Topological Quillen completion and localization of structured ring spectra

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Structured ring spectra are symmetric spectra with algebraic structures described as algebras over an operad $\mathcal{O}$, where $\mathcal{O}[0] = \ast$; such $\mathcal{O}$-algebras are non-unital. Topological Quillen homology (TQ-homology) naturally arises as the topological analog of Quillen homology. The canonical truncation of operad map $\mathcal{O} \to \tau_1 \mathcal{O}$ induces change of operads adjunction

$$\begin{array}{c}
\text{Alg}_\mathcal{O} \\ Q
\end{array} \leftrightarrow \begin{array}{c}
\text{Alg}_{\tau_1 \mathcal{O}} = \text{Mod}_{\mathcal{O}[1]}
\end{array} \quad U$$

with left adjoint on top, where $Q(X) := \tau_1 \mathcal{O} \circ \mathcal{O} (X)$ and $U$ is the forgetful functor.

**Definition**

Let $X$ be a cofibrant $\mathcal{O}$-algebra, its TQ-homology is $TQ(X) := UQ(X)$. The unit of the adjunction $(Q, U)$ is the TQ-Hurewicz map $X \to UQX = TQ(X)$.

For example, if $\mathcal{O}$ is the operad whose algebras are the non-unital commutative ring spectra, then $TQ(X) \simeq X/X^2$. 

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Question

Is it possible to recover an $\mathcal{O}$-algebra $X$ from its TQ-homology $TQ(X)$?

One can iterate the TQ-Hurewicz map to form a cosimplicial resolution of $X$

$$
X \rightarrow TQX \rightarrow TQ^2X \rightarrow TQ^3X \rightarrow \cdots
$$

Taking homotopy limit gives the TQ-completion map

$$
c: X \rightarrow X_{TQ}^\wedge
$$

Conjecture (Francis-Gaitsgory, 2012)

If $X$ is a homotopy pro-nilpotent $\mathcal{O}$-algebra, then the TQ-completion map $c: X \rightarrow X_{TQ}^\wedge$ is a weak equivalence.
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If $X$ is a homotopy pro-nilpotent $O$-algebra, then the TQ-completion map $c : X \rightarrow X^\wedge_{TQ}$ is a weak equivalence.

Definition

Let $X$ be an $O$-algebra and $n \geq 2$. We say that $X$ is $n$-nilpotent if all the $n$-ary and higher operations $O[t] \wedge X^\wedge t \rightarrow X$ are trivial (i.e., if these maps factor through $\ast$ for each $t \geq n$). We say that $X$ is nilpotent if there exists some $n \geq 2$ so that $X$ is $n$-nilpotent. We say that $X$ is homotopy pro-nilpotent if $X$ is the homotopy limit of a tower of nilpotent $O$-algebras.

For example, if $X$ is a non-unital commutative ring spectra, then $X/X^n$ is $n$-nilpotent. Nilpotent $O$-algebras are obviously homotopy pro-nilpotent.

Theorem

Assume that $O$ is $(-1)$-connected and $X$ is a 0-connected $O$-algebra.


Ching-Harper (2019): $c : X \rightarrow X^\wedge_{TQ}$ is a weak equivalence.
Definition

A fibrant $\mathcal{O}$-algebra $X$ is TQ-local if every TQ-homology equivalence $f : A \to B$ between cofibrant objects induces a weak equivalence $\text{Hom}(B, X) \to \text{Hom}(A, X)$ on mapping spaces in sSet.

Proposition

Let $X, Y$ be TQ-local $\mathcal{O}$-algebras. Then a map $X \to Y$ in $\text{Alg}_{\mathcal{O}}$ is a weak equivalence if and only if it is a TQ-homology equivalence.

Theorem (Harper-Z, 2019)

Let $X$ be a cofibrant $\mathcal{O}$-algebra, then there is a natural TQ-homology equivalence of the form $l : X \to L_{TQ}X$ with TQ-local codomain.
**Proposition**

Let $X$ be a cofibrant $O$-algebra. Then the $TQ$-completion map $c: X \to X_{TQ}^\wedge$ factors through the $TQ$-localization map $l: X \to L_{TQ}X$ in $\text{Alg}_O$

Furthermore, if $c$ is a weak equivalence, then $l, \xi$ are both weak equivalences.

**Theorem (Z)**

Let $X$ be a homotopy pro-nilpotent $O$-algebra. Then the $TQ$-localization map $l: X \to L_{TQ}X$ is a weak equivalence.

**Theorem (Z)**

A map $X \to Y$ between homotopy pro-nilpotent $O$-algebras is a weak equivalence if and only if it is a $TQ$-homology equivalence.