with three consumer types. Section 4 concludes. All the proofs not provided in the main text are relegated to the appendix.

2 The Base Model with Two Consumer Types

We consider a market with both horizontally and vertically differentiated products where consumers’ preferences differ in two dimensions. In the horizontal dimension, consumers have different tastes for different products (firms); in the vertical dimension consumers have different marginal utilities of quality. In the vertical dimension a consumer is either type $h$ (High) or type $l$ (Low), i.e., the vertical type $\theta \in \{\theta_h, \theta_l\}$, where $\theta_h > \theta_l > 0$. Without loss of generality, we normalize $\theta_h = 1$. The proportions of types $h$ and $l$ are $\alpha$ and $1 - \alpha$, respectively. We model the taste dimension as the horizontal “location” of a consumer on a unit-length circle representing the ideal product for that consumer; we adopt the Salop’s circular city model so that in the horizontal dimension, each type of consumer is uniformly distributed on a unit-length circumference. Consumers’ vertical and horizontal tastes are independently distributed. The total measure of consumers is 1.

We consider cases with one or two horizontal products. A horizontal product may offer two

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8For references please see Stole (2007).
goods of different qualities $q$. In the horizontal dimension, each consumer is characterized by $d$, the distance between his own (ideal) location and the location of a particular product (say, product 1). To sum up, each consumer is characterized by a two-dimensional type $(\theta, d)$. Each consumer has a unit demand for the good.\textsuperscript{9} If a type $\theta$ consumer consumes a product of quality $q$ that is located away from his own location by a distance $d$ and pays a transfer $t$, his utility is given by
\begin{equation}
 u(\theta, q, t, d) = \theta q - t - kd,
\end{equation}
where $k$ measures the degree of horizontal differentiation. The variable $k$ indicates consumers’ willingness to buy a good that is not exactly of his own taste and is the per-unit transportation cost in the standard Hotelling or Salop’s circular city models.

We assume that there is a minimum quality standard so that each firm can only produce $q \geq q^l$, $q \in (0, q^f b)$, where $q^f b$ is the first-best (full-information) quality provision to the low-type consumers. Such requirement is standard in various industries and is mainly due to government regulations for safety or externality considerations. For example, if the quality of a car is below some threshold level, it might not be safe to drive it. Another example is the new fuel-economy standards issued by the U.S. government for the automobile industry.\textsuperscript{10} The impacts of the minimum quality requirement on firms’ production decisions and consumers’ welfare are documented and studied in the literature.\textsuperscript{11}

If a firm sells a product of quality $q$ to a consumer, its profit from that sale is given by
\begin{equation}
 v(q, t) = t - c(q),
\end{equation}
where $c(q)$ is the cost of producing a good of quality $q$. $c(q)$ is assumed to be strictly increasing.

\textsuperscript{9}The unit demand assumption is standard in the nonlinear pricing literature (See, e.g., Mussa and Rosen, 1978; Armstrong and Vickers, 2001; and Rochet and Stole, 2002). This assumption is relevant in many markets (e.g., markets for durable goods such as TV or refrigerator). On the other hand this assumption enables us to focus on the product line adjustment along the quality dimension. Should this assumption fail, our analysis may change dramatically. To see this, assume the simplest case in which consumers have the same taste over quantity dimension but one cannot purchase from both firms. Based on the utility function in (1), when a consumer purchases $Q$ units, her utility is given by
\begin{equation}
 \int_0^Q (p(a, q; \theta) - t - kd) da,
\end{equation}
where the willingness to pay $p(a, q; \theta)$ decreases in $a$ and increases in $q$ and $\theta$. Since firms cannot engage in price discrimination in terms of quantities, the unit price $t$ only depends on the quality $q$. The maximizer $Q^*$ is determined by
\begin{equation}
 p(Q^*, q; \theta) = t + kd.
\end{equation}
Thus $Q^*$ is a function of both $q$ and $d$, which means that in the utility function, horizontal and vertical types are not separable. This usually requires a very different analytical framework, which is beyond the scope of this paper.

\textsuperscript{10}“U.S. sets higher fuel efficiency standards”, The New York Times, August 28, 2012. We thank an anonymous referee for suggesting this example to us.

\textsuperscript{11}See Armstrong and Sappington (2007) for references.
and convex in \( q \) with the following properties:

\[
\begin{align*}
c(0) &= 0, \lim_{q \to 0} c'(q) = 0, \text{ and } \lim_{q \to +\infty} c'(q) = +\infty.
\end{align*}
\] (2)

Joint surplus from trade for type \( \theta \), given quality provision \( q \), is given by \( S(q, \theta) = \theta q - c(q) \). When \( \theta \) is observable, the full-information first-best quality provision that maximizes the joint surplus is determined by the following conditions:

\[
c'(q_{fb}^h) = 1; \quad c'(q_{fb}^l) = \theta_l.
\]

We use \( S_{fb}^h \) and \( S_{fb}^l \) to denote the surplus generated from the first-best quality provision for types \( h \) and \( l \), respectively:

\[
S_{fb}^h \equiv q_{fb}^h - c(q_{fb}^h); \quad S_{fb}^l \equiv \theta_l q_{fb}^l - c(q_{fb}^l).
\]

Neither \( \theta \) nor \( d \) is observable to firms, but, as is obvious from (1), the single crossing property is only satisfied in the vertical dimension. As a result firms can only make offers to sort consumers with respect to their vertical types in our model.\(^{12}\) We are interested in how market structure affects the products offered in the vertical dimension (the contract variety). Specifically, we compare two different scenarios. The first scenario is a monopoly, where a single horizontal product is offered by a single firm. The second scenario is a duopoly, where two horizontal products are offered by two different firms, who evenly split the unit-length circle as illustrated by the following Salop’s circular city model.

In what follows we will focus on the case when, given any convex production cost function \( c(\cdot) \) that satisfies (2), \( q \) is not too large so that the following condition holds:

\[
(1 - \theta_l)q < \frac{S_{fb}^h}{2}.
\] (3)

It can be easily verified that, as a special case, when the production function takes the quadratic form, \( c(q) = q^2/2 \), condition (3) holds as \( q < \theta_l \).\(^{13}\) When condition (3) fails, the analysis would be much more messy but the main conclusion will not be altered.\(^{14}\)

\(^{12}\)For this reason we do not need to tackle the problem with multi-dimensional screening as in, e.g., Laffont, Maskin and Rochet (1987); McAfee and McMillan (1988); Armstrong (1996); and Rochet and Chone (1998).

\(^{13}\)With \( c(q) = q^2/2 \), we have \( (1 - \theta_l)q < (1 - \theta_l) \theta_l \leq 1/4 = S_{fb}^l/2 \).

\(^{14}\)The detailed analysis is available upon request.
2.1 Monopoly

The monopolist may offer a nonlinear pricing schedule \( P(q) \), or, equivalently, a menu of separate contracts, \((q_h, t_h)\) and \((q_l, t_l)\), targeting consumers with types \( h \) and \( l \), respectively. Associated with two contracts, the gross utility of a type \( i, i = h, l \), who chooses contract \((q_i, t_i)\), is given by 

\[
u_i = \theta_i q_i - t_i.
\]

Since it is more convenient to use \( u_i \) instead of \( t_i \), we write a contract as \((q_i, u_i)\).\(^{15}\)

A menu of (two) contracts is incentive compatible if and only if:

\[
(1 - \theta_l) q_h \geq u_h - u_l \geq (1 - \theta_l) q_l, 
\]

where the first inequality is the upward incentive compatibility (UIC) constraint requiring that type \( l \) have no incentive to mimic type \( h \) and the second inequality is the downward incentive compatibility (DIC) constraint requiring that type \( h \) have no incentive to mimics type \( l \). In what follows, \( u_h - u_l \), the difference in utilities gained by the two types, is referred to as the informational rent.\(^{15}\)

\(^{15}\)Here we follow the lead of Armstrong and Vickers (2001), who model firms as supplying utility directly to consumers.
Given \( u_i \), the (half) market share for each type, \( M(u_i, i) \),\(^{16}\) is given by,

\[
M(u_h, h) = \alpha \min \left\{ \frac{1}{2}, \frac{u_h}{k} \right\}, \quad \text{and} \quad M(u_l, l) = (1 - \alpha) \min \left\{ \frac{1}{2}, \frac{u_l}{k} \right\}.
\]

The monopolist has two options in terms of contract variety: offering one contract or offering two contracts. Without the minimum quality requirement \( q \geq q \), it can be readily shown that offering two contracts dominates offering only one contract. With the minimum quality requirement, the above property no longer holds (as offering the low contract may become too “costly” for the firm due to the minimum quality requirement). We assume a minimum quality requirement in our analysis so that the number of contracts offered is not necessarily two. When only one contract is offered, either only type \( h \) agents participate or both types participate (pooling). The following lemma establishes that pooling is never optimal.

**Lemma 1.** Suppose the monopolist offers a single contract with \( q \in [q, q_f^b] \) and both types participate. Then the monopolist can earn higher profit by offering two contracts.

Lemma 1 shows that we can focus on offering two contracts or only offering \( h \) contracts targeting type \( h \) when searching for optimal contract(s). When two contracts are offered, the UIC is always slack (see Lemma 1 in Rochet and Stole, 2002, for the details). Given that the UIC is slack (type \( l \) does not want to mimic type \( h \)), the quality provision for type \( h \) should be efficient, i.e., \( q_h = q_f^b \). Moreover, we only need to worry about the DIC. Because of horizontal differentiation, the DIC might be binding or slack: although \( k \) does not enter the DIC directly, it affects \( u_h \), and hence the DIC indirectly.

We first derive firms’ choices when the types are observable. These full-information results will serve as base results and are useful for derivations in the private-information case.

Let’s begin our discussion with the case when a single contract targeting type \( h \) is offered. In this case, the firm offers the first-best quality \( q_f^b \) and its maximization problem is given by:

\[
\max_{u_h} \alpha \min \left\{ \frac{1}{2}, \frac{u_h}{k} \right\} \left( S_f^b - u_h \right).
\]

It is easily verified that the optimal solution is given by

\[
u_f^b = \begin{cases} \frac{S_f^b}{2} & \text{if } k \in \left[ S_f^b, 2S_f^b \right] \\ \frac{k}{2} & \text{if } k \in \left( 0, S_f^b \right) \end{cases}.
\]

The resulting (half) market share for type \( h \) is either \( 1/2 \) if \( k \leq S_f^b \), or \( S_f^b/2k \) if \( k > S_f^b \). For

\(^{16}\)To ease exposition, we use half market share throughout the paper.
competition to be nontrivial in the duopoly case, we assume \( k < 2S_h^{f'} \), so that the market share for type \( h \) under duopoly is more than \( 1/4 \).

We next consider the case when two contracts are offered. The firm’s maximization problem is given by:

\[
\max_{(u_h, q_l, u_l)} \alpha \min \left\{ \frac{1}{2}, \frac{u_h}{k} \right\} \left( S_{h}^{f'} - u_h \right) + (1 - \alpha) \min \left\{ \frac{1}{2}, \frac{u_l}{k} \right\} \left( \theta_l q_l - c(q_l) - u_l \right)
\]

subject to: \( u_h \geq u_l + (1 - \theta_l) q_l \) (DIC)

\[ q_l \geq q, \ u_l \geq 0. \]

The (full-information) unconstrained solution is \( u_h = u_h^{f'} \), \( q_l = q_l^{f'} \) and

\[
u_{f}^{l} = \begin{cases} \frac{S_{l}^{f}}{2} & \text{if } k \in \left\{ S_{l}^{f}, 2S_{h}^{f} \right\} \\ \frac{k}{2} & \text{if } k \in \left( 0, S_{l}^{f} \right) \end{cases}
\]

Given the full-information results, we now turn our attention to the case in which types are private information. The unconstrained solution is not feasible if \( k \leq S_{l}^{f} \). It is not feasible either when

\[
\frac{k}{2} - \frac{S_{l}^{f}}{2} < (1 - \theta_l) q_{l}^{f} \quad \text{if } S_{l}^{f} < k \leq S_{h}^{f} \\
\frac{S_{l}^{f}}{2} - \frac{S_{l}^{f}}{2} < (1 - \theta_l) q_{l}^{f} \quad \text{if } S_{h}^{f} < k \leq 2S_{h}^{f}.
\]

Combining the above conditions, the unconstrained solution is not feasible if \( k < S_{l}^{f} + 2(1 - \theta_l)q_{l}^{f} \) in the case \( k \leq S_{h}^{f} \), or \( S_{h}^{f} - S_{l}^{f} < 2(1 - \theta_l)q_{l}^{f} \) in the case \( k \in \left( S_{h}^{f}, 2S_{h}^{f} \right) \). We consider the following cases in order.

**Case M1**: \( 0 < k \leq S_{h}^{f} \). Note that if \( (1 - \theta_l)q < \frac{k}{2} \), offering two contracts is better than offering the \( h \) contract alone. To see this, suppose the firm offers the full-information \( h \) contract \((q_{h}^{f}, u_{h}^{f})\) alone. Given that \((1 - \theta_l)q < \frac{k}{2}\), the firm can profitably offer the \( l \) contract \((q_{l}, u_{l})\) with \( q_{l} \geq q \), \( u_{l} \geq 0 \), and \( u_{l} + (1 - \theta_l) q_{l} \geq \frac{k}{2} \). Therefore, offering the \( h \) contract alone cannot be optimal.

Next suppose \((1 - \theta_l)q \geq \frac{k}{2}\). Note that this implies that the unconstrained solution is not feasible, so the DIC must bind.\(^{17}\) By the DIC, this implies that \( u_{h} \geq k/2 \) if two contracts are offered. The programming problem then becomes:\(^{18}\)

\[
\max_{(q_{l}, u_{l})} \alpha \left[ S_{h}^{f} - u_{l} - (1 - \theta_l) q_{l} \right] + (1 - \alpha) \frac{u_{l}}{k} \left[ \theta_l q_{l} - c(q_{l}) - u_{l} \right]
\]

\(^{17}\)This is because \((1 - \theta_l)q < (1 - \theta_l)q_{l}^{f} \) by the fact that \( q < q_{l}^{f} \).

\(^{18}\)In writing the following programming problem, we implicitly assume that \( u_{l} \leq k/2 \). This is justified as offering \( u_{l} \) more than \( k/2 \) will lead to a loss in profit.
subject to: \( u_l \geq 0, \ q_l \geq q \).

Let the Lagrangian multiplier for the first and second constraints be \( \mu \) and \( \lambda \), respectively. The first-order conditions are

\[
- \frac{\alpha}{2} + \frac{1 - \alpha}{k} (\theta_l q_l - c(q_l) - 2u_l) + \mu = 0; \ \mu \geq 0, \ \mu = 0 \text{ if } u_l > 0; \quad (5)
\]

\[
- \frac{\alpha}{2} (1 - \theta_l) + \frac{1 - \alpha}{k} u_l (\theta_l - c'(q_l)) + \lambda = 0; \ \lambda \geq 0, \ \lambda = 0 \text{ if } q_l > q. \quad (6)
\]

In general, it is hard to determine the necessary and sufficient conditions under which offering the \( h \) contract alone is optimal. For this reason we will identify sufficient conditions only and the detailed analysis is relegated to the Appendix.

**Case M2:** \( k \in \left( S_{f b}^h, 2S_{f b}^h \right] \). Since, by condition (3), \( (1 - \theta_l)q < \frac{S_{f b}^h}{2} \), offering two contracts is optimal. The reason is that the firm can always profitably add a low contract without raising \( u_h \) when the first-best \( h \) contract is offered. Therefore, when \( k \in \left( S_{f b}^h, 2S_{f b}^h \right] \), two contracts must be offered under a monopoly.

The following lemma summarizes our analysis in the monopoly case.

**Lemma 2.**

(i) \( k \in \left( 0, S_{f b}^h \right] \). If \( (1 - \theta_l)q < \frac{k}{2} \), then the monopolist offers two contracts. If \( (1 - \theta_l)q \geq \frac{k}{2} \), and the following two conditions are satisfied, then offering an \( h \) contract only is optimal for the monopolist:

\[
\frac{(\theta_l - c'(q_l))S_{f b}^h}{2 - \theta_l - c'(q)} < \frac{k \alpha}{2(1 - \alpha)^2}; \quad (7)
\]

\[
\theta_l q - c(q) < \frac{k \alpha}{2(1 - \alpha)} \quad (8)
\]

(ii) \( k \in \left[ S_{f b}^h, 2S_{f b}^h \right] \). Offering two contracts is optimal for the monopolist.

The result that offering an \( h \) contract alone is optimal is due to informational rent considerations (recall the IC condition (4)). If by offering a low contract too much informational rent needs to be given to high types (relative to the profit from low types), then the firm will optimally exclude low types by not offering the \( l \) contract. From the previous analysis, we see that exclusion is more likely to occur when \( \alpha \) is big and \( q \) is close to \( q_{f b}^l \). A bigger \( \alpha \) implies that the high type becomes more important. Moreover, when \( q \) is close to \( q_{f b}^l \), the low quality cannot be
distorted downward by a large amount, which makes the low-type contract more attractive to the high type. This makes the firm more reluctant to offer a low contract. Part (ii) of Lemma 2 shows that the exclusion of low types is only possible when $k$ is small. When $k$ is big, the firm is willing to give a high rent to type $h$ in order to penetrate enough into the market for the high type. As a result, informational rent consideration becomes less important and the exclusion of the low type becomes less likely.

2.2 Duopoly

Under duopoly, two firms compete by offering nonlinear pricing contracts $(q^j, u^j)$, $j = 1, 2$. We adopt Bertrand-Nash equilibrium as our solution concept. Specifically, $\{(q^1, u^1)(q^2, u^2)\}$ is an equilibrium if given $(q^{-j}, u^{-j})$, firm $j$ maximizes its own profit by choosing $(q^j, u^j)$, $j = 1, 2$. Since firms are symmetric, we will focus on symmetric equilibria in which both firms offer the same contract(s), i.e., $q^1 = q^2$ and $u^1 = u^2$.

The result of Lemma 1 can be readily extended to the duopoly setting: there is no equilibrium in which both firms offer one contract and both high and low type consumers are served. In constructing a profitable deviation, we can fix the other firm’s contract and let one firm offer another contract targeting either type $h$ or type $l$, which offers the same utility to the targeting type as the original contract. This means that pooling equilibria do not exist. Therefore, we can concentrate on two possible equilibria. In the first scenario, each firm offers contract $h$ only and thus only type $h$ consumers are served. In the second scenario, each firm offers two contracts targeting at types $h$ and $l$ separately. Given that $k < 2S^f_h$, the market for type $h$ will be fully covered in the horizontal dimension. Therefore, the market share for type $h$ of firm 1 becomes $\frac{1}{4} + \frac{u^1_h - u^2_h}{2k}$. On the other hand, the market for the low type might not be fully covered. As a result, the market share for type $l$ of firm 1 is $\min\left\{\frac{u^1_l}{k}, \frac{1}{4} + \frac{u^1_l - u^2_l}{2k}\right\}$.

As in the monopoly section, we will start with the full-information case and then move on to the private-information case. Superscript $^D$ will be used for the full-information solutions while its lower case $^d$ for the private-information solutions.

When both firms offer $h$ contracts only, the profit maximization problem for firm 1, given $(q^2_h, u^2_h)$, is as follows:

$$\max_{u^1_h} \alpha \left(\frac{1}{4} + \frac{u^1_h - u^2_h}{2k}\right) \left(S^f_h - u^1_h\right), \text{ if } u^1_h + u^2_h \geq \frac{k}{2};$$

$$\max_{u^1_h} \frac{u^1_h}{k} \left(S^f_h - u^1_h\right), \text{ if } u^1_h + u^2_h \leq \frac{k}{2}.$$
Note that, given $u^2_h$, firm 1’s objective function is not differentiable at $u^1_h = \frac{k}{2} - u^2_h$. Solving the first case in the above maximization problem, we have the equilibrium utility $u^D_h = S_{h}^{fb} - \frac{k}{2}$. However, if $k > \frac{4}{3} S_{h}^{fb}$, then $2u^D_h < k$. Thus the second case in the maximization problem applies and the solution is given by $u^D_h = S_{h}^{fb}/2$. But then $2u^D_h \geq k/2$ given $k \leq 2S_{h}^{fb}$, a contradiction. It turns out that we have a corner solution: $u^D_h = k/4$. To see this, suppose firm 2 offers $u^D_h = k/4$. Then it can be verified that firm 1’s profit decreases if $u^D_h$ increases from $k/4$ (the first case), and firm 1’s profit decreases as well if $u^D_h$ decreases from $k/4$ (the second case). To sum up, in the symmetric equilibrium, we have

$$u^D_h = \begin{cases} \frac{k}{4} & \text{if } k \in \left(\frac{4}{3} S_{h}^{fb}, 2S_{h}^{fb}\right] \\ S_{h}^{fb} - \frac{k}{2} & \text{if } k \in \left(0, \frac{4}{3} S_{h}^{fb}\right] \end{cases} \quad (9)$$

Note that when $k$ is large and we are in the first case, even though the market for type $h$ is fully covered, each firm has no incentive to steal the other firm’s market share. In other words, there is no competition between two firms.

Similarly, under full information, $q^D_l = q^l_{fb}$ and $u^D_l$ takes the following form:

$$u^D_l = \begin{cases} \frac{S_{l}^{fb}}{2} & \text{if } k \in \left(2S_{l}^{fb}, 2S_{h}^{fb}\right) \\ \frac{k}{4} & \text{if } k \in \left(\frac{4}{3} S_{l}^{fb}, 2S_{l}^{fb}\right] \\ S_{l}^{fb} - \frac{k}{2} & \text{if } k \in \left(0, \frac{4}{3} S_{l}^{fb}\right] \end{cases} \quad (10)$$

Competition occurs for type $l$ consumers only when $k < \frac{4}{3} S_{l}^{fb}$. When $k \in \left[\frac{4}{3} S_{l}^{fb}, 2S_{l}^{fb}\right]$, although type $l$ consumers are fully covered, there is no competition for type $l$ consumers.

Now suppose consumer types are private information and that both firms offer contracts $h$ and $l$. First consider the case $k \in \left(0, \frac{4}{3} S_{h}^{fb}\right]$. The profit maximization problem for firm 1, given $(q^2_i, u^2_i)$, $i \in \{h, l\}$, is as follows:

$$\max_{(u^1_i, q^1_i)} \alpha \left(\frac{1}{4} + \frac{u^1_h - u^2_h}{2k}\right) \left(S_{h}^{fb} - u^1_h\right) + (1 - \alpha) \min \left\{\frac{1}{4} + \frac{u^1_l - u^2_l}{2k}, \frac{u^1_l}{k}\right\} \left(\theta_l q^1_l - c(q^1_l) - u^1_l\right)$$

subject to: $u^1_h \geq u^1_l + (1 - \theta_l)q^1_l$ (DIC)

$q^1_l \geq q$; $u^1_l \geq 0$

**Case D1**: $k \in \left(0, S_{h}^{fb}\right]$. In this case, in duopoly equilibrium firms must offer two contracts. To see this, suppose in equilibrium each firm only offers an $h$ contract. From the previous analysis,
the full-information utility $u_h^D = S_h^f - k/2$. Now given that $k \leq S_h^f$, by condition (3) we have

$$u_h^D \geq \frac{S_h^f}{2} > (1-\theta_l)q,$$

and thus firm 1 can profitably offer a low contract $(q_1, u_l)$, with $u_l > 0, q_1 \geq q$, and $(1-\theta_l)q + u_l \leq u_h^D$ (the DIC is satisfied). Therefore, when $k \in (0, S_h^f)$, in the duopoly equilibrium both firms offer two contracts.

**Case D2:** $k \in \left[S_h^f, \frac{4}{3} S_h^f \right]$. There are two subcases. First when $k < 2S_h^f - 2(1-\theta_l)q$, the full-information utility $u_h^D = S_h^f - k/2$. Since $S_h^f - k/2 > (1-\theta_l)q$, two contracts will be offered in equilibrium.\(^\text{19}\)

Second, when $k \geq 2S_h^f - 2(1-\theta_l)q$, we have $(1-\theta_l)q \geq S_h^f - \frac{k}{2}$ and offering two contracts means that the DIC must bind. Note that this condition implies that $k > 2S_h^f$. This is because

$$k \geq 2 \left[ S_h^f - (1-\theta_l)q \right] > 2 q^f_i - c(q^f_i) - 2(1-\theta_l)q^f_i = 2S_h^f.$$  

When $k > 2S_h^f$, even under full information the market for type $l$ is not fully covered. Hence under private information, the market for type $l$ is not fully covered either. Given $(q_i^2, u_i^2)$, firm 1’s programming problem becomes

$$\max_{\{q_i^2, u_i^2\}} \left[ \frac{1}{4} + \frac{u_i^1 + (1-\theta_l)q_i^1 - u_h^1}{2k} \right] \left[ S_h^f - u_i^1 - (1-\theta_l)q_i^1 \right] + (1-\theta_l)q_i^1 - c(q_i^1) - u_i^1$$

subject to: $u_i^1 \geq 0, q_i^1 \geq q$;

$$u_h^D = u_i^d + (1-\theta_l)q_i^d$$

Note that the above characterization is based on the condition that $2S_h^f - 2(1-\theta_l)q \leq \frac{4}{3} S_h^f$, or equivalently $S_h^f \leq 3(1-\theta_l)q$. If instead $S_h^f > 3(1-\theta_l)q$, then by the fact that $k \leq \frac{4}{3} S_h^f$, we have

$$\frac{S_h^f - k}{2} \geq \frac{1}{3} S_h^f > (1-\theta_l)q,$$

which implies that two firms will surely offer two contracts in this case.

\(^{19}\)A subcase of $k \in [0, \frac{4}{3} S_h^f]$ is included in this case. To see this,

$$S_h^f - (1-\theta_l)q > S_h^f - (1-\theta_l)q^f_i + c(q^f_i) - c(q^f_i)$$

$$= S_h^f - (q^f_i - c(q^f_i)) + S_h^f > S_h^f.$$  

Furthermore, it can be verified that the DIC is slack in this subcase even under the full-information scenario. Therefore, in equilibrium two contracts will be offered.
**Case D3:** \( k \in \left( \frac{4}{3}S_h^f, 2S_h^f \right) \). In this case, recall that under full information \( u_D^h = \frac{k}{4} \), and, although the market for type \( h \) is fully covered, there is no competition for type \( h \). If \((1-\theta_l)q < \frac{k}{4}\), then in equilibrium two contracts must be offered, as offering some low contract will not violate the DIC.

Now suppose \((1-\theta_l)q \geq \frac{k}{4}\). Then the DIC must bind if two contracts are offered. Moreover, \( k > \frac{4}{3}S_h^f > \frac{4}{3}S_l^f \) implies that \( u_i^D \leq \frac{k}{4} \). This means that in the duopoly equilibrium, \( u_i^D \leq \frac{k}{4} \), i.e., there is no competition for type \( l \). Therefore, the programming problem is the same as before.

The following lemma summarizes our analysis in the duopoly case.

**Lemma 3.**

(i) \( k \in \left( 0, S_h^f \right] \). Both firms offer two contracts in the duopoly equilibrium.

(ii) \( k \in \left( S_h^f, 2S_h^f \right) \). Two contracts will be offered in equilibrium if \( q \) is smaller than some upper bound \( q^\dagger \).

Both firms will offer \( h \) contract alone if \( q \geq q^\dagger \) and the following two conditions are satisfied:

\[
\frac{\left( \theta_l - c'(q) \right) S_l^f}{2 - \theta_l - c'(q)} < \frac{\alpha}{1 - \alpha} \left( \frac{3}{8}k - \frac{S_h^f}{2} \right); \tag{11}
\]

\[
\theta_l q - c(q) < \frac{\alpha}{1 - \alpha} \left( \frac{3}{8}k - \frac{S_h^f}{2} \right). \tag{12}
\]

The case of offering two contracts should be easy to understand from our previous analysis. For the case of offering only the \( h \) contract, the set of sufficient conditions (11) and (12) is derived from the first order conditions (15) and (16) (listed in the appendix), which ensures that it is not profitable for firms to offer contract \( l \) in equilibrium. Roughly speaking, when it is too costly to serve the low type consumers (a bigger \( q^\dagger \)) and there are enough high type consumers (a bigger \( \alpha \)), then only one contract will be offered in equilibrium. A more detailed discussion of the intuition and driving forces of the results is provided in the next subsection.

### 2.3 Comparison

We are now ready to compare the duopoly equilibrium with the optimal solution under monopoly. First, note that when \( k < \frac{4}{3}S_h^f \), the full-information rent to type \( h \) under duopoly is higher than that under monopoly (\( u_D^h > u_h^f \)); when \( k > \frac{4}{3}S_h^f \), however, the relationship is reversed (\( u_D^h < u_h^f \)).

---

20 The derivation of \( q^\dagger \) is contained in the proof of Lemma 3, which is moved to the appendix.

21 Note that this set of sufficient conditions implies that \( k > \frac{4}{3}S_h^f \). When \( k \leq \frac{4}{3}S_h^f \), conditions (11) and (12) fail to hold and results are more sensitive to the specifications of primitives.
**Proposition 1.** When $k$ is sufficiently low, conditions can be identified so that competition will never lead to the removal of the low contract, as both firms always offer two contracts in the duopoly equilibrium, but competition may lead to the introduction of the low contract; When $k$ is sufficiently large, however, conditions can be identified so that competition will never lead to the introduction of the low contract, as two contracts are always offered under monopoly, but competition may lead to the removal of the low contract.

When the degree of horizontal differentiation is low, competition may lead to the introduction of the low contract. This is quite intuitive. Given that $k$ is low ($k \leq S_{fb}^f$), under monopoly, only an $h$ contract is offered (type $l$ is excluded) to reduce the informational rent to type $h$. On the other hand, under duopoly, competition for type $h$ leads to a higher rent to type $h$. This relaxes the incentive compatibility constraint along the vertical dimension. Hence informational rent consideration becomes less important as type $h$ secures higher rent due to competition. This implies that offering a contract to low-type consumers might be profitable, which turns out to be indeed the case when $k \leq S_{fb}^f$.

The reason that competition may lead to the removal of the low contract is that, besides the competition effect, there is a market share effect by moving from monopoly to duopoly. Competition from the entrant reduces the incumbent’s market share for type $h$, which tends to reduce $u_h$ as there is a smaller market to penetrate into. When $k$ is relatively large, under duopoly firms have lower incentives to steal each other’s market share for type $h$, thus entry leads to a lower $u_h$. This makes the incentive compatibility condition more binding in the vertical dimension. In the meantime, there is a mitigating effect that under duopoly the measure of $h$-type consumers served by each firm is less than that served under monopoly, which makes the low type relatively more important and that firms become more willing to increase the rent to the low type. When conditions (11) and (12) are satisfied, the first effect dominates: firms are more concerned about information rent under duopoly. As a result, an incumbent monopolist responds to entry by removing its low quality product targeting type $l$ (the low contract).

In Figure 2, given the primitives, we illustrate the results from Proposition 1 in the $k - \alpha$ space. The dash-dotted curve represents conditions (7) and (8), while the solid curve summarizes conditions (11) and (12). In the left panel with $k \leq S_{fb}^f$, competition will lead to the introduction of the new $l$ contract if the pair $(k, \alpha)$ falls into the area M1. In the right panel when $k > S_{fb}^f$, on the contrary, competition will incentivize firms to cut the $l$ contract if $(k, \alpha)$ is in D1.

In summary, we see that the introduction or removal of the low contract is more likely to occur when the proportion of $h$ type is relatively high and $q_i^{fb}$ is close to $q$. Depending on the degree
of horizontal differentiation, the two sets of conditions make the exclusion of the low type more likely under monopoly and duopoly, respectively. When the degree of horizontal differentiation is low, in the absence of competition, the market penetration for the high type is strong and the rent to the high type is low, which means that the exclusion of the low type is more likely. In duopoly, as discussed before, the fierce competition relaxes the incentive compatibility constraint and thus entry might lead to fighting brands. When the degree of horizontal differentiation is high, however, under monopoly a high rent is given to the high type in order to penetrate into the market for the high type, which means that exclusion of the low type is less likely. Meanwhile, the incumbent firm’s market share is significantly reduced (to half) by competition, which also reduces its incentive to penetrate into the market for the high type. As a result, entry reduces the rent to the high type and exclusion of the low type becomes more likely, which leads to the removal of the low contract.

When two contracts are offered under both monopoly and duopoly, it would be interesting to find out whether or not the quality distortion for the low type \( (q^f_b - q_l) \) would be reduced by entry.\(^{22}\) It turns out that we can indeed obtain an unambiguous comparison when we focus on

\(^{22}\) For the high type, quality provision is always the first-best so there is no distortion.
the case with a quadratic cost function and \( k \leq 1/2 \). Let the qualities of the low contracts under monopoly and duopoly be \( q^m \) and \( q^d \), respectively.

**Proposition 2.** Suppose \( c(q) = q^2/2 \) and \( S_h^{fb} = 1/2 \) and that two contracts are offered under both monopoly and duopoly. If \( k \leq 1/2 \), then \( q^m \leq q^d \).

The intuition, again, has to do with the interaction of the (vertical) DIC constraint and the (horizontal) market shares. When \( k \) is small, competition in duopoly leads to higher rent for type \( h \). This relaxes the DIC. On the other hand, the market share for type \( h \) becomes smaller in duopoly, which makes each firm more willing to increase the rent to the low type. These effects both contribute to a smaller downward quality distortion.

However, when \( k > 1/2 \) and two contracts are offered under both duopoly and monopoly, the quality distortion can be higher or lower under duopoly. This is because with a higher \( k \), competition for the \( h \) type under duopoly tends to reduce the rent to the \( h \) type. Thus the two effects mentioned above work in the opposite directions, leading to ambiguous results regarding quality distortion. The following examples illustrate both possibilities. Suppose \( k = 2/3 \) and \( \theta_l = 1/3 \). It can be verified that the first-best solution is feasible under monopoly, hence \( q^m = \theta_l = 1/3 \). Under duopoly \( u^D_h = 1/6 < \theta^2_l + (1-\theta_l)\theta_l \), thus the first-best solution is not feasible. The binding DIC dictates that \( q^d < \theta_l = q^m \). Next we provide an example in which the quality distortion is higher under monopoly. Suppose \( k = 0.54 \) and \( \theta_l = 0.9 \). Since \( k \leq \theta^2_l \), we know that the first-best solution is obtained in duopoly and \( q^d = \theta_l \). However, under monopoly it can be easily verified that the first-best solution is not feasible \( k < (1-\theta_l)\theta_l \). Thus we conclude that \( q^m < q^d = \theta_l \).

In the previous analysis, we always assume that the entrant has the same technology as the incumbent firm. This assumption might not hold in general. In our working paper version, we also study the case when the entrant is technologically inferior to the incumbent, that is, the upper bound of the quality range for the entrant’s products is lower than that of the incumbent. We first show that if both firms offer two contracts, the quality distortion of the low contract offered by the incumbent is smaller. We also provide comparative statics with respect to the degree of technology asymmetry, and show that when \( k \) is sufficiently small (so that there is effective competition for the high type), as the entrant becomes more technologically inferior, fighting brands becomes less likely and product pruning becomes more likely.

### 2.4 More Firms

In this subsection we study how further entry affects the product line, or the number of contracts offered by incumbent firms. Let \( n \) be the number of firms. In the horizontal dimension, the
location of \( n \) firms’ products evenly split the unit-length circle. As standard in the Salop circular city model, an increase in \( n \) in the \( n \)-firm model is equivalent to a decrease in \( k \) in the duopoly model, as competition may exist only between two adjacent firms (see Yang and Ye(2008), for a demonstration). For this reason we focus on the comparative statics of the duopoly model with respect to \( k \).

The following proposition shows that the impacts of a decrease in \( k \) on the menu of contracts offered depends on the initial level of \( k \).

**Proposition 3.** (i) If \( k \in (0, \frac{4}{3}S_b^h] \), then in the duopoly equilibrium a decrease in \( k \) can only lead to the introduction of low contracts. If two contracts are offered in the duopoly equilibrium under both \( k \) and \( k' \), where \( k > k' \), then \( q_d^l(k') \geq q_d^l(k) \). (ii) If \( k \in \left( \frac{4}{3}S_b^h, 2S_b^h \right] \), then in the duopoly equilibrium a decrease in \( k \) may result in either the introduction or removal of low contracts.

In the \( n \)-firm model, Proposition 3 implies that whether entry leads to the introduction or removal of the low contract depends on the initial degree of competition. When initial competition is fierce (\( k \) is small or the initial \( n \) is large), then further entry can only lead to the introduction of the low contract and a decrease in quality distortion. On the other hand, when initial competition is weak (\( k \) is big or the initial \( n \) is small), then further entry can lead to either the introduction or removal of the low contract. These results again come from the combined competition and market share effects from further entry. When the initial level of competition is high, entry leads to fierce competition for the high type, which increases the rent to the high type. Moreover, entry reduces incumbent firms’ effective market share for the high type, which makes the low type more important. These two effects work in the same direction, relax the DIC, and make it potentially profitable to introduce the low contracts. When the initial level of competition is low, the competition effect is absent. But reduced market share for the high type reduces the rent to the high type. This tends to make the DIC constraint more binding. On the other hand, a reduced market share for the high type implies that the low type becomes more important, which tends to relax the DIC. If the first effect dominates, entry can only lead to the removal of low contracts. However, if the second effect dominates, entry can only lead to the addition of low contracts.

Our results are consistent with the empirical findings of Seim and Viard (2011), who study how entry into local cellular phone market affects the number of calling plans offered by each incumbent firm. When the initial number of firms is small in a local market, entry reduces...
the number of calling plans offered by incumbents. However, when the initial number of firms is large, incumbent firms respond to entry by increasing the number of calling plans.

3 Three-type Model with Partial Pooling

We now consider a model with three vertical types. Suppose in the vertical dimension consumers have three types: $\theta_h, \theta_m, \text{and} \theta_l$, where $\theta_h = 1 > \theta_m > \theta_l$. The proportions of types are $\alpha_h, \alpha_m,$ and $\alpha_l$, respectively ($\alpha_h + \alpha_m + \alpha_l = 1$). All the other assumptions are the same as in the base model.

Since three contracts can be potentially offered, entry may lead to the introduction (or removal) of a middle quality product, a low quality product, or both middle and low products (contracts). The pattern with three types can be more complicated than in the two-type case where the adjustment only occurs at the lower-end product line. In particular, with three types, pooling of the middle and the low types may occur. So expansion or contraction of the set of contracts offered might occur for the middle product (contract).

3.1 Entry Leads to The Introduction of A Middle Contract

**Monopoly.** We start with the analysis of monopoly. Under monopoly, the full information solution is as follows: $q_i^{fb}$ is optimally determined by $c'(q_i^{fb}) = \theta_i, i = h, m, \text{or} l,$ and

$$u_i^{fb} = \begin{cases} \frac{S_i^h}{2} & \text{if } k \in (S_i^h, 2S_i^h] \\ \frac{k}{2} & \text{if } k \in (0, S_i^h] \end{cases}.$$ 

Similarly to the argument in Lemma 1, we can show that type $h$ is never pooled with the other two types. Overall, we have four cases to consider: only an $h$ contract is offered, only $h$ and $m$ contracts are offered and type $l$ is excluded, three contracts are offered (full separating), and two contracts are offered with types $m$ and $l$ pooling at the low contract (partial pooling). We are interested in the last case, as it is qualitatively different from the two-type base model. The relevant ICs are: $u_h - u_m \geq (1 - \theta_m)q_m \text{ (DIC}_{hm}), u_m - u_l \geq (\theta_m - \theta_l)q_l \text{ (DIC}_{ml}), u_h - u_l \geq (1 - \theta_l)q_l \text{ (DIC}_{hl}), \text{and } u_m - u_l \leq (\theta_m - \theta_l)q_m \text{ (UIC}_{lm})$. Note that when $q_m \geq q_l$, then DIC$_{hl}$ is redundant.

Under private information, the following lemma provides a set of sufficient conditions under which the monopolist will offer two contracts: one for the high type and one for both middle and low types.

**Lemma 4.** Suppose $k \leq S_h^{fb}$. If $k < S_m^{fb} + 2(1 - \theta_m)q_m^{fb}, S_m^{fb} - S_l^{fb} < 2(\theta_m - \theta_l)q_l^{fb}$, provided that the following restrictions regarding the type distribution hold, then under monopoly, the optimal menu of contracts exhibits partial pooling: two contracts are offered, with the high contract
targeting at type \( h \), and types \( m \) and \( l \) pooled at the low contract with \( q_l = q \):

\[
\begin{align*}
\frac{\alpha_h}{\alpha_m} &\geq \max \left\{ \frac{\theta_m - c(q)}{1 - \theta_m}, \frac{2}{k} \left( \theta_m q - c(q) \right) \right\}, \\
\alpha_h \left( 1 - c'(q) \right) &\geq \frac{2\alpha_m}{k} (\theta_m - \theta_l) S^f_l + \theta_l - c'(q),
\end{align*}
\]

(13)

and

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{1 - \alpha_h}{k} \left( \theta_l q - c(q) \right)^2 - \left( \frac{\alpha_h k}{2(1 - \alpha_h)} \right)^2 &- \alpha_h \left[ \left( \theta_l q - c(q) - \frac{\alpha_h k}{2(1 - \alpha_h)} \right) + 2(1 - \theta_l) q - k \right] > 0 \\
&\text{if } \theta_l q - c(q) \geq \frac{\alpha_h k}{2(1 - \alpha_h)}, \\
(1 - \theta_l) q &< \frac{k}{2} &\text{if } \theta_l q - c(q) < \frac{\alpha_h k}{2(1 - \alpha_h)}.
\end{array} \right.
\]

(14)

Condition (13) says that, compared to the proportion of the high type (\( \alpha_h \)), there cannot be too many middle consumers (\( \alpha_m \) is small). It is implied in condition (14) that either the proportion of the high type is bounded from above or the minimum quality \( q \) is low enough. Intuitively, since \( \alpha_m \) is small, it is too costly to price discriminate between the middle and low types. Meanwhile, when \( \alpha_h \) is not too large or \( q \) is sufficiently small, offering a second contract targeting at both the middle and low types would be profitable. Therefore partial pooling arises in monopoly equilibrium.\(^{25}\)

**Duopoly.** Under full information, the duopoly equilibrium contracts take the following form: \( q^D_i = q^f_i, i = h, m, \) or \( l \), and

\[
u^D_i = \begin{cases} 
S^f_i & \text{if } k \in [2S^f_i, 2S^f_h] \\
\frac{k}{4} & \text{if } k \in [\frac{3}{4}S^f_i, 2S^f_h) \\
S^f_i - \frac{k}{2} & \text{if } k \in (0, \frac{4}{3}S^f_i)
\end{cases}
\]

We are interested in the case \( k \leq S^f_h \) and will focus on the duopoly equilibrium in which the menu of contracts is fully separating.

Given that \( k \leq S^f_h \), \( u^D_h = S^f_h - \frac{k}{2} \). Under private information, since \( (1 - \theta_i) q < S^f_h / 2 \leq S^f_h - k / 2 \), in the duopoly equilibrium at least two contracts are offered. One sufficient condition to guarantee full separation is that \( k \leq \frac{4}{3} S^f_i \). In this case, competition exists for all three types. The full information solution always satisfies the DICs: for \( \theta_i > \theta_j \),

\[
u^D_i - u^D_j = S^f_i - S^f_j > (\theta_i - \theta_j) q^f_j.
\]

\(^{25}\)A concrete example will be introduced after the analysis of duopoly.
Therefore, the duopoly equilibrium exhibits full separation and no quality distortion.

We next identify another sufficient condition. Suppose that \( k \in (\frac{4}{3}S_f^b, 2S_f^b) \), and \( 4S_h^b - 3k \geq 4(1 - \theta_m)q^b_m \). By the first condition, \( v^D_m = k/4 \). By the second condition, the DIC_{hm} is slack in the full-information solution. Therefore, in the duopoly equilibrium we must have \( q^d_m = q^b_m \) (no quality distortion for type \( m \)). We further assume that \((\theta_m - \theta_l)q < k/4 \). This condition implies that if \( q_l \) is low enough, offering a contract to type \( l \) will not affect the DIC_{ml}. Therefore, in the duopoly equilibrium, each firm must offer three contracts (fully separating). The following lemma summarizes the results of duopoly.

**Lemma 5.** Suppose \( k \leq S_h^b \). (i) If \( k \leq \frac{4}{3}S_l^b \), then in the duopoly equilibrium the full-information solution is feasible: each firm offers three contracts without quality distortion. (ii) If \( k \in (\frac{4}{3}S_m^b, 2S_m^b) \), \( 4S_h^b - 3k \geq 4(1 - \theta_m)q^b_m \), and \((\theta_m - \theta_l)q < k/4 \), then in the duopoly equilibrium each firm offers three contracts, with \( q^d_m = q^b_m \).

Combining Lemma 4 and Lemma 5, we have the following result.

**Proposition 4.** Let \( k \leq S_h^b \). If the parameter values are such that all the conditions in Lemma 4 are satisfied, and either \( k \leq \frac{4}{3}S_l^b \) or the conditions in part (ii) of Lemma 5 are satisfied, then under monopoly two contracts are offered, with the low and middle types pooled at \( q \), while in the duopoly equilibrium each firm offers three contracts (fully separating).

It is easy to see that there are parameter values such that both conditions in Lemma 4 and Lemma 5 are satisfied. This is because the conditions in Lemma 5 have nothing to do with the distribution of types. So we can choose \( \alpha \)'s freely to satisfy the conditions in Lemma 4.\(^{26}\)

Proposition 4 illustrates that entry can expand the incumbent’s menu of contracts by converting a partial pooling equilibrium to a fully separating equilibrium. We should emphasize that this scenario is different from the introduction of low contracts in the two-type model prompted by entry. In the scenario described by Proposition 4, the low quality good (contract) is offered under monopoly, and entry leads to an introduction of a middle quality good (contract).\(^{27}\) Our analysis thus identifies a new pattern of product line expansion other than fighting brands or product pruning. Such a pattern is consistent with, for example, a finding in Seim and Viard (2011) that with more entry, firms may spread their calling plans more evenly over the usage spectrum. As another example, following the release of TomTom’s first GPS series GO in March

\(^{26}\)The following parameter values satisfy all the conditions in Lemma 4 and the conditions in part (ii) of Lemma 5: \( c(q) = q^2/2, k = 0.3, \theta_m = 0.6, \theta_l = 0.5, q = 0.4, a_h = 0.19, a_m = 0.1, a_l = 0.71 \).

\(^{27}\)Note that in the duopoly equilibrium the middle quality is strictly higher and the low quality is weakly higher than the low quality under monopoly. In this sense, entry leads to the addition of the middle contract (quality) instead of the low contract (quality).
2004, the incumbent Garmin introduced the Quest series as a medium-level product which featured a 2.7" 240x160 non-touch-sensitive color screen.\textsuperscript{28}

The driving force behind Proposition 4 is again the interaction between horizontal competition and vertical screening. When $k$ is small, competition for high types after entry leads to higher rent for high types. This relaxes the sorting constraint (i.e., the incentive compatibility constraint (4)) and makes informational rent consideration along the vertical dimension less important. As a result, the incumbent has less incentive to exclude low types or to pool the low types.

### 3.2 Entry Leads to The Removal of A Middle Contract

In this subsection we provide an analysis of the opposite case, which exhibits fully separating under monopoly but partial pooling under duopoly. In effect we will identify conditions under which entry leads to fewer contracts offered. We restrict attention to the case that $S^f_h < k < \frac{4}{3}S^f_h$.

**Monopoly.** Under monopoly, $u^f_h = \frac{S^f_h}{2}$, $u^f_m = \frac{S^f_m}{2}$, and $u^f_l = \frac{S^f_l}{2}$. When $u^f_m + (1 - \theta_m)q^f_m \leq u^f_h$, the DIC$_{hm}$ is slack under full information. When $u^f_m > (\theta_m - \theta_l)q$, it is always profitable to offer a low contract. Overall, we conclude that if conditions $S^f_h - S^f_m \geq 2(1 - \theta_m)q^f_m$ and $S^f_m > 2(\theta_m - \theta_l)q^f_l$ hold, it is optimal to offer three separate contracts under monopoly.\textsuperscript{29}

**Duopoly.** In order for partial pooling to arise in duopoly, we further restrict our attention to the case when $S^f_h \geq 2S^f_m$. In that case, $u^D_h = S^f_h - \frac{k}{2}$, $u^D_m = \frac{S^f_m}{2}$, and $u^D_l = \frac{S^f_l}{2}$. When $u^D_m + (1 - \theta_m)q^f_m > u^D_h$, DIC$_{hm}$ binds; when $u^D_m < u^D_l + (\theta_m - \theta_l)q^f_l$, the DIC$_{ml}$ also binds. Combining these two conditions, we have that if $\frac{k}{2} > S^f_h - \frac{S^f_m}{2} - (1 - \theta_m)q^f_m$ and $S^f_m - S^f_l < 2(\theta_m - \theta_l)q^f_l$, both DICs bind.

The analysis in this section is quite tedious. To ease exposition, we move most of the discussion to the Appendix.

**Proposition 5.** When $S^f_h < k < \frac{4}{3}S^f_h$, $S^f_h - S^f_m \geq 2(1 - \theta_m)q^f_m$, $S^f_m > 2(\theta_m - \theta_l)q$, $S^f_h \geq 2S^f_m$, and $S^f_m - S^f_l < 2(\theta_m - \theta_l)q^f_l$, if conditions (28), (29), (33), and (34) (listed in the Appendix) hold,\textsuperscript{28} TomTom unveils TomTom GO All-in-one navigation device is the easiest to use and most portable car navigation tool ever", PR Newswire, March 22, 2004, and "Quest(TM): Garmin’s new pocket-sized street navigator proof that big things come in small packages", PR Newswire, July 12, 2004.

\textsuperscript{29} Note that these two conditions together are stronger than the assumption $(1 - \theta_l)q < \frac{S^f_h}{2}$ as it is implied from these two conditions.
then the firm will offer three separate contracts under monopoly while partial pooling of middle and low types would take place under duopoly.

Proposition 5 shows that when $k$ is relatively large, $\alpha_m$ is sufficiently small, and type $m$ and type $l$ are fairly close to each other but rather far away from type $h$, then the incumbent monopolist responds to entry by removing the middle contract. The rough intuition is as follows. A relatively large $k$ makes the monopolist willing to give the high type a high rent in order to penetrate into its market. This means that the IC constraints in the vertical dimension are fairly relaxed, leading to a fully separating equilibrium under monopoly. On the other hand, a relatively large $k$ under duopoly leads to a lower rent for the high type, which makes the IC constraints in the vertical dimension more stringent. Given that $\alpha_m$ is sufficiently small, and type $m$ and type $l$ are fairly close to each other, entry makes it too costly for firms to offer a separate contract to the middle type, thus the middle contract of the incumbent is removed. We can easily choose $\alpha_h$ to satisfy the conditions in the proposition. One such choice, when $c(q) = q^2/2$, is the following combination of parameters: $k = 0.62$, $\theta_h = 1$, $\theta_m = 0.33$, $\theta_l = 0.32$, $q = 0.31$, and $\alpha_h = 0.62$, $\alpha_m = 0.10$, $\alpha_l = 0.28$.

The practice of removing some middle contracts (or middle-ranged quality product lines) in response to entry is also documented. For example, following the entry of Toyota into North American market, Buick reduced the number of its mid-size models offered from two (Special and Skylark) to one (Skylark) in 1970. Ford also reduced its mid-size model line from two models, Fairlane and Torino, to just one model, Torino. In response to competition from more downsized vehicles, in 1980 Ford cut its largest mid-size car, LTD II, from its product line.

4 Conclusion

In this paper, we study the effects of entry or increased competition on the product line or variety of contracts offered in a standard Salop circular city model, with both horizontally and vertically differentiated products. In our base model with two types of consumers, we show that when the degree of horizontal differentiation is low or the horizontal competition level is keen, entry will typically lead to the introduction of the lower end product under certain conditions; when the degree of horizontal differentiation is high or the horizontal competition level is weak, however, entry will typically lead to the removal of the lower end product under certain conditions. The

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conditions we identify are intuitive, as they can all be explained by the interactions between horizontal differentiation (competition) and vertical screening. The extension to three types of consumers further confirms the general insights obtained from our base model. Our analysis reveals an interesting pattern between fully separating and partial pooling equilibria, and offers an explanation for why incumbent firms may adjust the middle range of a product line (middle contracts) in response to competition.

Our results offer an alternative explanation for fighting brands and product pruning (Johnson and Myatt, 2003). Our paper differs from JM in the following aspects. First, in JM products are only vertically differentiated and firms compete in quantities, while in our model products are both vertically and horizontally differentiated and firms compete by offering a menu of contracts (or, equivalently, price-quality schedules). Second, in JM, whether fighting brand or product line pruning will occur depends on the shape of the marginal revenue curve, which in turn depends on the distribution of consumer types. In our model, it is the degree of horizontal differentiation (intensity of competition) that determines whether fighting brands or product line pruning will occur. Finally, in JM the changes in product line always happen in the low-end, while in our three-type model we demonstrate that the changes in product line (or contract variety) can occur at the middle range, which accounts for another type of product line adjustment in response to competition. Our result thus points to some more subtle effects of entry or increased competition on the product line or variety of contracts offered.

From both theoretical and practical points of view, it would be desirable to work out a more general model allowing for any finite number of types. However, doing so presents some technical difficulty, as the incentive comparability constraints along the vertical dimension will become too involved to analyze. While we believe that the main insights obtained from our current model is quite robust, further generalizing our analysis is left for future research.

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33 They also assume that the set of product qualities available to firms is fixed, which is not the case in our model.
34 Johnson and Myatt (2006) extend JM into n-firm setting, which is comparable to our n-firm case.
APPENDIX

Proof of Lemma 1: Let \( t \) be the transfer under the single contract. First consider the case \( q \in [q, q_{hf}^b) \). Suppose the monopolist introduces another contract targeting type \( h \): \( q_h = q_{hf}^b \) and \( t_h = t + (q_{hf}^b - q) \). By construction, it can be verified that \( u_h(q, t) = q - t = q_h - t_h = u_h(q_h, t_h) \). Thus type \( h \) will accept contract \( h \) and its market coverage does not change. On the other hand, \( u_l(q, t) = \theta_l(q - t) > \theta_l(q_h - t_h) = u_l(q_h, t_h) \). Hence type \( l \) will still buy the original contract and the firm's profit from type \( l \) does not change. However, the profit per consumer from type \( h \) increases under contract \( h \): under the original contract the profit margin is \( t - c(q) \), and under contract \( h \) it becomes \( t + (q_{hf}^b - q) - c(q_{hf}^b) \), which is strictly greater than \( t - c(q) \) because, due to the convexity of \( c(\cdot) \), we have

\[
\frac{c(q_{hf}^b) - c(q)}{q_{hf}^b - q} < c'(q_{hf}^b) = 1.
\]

Since the market share for type \( h \) remains the same, the introduction of contract \( h \) strictly increases the firm's profit.

Next consider the case \( q = q_{hf}^b \). Suppose the monopolist introduces another contract targeting type \( l \): \( q_l = q_{lf}^b \) and \( t_l = t - \theta_l(q - q_{lf}^b) \). By construction, type \( l \) is indifferent between the original contract and contract \( l \). Thus type \( l \) selects the \( l \) contract and the market share for type \( l \) does not change. It can be verified that type \( h \) prefers the original contract: \( (q - t) - (\theta_l - t_l) = (1 + \theta_l)(q_{hf}^b - q_{lf}^b) > 0 \). Thus type \( h \) will stick to the old contract and the profit from type \( h \) agents does not change. However, since

\[
\frac{c(q) - c(q_{lf}^b)}{q - q_{lf}^b} > c'(q_{lf}^b) = \theta_l,
\]

the profit margin from type \( l \) becomes higher:

\[
t - \theta_l(q - q_{lf}^b) - c(q_{lf}^b) > t - c(q).
\]

Therefore, the introduction of contract \( l \) strictly raises the firm's profit.

Proof of Lemma 2: First note that the optimal quality \( q_l \) must be less than \( q_{lf}^b \) (which can be easily verified by plugging \( q_l = q_{lf}^b \) into equation (6)).

Suppose two contracts are offered. Note that \( \theta_l q_l - c(q_l) \leq S_{lf}^b \). Therefore from condition (5) we have

\[
2u_l \geq S_{lf}^b - \frac{ka}{2(1 - \alpha)}.
\]
Therefore, if the following condition holds, the LHS of (6) (excluding $\lambda$) is negative and thus $q_l = q$ is binding:

$$a(1 - \theta_l) > \frac{1 - a}{k} \left[ S_{f}^{b} - \frac{ka}{2(1 - a)} \right] \left( \theta_l - c'(q) \right).$$

Rearrange and we have condition (7). Given that $q_l = q$, condition (8) implies that the LHS of (5) is negative. Therefore, $u_l = 0$ and the firm has no incentive to offer an $l$ contract.

**Proof of Lemma 3:** We only need to prove the second half of the result, as part (i) has been shown in the previous analysis.

First we find out the value of $q^\dagger$. From Case D2, we see that two contracts will be offered in equilibrium if either

$$q < \frac{2S_{f}^{b} - k}{2(1 - \theta_l)}$$

or

$$q < \frac{S_{h}^{l}}{3(1 - \theta_l)}.$$

In Case D3, we identify another condition: $q < \frac{k}{4(1 - \theta_l)}$. Combining these two cases, it is easy to see that if we set

$$q^\dagger = \max \left\{ \frac{2S_{f}^{b} - k}{2(1 - \theta_l), \frac{k}{4(1 - \theta_l)} \right\},$$

then two contracts will be offered when $q < q^\dagger$.

Now consider conditions (11) and (12). We first write down the first order conditions for the maximization problem stated in Case D2. Let the Lagrangian multipliers of the first and second constraints be $\mu_D$ and $\lambda_D$, respectively. The symmetric equilibrium is characterized by the following first-order conditions:

$$\begin{align*}
\alpha \left[ \frac{1}{2k} \left( S_{f}^{b} - u_{h}^{d} \right) - \frac{1}{4} \right] + \frac{1 - a}{k} \left[ \theta_l q_i^d - c(q_i^d) - 2u_i^d \right] + \mu_D &= 0; \mu_D \geq 0, \mu_D = 0 \text{ if } u_i^d > 0 \quad (15) \\
\alpha(1 - \theta_l) \left[ \frac{1}{2k} \left( S_{h}^{f} - u_{h}^{d} \right) - \frac{1}{4} \right] + \frac{1 - a}{k} u_i^d \left( \theta_l - c'(q_i^d) \right) + \lambda_D &= 0; \lambda_D \geq 0, \lambda_D = 0 \text{ if } q_i^d < q. \quad (16)
\end{align*}$$

Suppose two contracts are offered for both firms. Consider the first order condition (15) that characterizes the symmetric equilibrium. Note that in the LHS of condition (15), $\frac{1}{2k} (S_{f}^{b} - u_{h}^{d}) - \frac{1}{4} \leq \frac{S_{f}^{b}}{2k} - \frac{3}{8}$ since $u_{h}^{d} \geq u_{h}^{o} \geq \frac{k}{4}$. Therefore

$$2u_i^d \leq \frac{k \alpha}{1 - a} \left( \frac{S_{f}^{b}}{2k} - \frac{3}{8} \right) + \theta_l q_i - c(q_i)$$
\[
\leq \frac{k \alpha}{1 - \alpha} \left( \frac{S_h^f}{2k} - \frac{3}{8} \right) + S_l^f.
\]

As a result, the following condition ensures that the LHS of (16) is negative:

\[
\alpha(1 - \theta_l)(\frac{S_h^f}{k} - \frac{3}{4}) + \frac{1 - \alpha}{\alpha k} [\theta_l - c'(q)] [S_l^f - \frac{\alpha}{1 - \alpha} (\frac{3}{8} - \frac{S_h^f}{2k})] < 0,
\]

which means in equilibrium \( q^d_l = q \). Rearrange and we have condition (11). With \( q^d_l = q \), condition (12) ensures that the LHS of (15) is negative, thus \( u^d_l = 0 \).

**Proof of Proposition 2:** First we show that if the DIC does not bind (the full-information solution is feasible) under monopoly, then it does not bind under duopoly either. From the previous analysis, when \( k \leq 1/2 \), the DIC does not bind under monopoly if and only if

\[
(1 - \theta_l)\theta_l \leq \frac{1}{2} - \frac{\theta_l^2}{4}.
\]

(17)

On the other hand, when \( k \leq 1/2 \), the DIC does not bind under duopoly if and only if one of the following three conditions hold: (i) \( k \in (0, \frac{2}{3}\theta_l^2) \),

(ii) \( 1 - \frac{k}{2} - \frac{\theta_l^2}{4} \geq (1 - \theta_l)\theta_l \) if \( k \in [\frac{2}{3}\theta_l^2, \theta_l^2] \), and (iii) \( 1 - \frac{k}{2} - \frac{\theta_l^2}{4} \geq (1 - \theta_l)\theta_l \) if \( k \in [\theta_l^2, 1] \).

(18)

Comparing (17) and (18), we see that if (17) is satisfied then (18) must be satisfied. This result implies that whenever the DIC is slack under duopoly, we have \( q^m_l \leq q^d_l \).

What remains to be shown is that \( q^m_l \leq q^d_l \) when the DIC binds under both monopoly and duopoly. If \( q^m_l = q \), then \( q^d_l \geq q^m_l \) holds trivially. So we focus on the case that \( q^m_l > q \). Let the optimal solution under monopoly be \((q^m_l, u^m_l)\). Suppose \((q^d_l, u^d_l) = (q^m_l, u^m_l)\). We will compare the LHS of the first-order conditions (5) and (15) with both \( \mu \) and \( \mu_D \) being 0. Since \( u^d_h \leq 1/2 \) (the maximum social surplus of the high type), the first term in (5) is strictly less than that in (15). Given that (5) holds, the LHS of (15) must be strictly positive when \((q^d_l, u^d_l) = (q^m_l, u^m_l)\). Following the same procedure, we can show that the LHS of (16) is strictly positive when \((q^d_l, u^d_l) = (q^m_l, u^m_l)\). This means that each firm can increase its profit by offering \((q^d_l, u^d_l) > (q^m_l, u^m_l)\). This proves that \( q^m_l \leq q^d_l \) when DIC binds under both monopoly and duopoly.

**Proof of Proposition 3:** First consider the case \( k \in \left[0, \frac{4}{3}S_h^f\right] \). Note that the full-information \( u^D_h \) is increasing in \( k \). Moreover, as \( k \) becomes smaller, the DIC is less likely to bind. Therefore, what remains to be shown is that the result of part (i) holds when the DIC binds under both
Suppose two contracts are offered in the duopoly equilibrium under $k$. Then it must be the case that either (a) $(1-\theta_l)q < S^f_h - \frac{k}{2}$, or (b) $(1-\theta_l)q \geq S^f_h - \frac{k}{2}$, and (15) and (16) have a solution $(q^d_i(k), u^d_i(k))$ with $q^d_i(k) \in [q, q^f_i]$ and $u^d_i(k) > 0$. Consider case (a). Since $k' < k$, we also have $(1-\theta_l)q < S^f_h - \frac{k'}{2}$. Therefore, two contracts must be offered in the duopoly equilibrium under $k'$. Now consider case (b). Substituting $q^d_i(k)$ and $u^d_i(k)$ into (15) and (16) under $k'$, we have that the LHS of both (15) and (16) are strictly greater than 0, which is the LHS of FOC’s under $k$. Therefore, (15) and (16) have a solution $(q^d_i(k'), u^d_i(k'))$ with $q^d_i(k') \geq q^d_i(k) \geq q$ and $u^d_i(k') > u^d_i(k) > 0$, hence in the duopoly equilibrium under $k'$, two contracts must be offered and the quality distortion decreases. This proves part (i).

To show part (ii), we find two examples in which a decrease in $k$ leads to the introduction or removal of the low contract, respectively. Assume $c(q) = q^2/2$ and $k' \geq 2/3$. First, we provide an example in which the low contract is removed. Consider the parameter space such that the following conditions hold: $\frac{k'}{4} \leq (1-\theta_l)q < \frac{k}{4}$ and $\theta^2_l < \frac{\alpha}{1-\alpha}(\frac{3}{4}k' - \frac{1}{2})$. Then by Lemma 3, in the duopoly equilibrium under $k$, both contracts are offered and, in the duopoly equilibrium under $k'$, only the $h$ contract is offered. Thus a decrease in $k$ leads to removal of the low contract. Next, we provide an example in which the number of contracts increases. Consider the parameter space such that the following conditions hold: $(1-\theta_l)q \geq \frac{k}{4}, \theta^2_l \leq \frac{\alpha}{1-\alpha}(\frac{3}{4}k' - \frac{1}{2})$, and

$$\theta_l q - \frac{1}{2}q^2 > \frac{\alpha}{1-\alpha}(\frac{k}{8} + \frac{k'}{4} - \frac{1}{4}). \tag{19}$$

By Lemma 3, the first two conditions ensure that in the duopoly equilibrium under $k$ only the $h$ contract is offered. Now consider the LHS of (15) under $k'$. Condition (19) implies that when $q_l = q$ and $u^d_h = \frac{k}{4} > \frac{k'}{4} = u^d_h(k')$, the LHS is strictly greater than 0. Therefore, under $k'$ the equations (15) and (16) have a solution with $u^d_i > 0$. Hence two contracts must be offered in the duopoly equilibrium. Thus a decrease in $k$ leads to fighting brands.

**Proof of Lemma 4:** When $k \leq S^f_h$, type $h$ is fully covered. When $k \leq S^f_l$, the DIC's must bind. When $S^f_l < k \leq S^f_h$, the DIC$_{hm}$ must bind under full information, and the DIC$_{ml}$ binds if $\frac{k}{2} < S^f_h - (\theta_m - \theta_l)q^f_i$. When $k > S^f_m$, the DIC$_{hm}$ binds if $\frac{k}{2} < S^f_h - (1-\theta_m)q^f_m$. Similarly, the DIC$_{ml}$ binds if $S^f_m - S^f_i < 2(\theta_m - \theta_l)q^f_i$. Therefore, a set of sufficient conditions for both DIC's to bind is $S^f_m - S^f_i < 2(\theta_m - \theta_l)q^f_i$ and $\frac{k}{2} < S^f_h - (1-\theta_m)q^f_m$. We hence maintain these two assumptions in this subsection.

We first look at the case of fully separating contracts. Similar to the base model, it is not

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35To offer separating contracts, $u_m$ must increase which makes DIC$_{hm}$ even more binding.

36When DIC$_{hm}$ binds, to offer separating contracts, $u_m$ needs to be reduced which makes DIC$_{ml}$ more binding.
possible that the low contract covers the whole market. If \( \theta_m \) is close enough to \( \theta_l \), then the middle type will not be fully covered either. The monopolist’s programming program is given by:

\[
\begin{align*}
\max_{(u_l, q_l; q_m)} & \quad \frac{\alpha_h}{2} \left[ S_h^b - u_l - (\theta_m - \theta_l)q_l - (1 - \theta_m)q_m \right] + \alpha_m \frac{u_l + (\theta_m - \theta_l)q_l}{k} \left( \theta_m q_m - c(q_m) - u_m \right) \\
& + \alpha_l \frac{u_l}{k} (\theta_l q_l - c(q_l) - u_l) \\
\text{subject to: } & u_l \geq 0; \ q_m \geq q; \ q_l \geq q
\end{align*}
\]

Let the Lagrangian multipliers of the three constraints be \( \mu, \lambda_m, \) and \( \lambda_l \) respectively. The FOCs are as follows:

\[
\begin{align*}
- \frac{\alpha_h}{2} + \frac{\alpha_m}{k} (\theta_m q_m - c(q_m) - 2u_m) + \frac{\alpha_l}{k} (\theta_l q_l - c(q_l) - 2u_l) + \mu &= 0, \\
& \quad \mu \geq 0, \ \mu = 0 \text{ if } u_l > 0; \quad (20) \\
- \frac{\alpha_h}{2} (1 - \theta_m) + \frac{\alpha_m}{k} u_m (\theta_m - c'(q_m)) + \lambda_m &= 0, \\
& \quad \lambda_m \geq 0, \ \lambda_m = 0 \text{ if } q_m > q; \quad (21) \\
- \frac{\alpha_h}{2} (\theta_m - \theta_l) + \frac{\alpha_m}{k} (\theta_m q_m - c(q_m) - 2u_m) (\theta_m - \theta_l) + \frac{\alpha_l}{k} u_l (\theta_l - c'(q_l)) + \lambda_l &= 0, \\
& \quad \lambda_l \geq 0, \ \lambda_l = 0 \text{ if } q_l > q. \quad (22)
\end{align*}
\]

From (21), we can see that if \( \alpha_h (1 - \theta_m) \geq \alpha_m (\theta_m - c'(q)) \), then \( \lambda_m > 0 \) and \( q_m = q \) (since \( u_m/k \leq 1/2 \)). Therefore, \( \alpha_h (1 - \theta_m) \geq \alpha_m (\theta_m - c'(q)) \) implies that fully separating is not optimal. Moreover, if \( h \) and \( m \) contracts are offered only, \( q_m = q \).

Now consider the case of partial pooling (types \( m \) and \( l \) pool at the low contract). The programming problem is as follows:

\[
\begin{align*}
\max_{(u_l, q_l; q_m)} & \quad \frac{\alpha_h}{2} \left[ S_h^b - u_l - (1 - \theta_l)q_l \right] + \left[ \alpha_m \frac{u_l + (\theta_m - \theta_l)q_l}{k} + \alpha_l \frac{u_l}{k} \right] (\theta_l q_l - c(q_l) - u_l) \\
\text{subject to: } & u_l \geq 0; \ q_l \geq q
\end{align*}
\]

The FOCs are as follows:

\[
\begin{align*}
- \frac{\alpha_h}{2} + \frac{\alpha_m}{k} [\theta_l q_l - c(q_l) - u_l - u_m] + \frac{\alpha_l}{k} [\theta_l q_l - c(q_l) - 2u_l] + \mu &= 0, \\
& \quad \mu \geq 0, \ \mu = 0 \text{ if } u_l > 0; \quad (23) \\
- \frac{\alpha_h}{2} (1 - \theta_l) + \frac{\alpha_m}{k} (\theta_m - \theta_l) [\theta_l q_l - c(q_l) - u_l] + \left[ \frac{\alpha_m}{k} u_m + \frac{\alpha_l}{k} u_l \right] (\theta_l - c'(q_l)) + \lambda &= 0,
\end{align*}
\]
\[ \lambda \geq 0, \lambda = 0 \text{ if } q_l > q. \quad (24) \]

From (24), we can see that if \( \alpha_h(1 - \theta_l) \geq \frac{2a_m}{k}(\theta_m - \theta_l)S^b_l + (1 - \alpha_h)(\theta_l - c'(q)), \) then \( \lambda > 0 \) and \( q_l = q. \)

To establish that partial pooling is optimal, we must show that partial pooling dominates exclusion (that is, offering an \( h \) contract only or only offering \( h \) and \( m \) contracts). Offering an \( h \) contract alone leads to a (half) profit of

\[ \pi_h = \alpha_h^2 \left( S^h_f - \frac{k}{2} \right). \]

When offering \( h \) and \( m \) contracts only, the optimal \( u_m = \frac{1}{2}(\theta_m q - c(q) - \frac{\alpha_h k}{2}). \) Let the corresponding profit be \( \pi_{hm}. \) If

\[ \frac{\alpha_h}{a_m} \geq \frac{2}{k}(\theta_m q - c(q)), \quad (25) \]

then \( u_m \leq 0, \) which means that \( \pi_{hm} < \pi_h. \) In the case of partial pooling, the optimal \( u_l \) is given by

\[ u_l = \frac{1}{2} \left[ \theta_l q - c(q) - \frac{\alpha_h k}{1 - \alpha_h} - \frac{a_m}{1 - \alpha_h}(\theta_m - \theta_l)q \right]. \]

Let the corresponding total profit be \( \pi_{h(ml)}. \) For all \( u_l, \) we must have

\[ \pi_{h(ml)} - \pi_h > - \frac{\alpha_h}{2} \left[ u_l + (1 - \theta_l)q - \frac{k}{2} \right] + (1 - \alpha_h) \frac{u_l}{k} \left( \theta_l q - c(q) - u_l \right) \equiv f(u_l). \]

If the maximizer \( u^{*}_l \geq 0, \) that is

\[ \theta_l q - c(q) \geq \frac{\alpha_h k}{1 - \alpha_h} \]

then if

\[ f(u^{*}_l) = (1 - \alpha_h) \frac{(\theta_l q - c(q))^2 - \left( \frac{\alpha_h k}{2} \right)^2}{4k} - \frac{\alpha_h}{2} \left[ \frac{1}{2}(\theta_l q - c(q) - \frac{\alpha_h k}{2}) + (1 - \theta_l)q - \frac{k}{2} \right] > 0, \quad (26) \]

we have \( \pi_{h(ml)} > \pi_h. \)

If \( u^{*}_l \geq 0, \) then we plug \( u_l = 0 \) and require

\[ (1 - \theta_l)q < \frac{k}{2} \]

so that \( \pi_{h(ml)} > \pi_h. \)

**Proof of Proposition 5:** Now suppose both firms offer three separate contracts. The problem becomes:
max $\alpha_h \left[ \frac{1}{4} + \frac{u_i + (\theta_m - \theta_l)q_l + (1 - \theta_m)q_m - u_h^I}{2k} \right] \left( S_f^b - u_l - (\theta_m - \theta_l)q_l - (1 - \theta_m)q_m \right) + \alpha_m \frac{u_i + (\theta_m - \theta_l)q_l}{k} \left[ \theta_m q_m - c(q_m) - u_l - (\theta_m - \theta_l)q_l \right] + \alpha_l \frac{u_i}{k} \left[ \theta_l q_l - c(q_l) - u_l \right]$

s.t. $u_l \geq 0, q_m \geq q, q_l \geq q$

Here $u_h^I$ denotes the utility offered to the high type by the other firm.

The LHS (excluding $\lambda_m$) of the FOC for $q_m$ is:

$$(u_l + (\theta_m - \theta_l)q_l) \left[ \frac{\alpha_m}{k} (\theta_m - c'(q_m)) - \alpha_h \frac{1 - \theta_m}{2k} \right] + \alpha_h \frac{1 - \theta_m}{2k} \left[ S_f^b - \frac{k}{2} (1 - \theta_m)q_m \right].$$

If

$$\frac{\alpha_m}{k} (\theta_m - c'(q)) - \alpha_h \frac{1 - \theta_m}{2k} < 0,$$

and

$$S_f^b - \frac{k}{2} (1 - \theta_m)q < 0,$$

then $q_m = q$, which means that fully separating is not optimal.

Next consider the case where partial pooling occurs. The problem now becomes:

$$\max_{(u_l, q_l)} \alpha_h \left[ \frac{1}{4} + \frac{u_i + (1 - \theta_l)q_l - u_h^I}{2k} \right] \left( S_f^b - u_l - (1 - \theta_l)q_l \right) + \alpha_m \frac{u_i + (\theta_m - \theta_l)q_l}{k} + \alpha_l \frac{u_i}{k} \left[ \theta_l q_l - c(q_l) - u_l \right]$$

s.t. $u_l \geq 0, q_m \geq q, q_l \geq q$

The FOC for $u_l$ is as follows:

$$\frac{\alpha_h}{2k} \left[ S_f^b - \frac{k}{2} (1 - \theta_l)q_l \right] + \frac{1 - \alpha_h}{k} (\theta_l q_l - c(q_l)) - \frac{\alpha_m}{k} (\theta_m - \theta_l)q_l - \left[ \frac{\alpha_h}{2k} + 2 \frac{1 - \alpha_h}{k} \right] u_l + \mu = 0$$

And the FOC for $q_l$ is:

$$\alpha_h \frac{1 - \theta_l}{2k} \left[ S_f^b - \frac{k}{2} (1 - \theta_l)q_l \right] + \alpha_m \frac{\theta_m - \theta_l}{k} (2\theta_l q_l - c(q_l) - q_l c'(q_l))$$

$$+ \left[ \frac{1 - \alpha_h}{k} (\theta_l - c'(q_l)) - \alpha_h \frac{1 - \theta_l}{2k} - \frac{\theta_m - \theta_l}{k} \right] u_l + \lambda = 0$$

If

$$(1 - \alpha_h)(\theta_l - c'(q_l)) - \frac{\alpha_h}{2} (1 - \theta_l) - \alpha_m (\theta_m - \theta_l) < 0,$$

which is equivalent to:

$$(\theta_l - c'(q_l)) - \alpha_h \left( \frac{\theta_l + 1}{2} - c'(q_l) \right) < \alpha_m (\theta_m - \theta_l),$$

(29)
then the LHS of the FOC for \( q \) is less than or equal to
\[
\alpha_h \frac{1-\theta_l}{2k} \left[ S_h^f - \frac{k}{2} - (1-\theta_l)q \right] + \alpha_m \frac{\theta_m - \theta_l}{k} \left( 2\theta_l q - c(q) - q c'(q) \right).
\]

Define \( A = \alpha_h \frac{1-\theta_l}{2} \) and \( B = \alpha_m (\theta_m - \theta_l) \). Then the above expression is decreasing in \( q \) if \( 2\theta_l B - (1-\theta_l)A < 0 \), or more explicitly,
\[
4\theta_l (\theta_m - \theta_l) \alpha_m < (1-\theta_l)^2 \alpha_h. \tag{30}
\]

Therefore, if condition (30) and the following condition hold,
\[
\alpha_h \frac{1-\theta_l}{2k} \left[ S_h^f - \frac{k}{2} - (1-\theta_l)q \right] + \alpha_m \frac{\theta_m - \theta_l}{k} \left( 2\theta_l q - c(q) - q c'(q) \right) < 0, \tag{31}
\]
then we have \( q = q \).

Next we will compare the expected profit from partial pooling with those from offering high contract only and offering both high and middle contracts.

If only the high contract is offered, the expected profit would be \( \pi_h = \frac{\alpha_h k}{8} \). If both high and middle contracts are offered, \( q_m = q \), the LHS (excluding multiplier) of the FOC for \( u_m \) is
\[
\alpha_h \frac{1-\theta_l}{2k} \left[ S_h^f - \frac{k}{2} - u_m - (1-\theta_m)q \right] + \alpha_m \frac{\theta_m - \theta_l}{k} \left( \theta_l q - c(q) - 2u_m \right).
\]

If
\[
\frac{\alpha_h}{2} \left( S_h^f - \frac{k}{2} - (1-\theta_m)q \right) + \alpha_m (\theta_m q - c(q)^2) < 0, \tag{32}
\]
then the optimal \( u_m = 0 \) which means that it is not profitable to offer a middle contract along with a high contract.

With partial pooling, we denote expected profit as \( \pi_{h(m)} \). From the previous discussion, we know that
\[
\pi_{h(m)} - \pi_h > \frac{\alpha_h}{4} \left[ S_h^f - \frac{k}{2} - u_l - (1-\theta_l)q \right] + (1-\alpha_h) \frac{u_l}{k} \left( \theta_l q - c(q) - u_l \right).
\]

When
\[
\max g(u_l) = \frac{1-\alpha_h}{4k} \left( \theta_l q - c(q)^2 \right) - \left( \frac{\alpha_h k}{4(1-\alpha_h)} \right)^2
\]

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partial pooling is optimal in duopoly.

We can simplify the conditions a little bit. First, condition (28) implies that \( \frac{k}{2} > S_{fb}^{h} - S_{fb}^{m} - (1 - \theta_m)q_m \), which is one sufficient condition for binding DICs. Second, condition (27), (30), (31), and (32) are all about the proportions of high type and middle type, and they can be summarized by the following condition:

\[
\alpha_h > \delta \alpha_m,
\]

where

\[
\delta = \max \left\{ \frac{2(\theta_m - c'(q))}{1 - \theta_m}, \frac{4\theta_l(\theta_m - \theta_l)}{(1 - \theta_l)^2}, \frac{2(\theta_m - \theta_l)(2\theta_lq - c(q) - qc'(q))}{(1 - \theta_l)(1 - \theta_l)q - S_{fb}^{h} + \frac{k}{2}}, \frac{2(\theta_m q - c(q))}{(1 - \theta_m)(1 - \theta_l)q - S_{fb}^{h} + \frac{k}{2}} \right\}.
\]
REFERENCES


